BAYESIAN INFERENCE FOR A MULTIVARIATE
FAILURE DISTRIBUTION

BY

S. J. PRESS AND A. A. SHAMSOLDIN

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Abstract

The purpose of this paper is to develop Bayesian statistical inferences for a \((K+1)\)-parameter version of the MVE distribution of Marshall and Olkin (1967) to be applied as a failure model for multicomponent parallel systems. Our concern will be in estimating the MVE parameter vector by adopting the Bayesian method.

A quadratic loss function is used in deriving optimal closed form estimators for the MVE parameters in many situations of prior information. Approximated confidence intervals for the individual parameters are developed by utilizing a gamma approximation to marginal posterior densities.
1. Introduction

Recently, it has been realized that common cause failures can decrease or eliminate the usefulness of any degree of redundancy employed in a system (Eppler, 1969; Apostolakis, 1976). In engineering applications, failure of more than one component simultaneously due to a single cause is identified as a common cause failure and single component failure is called chance failure (Vesely, 1977; Apostolakis, 1976). The redundancy principle is extensively used in the study of safety of complex systems (for example, the redundant channels of a protection system of a nuclear power plant). Precautions, however, should be taken for the incidence of a catastrophic failure, the most critical common cause failure that destroys all system components simultaneously, e.g., fires, earthquakes, human errors, etc.

The considered (k+l)-parameter MVE is applicable as a failure model for complex systems when there exists positive probability for a catastrophic failure in addition to the chances for individual component failure. Individual and simultaneous failures are assumed to be governed by Poisson shocks.

Apostolakis (1976) has investigated the importance of common cause failure on reliability of redundant systems with applications on some nuclear engineering safety problems. Vesely (1977) has suggested classical estimation techniques for estimating common failure probabilities in reliability and risk analysis, when the failure distribution is the MVE of Marshall and Olkin. Other related studies in which the MVE distribution is being recommended as a promising multivariate failure model may be found in WASH-1400; (1975).

Proschan and Sullo (1976) have studied the classical maximum likelihood method for estimating the parameters of MVE \((k+l|\lambda)\) distribution. Since the
MLE estimator can not be derived in a closed form, they proposed another estimator called INT, derived from an intuitive principle. They also suggested the INT estimator to be the first iterate in an iteration method used for solving the nonlinear likelihood equations. They realized, however, that in some sampling circumstances the MLE estimator does not exist or is not uniquely determined. It is also noted that when the MLE estimator for the parameter vector exists, classical confidence intervals for the individual parameters cannot be constructed. The above difficulties encountered in the maximum likelihood method, in addition to the general plausibility features of the Bayesian approach, directed our attention toward the present study.

2. **Problem Statement**

A major system consists of k-parallel subsystems (or components) which are subjected to exponential failures, the individual (chance) failure associated with each component and the catastrophic failure that simultaneously destroys all components. Let \( T_i \) represent the random variable of the time to failure of component \( i, i = 1, 2, \ldots, k \) and let \( T = (T_1, T_2, \ldots, T_k) \) be a \((k+1)\)-parameter MVE random vector representing the system failure behavior. MVE \((k+1|\lambda)\) distribution is characterized by the conditional joint survival probability function

\[
P(T_1 > t_1, T_2 > t_2, \ldots, T_k > t_k | \lambda) = \bar{F}(t_1, t_2, \ldots, t_k | \lambda)
= \exp\left[-\sum_{i=1}^{k} \lambda_i t_i - \lambda_0 \max(t_1, t_2, \ldots, t_k)\right],
\]

\[t_i \geq 0, \; i = 1, 2, \ldots, k,
\]  

(2.1)
where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_0)$ is a $(k+1)$-dimensional vector representing a realization (a random determination) of the stochastic parameter vector $A = (A_1, A_2, \ldots, A_k, A_0)$ defined on the set $\Omega$ given by $\Omega = \{\lambda: 0 \leq \lambda_i < \infty, i = 0, 1, \ldots, k; \lambda_0 + \lambda_j > 0, j = 1, 2, \ldots, k\}$.

Although the MVE parameter vector is treated as a random vector in the general Bayesian context, life testing observations are assumed to correspond to a particular unknown realization $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_0)$.

The problem to be studied is that of developing Bayesian estimates for the MVE parameter vector.

3. **Likelihood Function**

Let $\{T_{ij} = T_{1j}, T_{2j}, \ldots, T_{kj}\}_{i,j}^n$ denote a random sample of size $n$ from MVE $(k+1 | \lambda)$ distribution and let $\{t_j\}_{j=1}^n$ represent the corresponding set of sample values. In the current situation of parallel sampling, the systems on test are observed until all $kn$ components have failed. The random variables $\min(T_1, T_2, \ldots, T_k)$ and the $\max(T_1, T_2, \ldots, T_k)$ are observable.

All sources of component failures except the last one can be always identified. The last failure, however, is identifiable only if it is a more than one component failure, and in this case it will be recognized as a catastrophic failure. As in the two dimensional case, the difficulty in identifying the last failure is transferred into the likelihood function in the form of a superposition of two Poisson processes.

To count frequencies of all modes of failure in a life time testing of the $nk$-component parallel systems, we define the following counting statistics: For $i = 1, 2, \ldots, k$, let $N_i$ be the number of times component $i$ is observed to fail strictly before the last failure and $N_i^c$ be the number of times
component \( i \) is observed to fail last but not simultaneously with any other component. Let \( N_0 \) be the number of times simultaneous failure of at least two components is observed. Sullo (1973) has shown that \( D = (\frac{n}{j=1} T(k)j, \sum_{j=1}^{n} T_{ij}, N_1, N_0^c; i = 1, \ldots, k) \) constitute a set of minimal sufficient statistics for MVE \((k+1|\lambda)\). Let \( d = (\sum_{j=1}^{n} t(k)j, \sum_{j=1}^{n} t_{ij}, n_1, n_0^c; i = 1, \ldots, k) \) be the corresponding set of experimental values of \( D \). The likelihood function as a function of \( \lambda \) given \( d \), can be written in the form

\[
\ell(\lambda|d) = \lambda_0^{n_0} \prod_{i=1}^{k} \lambda_i^{n_i}(\lambda_i + \lambda_0)^{n_i^c} \cdot \exp\left[-\sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i t_{ij} - \lambda_0 \sum_{j=1}^{n} t(k)j\right]. \tag{3.1}
\]

4. Prior Distributions

4.1 Non-informative prior

We assume that our knowledge about \( \Lambda \) is only vague and use the non-informative prior:

\[
g(\Lambda) \propto \frac{1}{k^{\prod_{i=0}^{k} \lambda_i}} \tag{4.1.1}
\]

In this prior, it assumed that \( \Lambda_1, \Lambda_2, \ldots, \Lambda_k \) and \( \Lambda_0 \) are independently distributed and each has the logarithmic uniform density that satisfies the invariance principle stated in Villegas (1977).

Now, after the assessment of a prior density, the joint posterior distribution of the parameters \( \Lambda_1, \ldots, \Lambda_k, \Lambda_0 \) can be derived by applying Bayes theorem. Let \( \Pi(\Lambda|d) \) represent the joint posterior density for \( \Lambda|d \).
The kernel of the likelihood function (3.1) can be written as

$$
\frac{n_0^k \prod_{i=1}^{k} \frac{n_i}{\lambda_i} (\lambda_i + \lambda_0) n_i^c}{\exp[-\sum_{i=1}^{k} \lambda_i \tau_i - \lambda_0 \tau_0]},
$$

(4.1.2)

where $\tau_i = \frac{n}{\sum_{j=1}^{n} t_{ij}}$ and $\tau_0 = \frac{n}{\max(t_{1j}, t_{2j}, \ldots, t_{kj})}$.

By Bayes' theorem, the joint posterior density is proportional to the likelihood kernel times the joint prior density. Therefore,

$$
\Pi_1(\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_0 | d) \propto \frac{n_0^{-1} k \prod_{i=1}^{k} \frac{n_i}{\lambda_i} (\lambda_i + \lambda_0) n_i^c}{\exp[-\sum_{i=1}^{k} \lambda_i \tau_i - \lambda_0 \tau_0]},
$$

(4.1.3)

where the index 1 stands for the non-informative prior case.

The constant of proportionality omitted from the above posterior density is denoted by $c_1(d)$ and is given by

$$
c_1(d) = \int_0^\infty \ldots \int_0^\infty \frac{n_0^{-1} k \prod_{i=1}^{k} \frac{n_i}{\lambda_i} (\lambda_i + \lambda_0) n_i^c}{\exp[-\sum_{i=1}^{k} \lambda_i \tau_i - \lambda_0 \tau_0]} d\lambda_1 d\lambda_2 \ldots d\lambda_k d\lambda_0}^{-1}.
$$

(4.1.4)

To find the constant term $c_1(d)$ required for $\Pi_1(\lambda | d)$ to integrate to one, we first integrate with respect to $\lambda_1$, then with respect to $\lambda_2$, and so on to $\lambda_k$, finally integrating with respect to $\lambda_0$.

The joint posterior density of $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_0)$ with respect to the non-informative prior given by (4.1.1) can be obtained by generalizing
the method used in the two dimensional case (see Shamseldin and Press, 1981), the result is

$$
\Pi_1(\lambda|d) = c_1(d) \sum_{s=0}^{\infty} \left( \sum_{i_0 > 0} \sum_{i_1 \geq 0} \ldots \sum_{i_k \geq 0} \prod_{s=0}^{i_0} \prod_{i=1}^{k} \lambda_i^{n_i + i_i - 1} \mu_i^d \tau_0^{n_0} \right) \lambda_0^{n-s-1} e^{-\tau_0 \lambda_0}, \lambda \in \Omega^+
$$

(4.1.5)

where $\sigma_n^c = n_1^c + n_2^c + \ldots + n_k^c$, $i_0 \geq 0 \equiv i_1, i_2, \ldots, i_k \geq 0$, $\sigma_i = s \equiv i_1 + i_2 + \ldots + i_k = s$ and $c_1(d)$ is given by

$$
c_1(d) = \sum_{s=0}^{\infty} \left( \sum_{i_0 > 0} \sum_{i_1 \geq 0} \ldots \sum_{i_k \geq 0} \prod_{s=0}^{i_0} \prod_{i=1}^{k} \frac{\Gamma(n_i + i_i)}{n_i^c + i_i} \frac{\Gamma(n-s)}{\tau_0^{n-s}} \right).
$$

(4.1.6)

The index $1$, in $\Pi_1(\lambda|d)$ and $c_1(d)$, refers to the non-informative prior given by equation (4.1.1).

It is clear now that $\Pi_1(\lambda|d)$ given by (4.1.5) is a convex combination of $(k+1)$th-product of gamma densities. It is also noted that integrating with respect to an arbitrary set of $\lambda$'s would also result in a marginal density of a convex combination of products of gamma densities, with the same weighting coefficients as the original joint posterior density. Therefore, any univariate marginal posterior density of an individual parameter has a mixture type of gamma densities with the same scale parameter. This density is complicated in form for numerical purposes. Fortunately, however, there is an excellent approximation available (see Section 5.2) matching moments to a single gamma density.
The marginal posterior density of $\Lambda_0$ is developed by integrating out $\lambda_1, \lambda_2, \ldots, \lambda_k$. Let $\Pi_{1i}^m(\lambda_0 | d)$ denote the marginal posterior density of $\lambda_i$, $i = 0, 1, \ldots, k$. Thus,

$$\Pi_{10}^m(\lambda_0 | d) = c_1(d) \sum_{s=0}^{\infty} \sum_{j \geq 0} \Gamma(n_{j} + i_j) \frac{\Gamma(n_{j} + i_j)}{n_{j} - i_j} \lambda_0^{n_s - 1} e^{-\lambda_0 \lambda_0} \left(0 < \lambda_0 < \infty\right).$$

(4.1.7)

The marginal posterior density of $\Lambda_j; j \neq 0$, can be similarly derived.

The result is

$$\Pi_{1j}^m(\lambda_j | d) = c_1(d) \sum_{s=0}^{\infty} \sum_{i=0}^{\infty} \frac{\Gamma(n_j + i_j)}{n_j - i_j} \lambda_j^{n_s - 1} e^{-\lambda_j \lambda_j} \left(0 < \lambda_j < \infty\right),$$

(4.1.8)

where $L_j = \{1, 2, \ldots, j-1, j+1, \ldots, k\}$.

These marginal densities are in the same family as the joint density, namely, a convex mixture of gamma densities with the same scale parameters.

For a squared-error loss function, the Bayes estimator of $\lambda_j$, $j=0, 1, \ldots, k$, is the corresponding marginal posterior mean. So for the parameter $\lambda_0$, the marginal posterior mean is given by
\begin{align*}
\tilde{\lambda}_0 &= E(\Lambda_0 | \mathbf{d}) = \int_0^\infty \lambda_0 \prod_{i=0}^m \lambda_0 | \mathbf{d} \, d\lambda_0 \\
&= c_1(\mathbf{d}) \sum_{s=0}^{\sigma n^c} \left[ \sum_{i=0}^{k} \left( \prod_{l=1}^{n_i} \frac{\Gamma(n_l + i_j)}{n_l + i_j} \right) \frac{\Gamma(n_{s-l} + i_j + \delta^j)}{\tau_{s-l} \tau_0} \right] , 
\end{align*}

where \( c_1(\mathbf{d}) \) is given by (4.1.6).

Moreover, for \( \lambda_j, j=1,2,\ldots,k \).

\begin{align*}
\tilde{\lambda}_j &= E(\Lambda_j | \mathbf{d}) = c_1(\mathbf{d}) \sum_{s=0}^{\sigma n^c} \left[ \sum_{i=0}^{k} \left( \prod_{l=1}^{n_i} \frac{\Gamma(n_l + i_j + \delta^j)}{n_l + i_j + \delta^j} \right) \frac{\Gamma(n_{s-l})}{\tau_{s-l} \tau_0} \right] , 
\end{align*}

where \( \delta^j \) is defined by

\begin{align*}
\delta^j = \begin{cases} 
1 & \text{if } l=j \\
0 & \text{otherwise} 
\end{cases} .
\end{align*}

The marginal posterior variance of \( \Lambda_i | \mathbf{d}, i=0,1,\ldots,k \), can also be computed. For the variance of \( \Lambda_0 | \mathbf{d} \), let us first evaluate \( E(\lambda_0^2 | \mathbf{d}) \). Thus,

\begin{align*}
E(\lambda_0^2 | \mathbf{d}) &= \int_0^\infty \lambda_0^2 \prod_{i=0}^m \lambda_0 | \mathbf{d} \, d\lambda_0 \\
&= c_1(\mathbf{d}) \sum_{s=0}^{\sigma n^c} \left[ \sum_{i=0}^{k} \left( \prod_{l=1}^{n_i} \frac{\Gamma(n_l + i_j)}{n_l + i_j} \right) \frac{\Gamma(n_{s-l} + i_j + \delta^j)}{\tau_{s-l} \tau_0} \right] ,
\end{align*}

and so

\begin{align*}
\text{Var}(\Lambda_0 | \mathbf{d}) &= c_1(\mathbf{d}) \sum_{s=0}^{\sigma n^c} \left[ \sum_{i=0}^{k} \left( \prod_{l=1}^{n_i} \frac{\Gamma(n_l + i_j)}{n_l + i_j} \right) \frac{\Gamma(n_{s-l} + i_j + \delta^j)}{\tau_{s-l} \tau_0} \right] - \tilde{\lambda}_0^2 .
\end{align*}
In a similar way, the marginal posterior variance of $\Lambda_j$, $j = 1, 2, \ldots, k$ can be obtained. The result is

$$
\text{Var}(\Lambda_j | d) = c_1(d) \sum_{s=0}^{s_n} \left[ \sum_{i=0}^{k} \frac{\Gamma(n_s+i_j+2\delta)}{\Gamma(n_s+i_j+\delta)} \right] \frac{\Gamma(n-s)}{\tau_0}.
$$

(4.1.14)

4.2 Informative Prior

In this section, we shall consider two informative prior families of distributions.

(A) Proper Uniform Priors:

The prior knowledge situation that will be treated here is characterized by the joint prior density given by

$$
g(\lambda) = \prod_{i=0}^{k} g(\lambda_i)
$$

(4.2.1)

where

$$
g(\lambda_i) = \begin{cases} \frac{1}{\beta_i-a_i}, & a_i \leq \lambda_i \leq b_i; \ i=0,1,\ldots,k \\ 0 & \text{otherwise} \end{cases}
$$

Recall from section (3.1) that the likelihood function is given by

$$
\mathcal{L}(\lambda | d) = \lambda_0^n \prod_{i=1}^{k} \lambda_i^{n_i} \exp\left[ -\sum_{i=0}^{k} \tau_i \lambda_i \right].
$$

(4.2.2)

Then, by applying Bayes theorem on the above likelihood, the joint posterior density of $\Lambda$ with respect to the joint prior (4.2.1) can be obtained as
\[
\Pi_2(\lambda | d) = \frac{\sum_{s=0}^{n_c} \left( \prod_{i=1}^k (n_i + \lambda_i + \tau_i \lambda_i) \right) e^{n_i \tau_i \lambda_i} \exp[-\tau_i \lambda_i] d_{\lambda_i}}{\prod_{i=1}^k \lambda_i^{n_i} \exp[-\tau_i \lambda_i] \prod_{i=1}^k \lambda_i^{n_i} \exp[-\tau_i \lambda_i] d_{\lambda_i}}, \quad (a_i \leq \lambda_i \leq \beta_i),
\]

which gives
\[
\Pi_2(\lambda | d) = \frac{\sum_{s=0}^{n_c} \left( \prod_{i=1}^k (n_i + \lambda_i + \tau_i \lambda_i) \right) e^{n_i \tau_i \lambda_i} \exp[-\tau_i \lambda_i] \prod_{i=1}^k \lambda_i^{n_i} \exp[-\tau_i \lambda_i] d_{\lambda_i}}{\sum_{s=0}^{n_c} \left( \prod_{i=1}^k (n_i + \lambda_i + \tau_i \lambda_i) \right) e^{n_i \tau_i \lambda_i} \prod_{i=1}^k \lambda_i^{n_i} \exp[-\tau_i \lambda_i] \prod_{i=1}^k \lambda_i^{n_i} \exp[-\tau_i \lambda_i] d_{\lambda_i}} \quad (4.2.3)
\]

where \( \gamma'(n_i + \lambda_i + \tau_i \lambda_i) = \gamma(n_i + \lambda_i + \tau_i \lambda_i) - \gamma(n_i + \lambda_i + \tau_i \lambda_i) \),

while \( \gamma(a, z) \) is the incomplete gamma function defined by
\[
\gamma(a, z) = \int_0^z e^{-u} u^{a-1} \, du, \quad a > 0,
\]

and \( \gamma'(n - s + 1, \tau_0) \) is defined similarly.

Let \( \Pi_{2j}^m(\lambda_j | d) \) denote the univariate marginal posterior density of the parameter \( \lambda_j, j = 0, 1, \ldots, k \), with respect to the proper uniform prior (4.2.1).

The marginal posterior density of \( \lambda_j, j = 0, 1, \ldots, k \) given \( d \) is obtained by integrating out the other parameters from \( \Pi_2(\lambda | d) \). The result is, for \( \lambda_0 \),
\[
\Pi_{20}^m(\lambda_0 | d) = c_2(d) \sum_{s=0}^{n_c} \left( \prod_{i=1}^k (n_i + \lambda_i + \tau_i \lambda_i) \right) e^{n_i \tau_i \lambda_i} \exp[-\tau_i \lambda_i] \prod_{i=1}^k \lambda_i^{n_i} \exp[-\tau_i \lambda_i] \prod_{i=1}^k \lambda_i^{n_i} \exp[-\tau_i \lambda_i] d_{\lambda_i}, \quad 0 < \lambda_0 < \infty, \quad (4.2.4)
\]
and for $\Lambda_j$, $j = 1, 2, \ldots, k$,

$$
\Pi_{2j} m(\lambda_j | d) = c_2(d) \sum_{s=0}^{\sigma n_c} \left\{ \sum_{i \geq 0} \left( \prod_{i \in L_j} \binom{n_i}{i_j} \frac{n_j+i_j+1}{\tau_j} \right) \frac{\gamma'(n_j+i_j+1, \tau_j)}{\tau_j} \right\} \frac{\gamma'(n-s+1, \tau_0)}{\tau_0} (0 < \lambda_j < \infty),
$$

(4.2.5)

where $L_j = \{1, 2, \ldots, j-1, j+1, \ldots, k\}$ and $c_2(d)$ is given by

$$
c_2^{-1}(d) = \sum_{s=0}^{\sigma n_c} \left\{ \sum_{i \geq 0} \left( \prod_{i \in L} \binom{n_i}{i_x} \frac{n_x+i_x+1}{\tau_x} \right) \frac{\gamma'(n-s+1, \tau_0)}{\tau_0} \right\}.
$$

(4.2.6)

Optimal Bayes estimators of $\Lambda$ with respect to a quadratic loss function is

$$
\tilde{\Lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_k, \tilde{\lambda}_0)
$$

which is given by

$$
\tilde{\lambda}_0 = E(\Lambda_0 | d) = \int_{\alpha_0}^{\beta_0} \lambda_0 \Pi_{20} m(\lambda_0 | d) d\lambda_0
$$

$$
= c_2(d) \sum_{s=0}^{\sigma n_c} \left\{ \sum_{i \geq 0} \left( \prod_{i \in L} \binom{n_i}{i_x} \frac{n_x+i_x+1}{\tau_x} \right) \frac{\gamma'(n-s+1, \tau_0)}{\tau_0} \right\} \frac{\gamma(n-s+2)}{\tau_0}.
$$

(4.2.7)
and

\[ \lambda_j = E(\Lambda_j | \mathbf{d}) \]

\[ = c_2(d) \sum_{s=0}^{\infty} \left\{ \sum_{i \geq s} \left( \prod_{l=1}^{i} \frac{n_{l,s} \cdot \delta_l^j}{n_{l,s} + i - 1} \right) \frac{\gamma'(n_{s+1})}{\gamma'(n_{s+1} + \tau_0)} \right\} \frac{n_{s+1}}{\tau_0} . \quad (4.2.8) \]

Notice that the above estimators are also well defined in the sampling circumstances where some of the failure frequencies are observed zero during the life testing period.

The marginal posterior variances of \( \Lambda_j | \mathbf{d} \) are derived as usual.

4.3 Natural Conjugate priors

In this case, we shall use a member of an "extended natural conjugate" family to express mathematically our prior knowledge about the vector parameter \( \Lambda \). This family is characterized by

\[
8(\lambda) = \left\{ \begin{array}{ll}
\frac{\nu_0^{-1} e^{-\alpha_0 \lambda_0}}{\eta_0} \prod_{i=1}^{k} \frac{\nu_i^{-1}}{\nu_i + \alpha_i} \frac{\lambda_i}{\alpha_i} \exp\left[-\alpha_i \lambda_i\right] & \quad \alpha_i > 0, \nu_i > 0, \lambda_i > 0 \\
0 & \quad \text{otherwise}.
\end{array} \right.
\]

\[ \begin{array}{ll}
\eta_i = 0, 1, \ldots; \nu_i = 0, 1, \ldots, k.
\end{array} \quad (4.3.1) \]
where $\nu_{k}, \alpha_{k}; i = 0,1,\ldots,k$ and $m_{j}; j = 1,2,\ldots,k$, are hyperparameters and $N(d)$ is a normalized constant for $g(\lambda)$.

With respect to the joint prior distribution (4.3.1), the joint posterior density of $A_{d}$ given $d$ is obtained by applying Bayes theorem on the likelihood function (3.1). The result is

$$
\Pi_{3}(d|d) = \sum_{c_{d}'} \frac{c_{3}(d)}{\chi_{k-1}} \sum_{s=0}^{k} \left\{ \sum_{i=0}^{k} \left( \prod_{i \leq 1} n_{i} \right) \lambda_{i} \lambda_{i} \lambda_{i} e^{-\lambda_{i}\lambda_{i}} \right\}
$$

$$
\lambda_{i} \sum_{s=0}^{n'_{i}-s-1} \left( \chi_{0} \right) e^{-\chi_{0}} (0 < \lambda_{i} < \infty, i = 0,1,\ldots,k)
$$

$$
0 \text{ otherwise}
$$

where

$$
n_{c} = n_{c} + m_{c}, n'_{c} = n_{c} + \nu_{c}, n_{i} + \nu_{0} + \sum_{i=1}^{k} m_{i} \text{ and } c_{3}(d),
$$

is a normalized constant which is given by

$$
c_{3}^{-1}(d) = \sum_{s=0}^{n'_{c}'} \left\{ \sum_{i=0}^{n'_{c}'} \left( \prod_{i \leq 1} \frac{n_{c}^{i}}{\lambda_{i}^{i}} \right) \frac{\Gamma(n_{c}^{i}+i_{c})}{\tau_{i}^{i}} \frac{\Gamma(n_{c}^{i})}{\tau_{0}^{i}} \right\}.
$$

Therefore, marginal posterior densities of $A_{d}$, $j = 0,1,\ldots,k$, can be obtained from (4.1.7) and (4.1.8) by replacing the $n_{c}^{c}, n_{c}, \tau_{c}; c = 1,2,\ldots,k,$ and $n, \tau_{0}$ by the corresponding values $n_{c}^{c'}, n_{c}', \tau_{c}'; c = 1,2,\ldots,k$ and $n', \tau_{0}'$. In a similar way, Bayes estimators of $A$ can be derived from (4.1.9) and (4.1.10).
5. **Bayesian Credibility Intervals**

Bayesian probability intervals for an individual parameter can be
derived from the corresponding marginal posterior density by numerical
integration methods. Approximate Bayesian credibility intervals for
\( \Lambda_i | d, i = 0, 1, \ldots, k \) will be developed by approximating the corresponding
marginal posterior distributions. From (4.1.7) and (4.1.8), it can be seen
that all univariate marginal posterior densities can be expressed as a
weighted finite sum of gamma densities. We shall now construct Bayesian
credibility intervals for \( \Lambda_0 | d \). Credibility intervals for \( \Lambda_j | d, j = 1, 2, \ldots, k \) can be constructed analogously. As is shown by (4.1.7), the
marginal posterior density of \( \Lambda_0 \) can be written in the following form of
a weighted finite sum of gamma densities:

\[
\prod_{i=0}^{m} \left( \lambda_0 | d \right) = \sum_{s=0}^{\infty} \left\{ \sum_{i=s}^{n} \left( A_{i_1, i_2, \ldots, i_k}^{(d)} \right) d \Gamma(\lambda_0 | n-s, \tau_0) \right\} (5.1)
\]

where

\[
A_{i_1, i_2, \ldots, i_k}^{(d)} = \frac{\prod_{k=1}^{n} \left( \frac{i_k - i_k}{n_k + i_k} \right) \Gamma(n-s)}{\sum_{s=0}^{\infty} \left\{ \sum_{i=s}^{n} \left( \prod_{k=1}^{n} \left( \frac{i_k}{n_k + i_k} \right) \frac{\Gamma(n-s)}{\tau_0} \right) \right\}}
\]

are positive weights added to one, and
\[ dG(\lambda_0 | n-s, \tau_0) = \frac{\tau_0^{n-s}}{\Gamma(n-s)} \lambda_0^{n-s-1} e^{-\tau_0 \lambda_0} \, d\lambda_0, \quad (5.2) \]

is a gamma density function with \((n-s), \tau_0\) as shape and scale parameters, respectively.

It is natural now to try to approximate the above density (5.1) by that of a single gamma density. Let \( G(\nu_0, \alpha_0) \) be a candidate for approximating \( \Lambda_0 | d \) where \( \nu_0 \) and \( \alpha_0 \) has to be evaluated so that the corresponding mean and variances of the two distributions are equal.

By using the computed mean and variance of \( \Lambda_0 | d \) given by (4.1.10) and (4.1.13) gives

\[ \nu_0 = \frac{E^2(\Lambda_0 | d)}{\text{var}(\Lambda_0 | d)} \quad \text{and} \quad \alpha_0 = \frac{E(\Lambda_0 | d)}{\text{var}(\Lambda_0 | d)} . \quad (5.3) \]

Thus, upper bound credibility interval for a level of confidence \( 1 - \alpha \) is given by

\[ P_r(0 \leq \Lambda_0 \leq C_0 | d) = 1 - \alpha , \quad (5.4) \]

where \( C_0 \) is determined uniquely for the preassigned \( \alpha \) and the computed \( \nu_0 \) and \( \alpha_0 \). Approximated credibility intervals for \( \Lambda_j, j = 1, \ldots, k \) can similarly be constructed. The result would be

\[ P_r(0 \leq \Lambda_j \leq C_j | d) = 1 - \alpha , \quad (5.5) \]

where \( C_j \) is a uniquely determined constant.
6. A Numerical Example

Without loss of generality, we shall present in this section a numerical example of the BVE distribution of three parameters. A simulated complete sample life testing experiment is conducted on a random sample of size 10 of two-component parallel systems in which the times to failure in each is the BVE \( \lambda_1, \lambda_2, \lambda_0 \) and the following observations are obtained (Table 1).

**TABLE 1**

**OBSERVED VALUES OF FAILURE TIMES** \((T_1, T_2)\) **FOR 12 SIMULATED BVE \( \lambda_1, \lambda_2, \lambda_0 \) PARALLEL SYSTEMS**

<table>
<thead>
<tr>
<th>Sample</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.13670</td>
<td>0.13670</td>
</tr>
<tr>
<td>2</td>
<td>0.66087</td>
<td>0.25871</td>
</tr>
<tr>
<td>3</td>
<td>0.14712</td>
<td>0.01576</td>
</tr>
<tr>
<td>4</td>
<td>0.01936</td>
<td>0.03236</td>
</tr>
<tr>
<td>5</td>
<td>0.06325</td>
<td>0.25215</td>
</tr>
<tr>
<td>6</td>
<td>0.15241</td>
<td>0.15241</td>
</tr>
<tr>
<td>7</td>
<td>0.03018</td>
<td>0.07768</td>
</tr>
<tr>
<td>8</td>
<td>0.02798</td>
<td>0.06758</td>
</tr>
<tr>
<td>9</td>
<td>0.02617</td>
<td>0.02617</td>
</tr>
<tr>
<td>10</td>
<td>0.01516</td>
<td>0.01720</td>
</tr>
<tr>
<td>11</td>
<td>0.04085</td>
<td>0.00941</td>
</tr>
<tr>
<td>12</td>
<td>0.05456</td>
<td>0.25565</td>
</tr>
</tbody>
</table>

From this table the following statistics are computed:

\[
\tau_1 = 1.3746, \quad \tau_2 = 1.30178, \quad \tau_0 = 1.86669; \\
\eta_1 = 6, \quad \eta_2 = 3, \quad \eta_0 = 3.
\]
For the non-informative prior case, the Bayes point estimate of \( \lambda = (\lambda_0, \lambda_1, \lambda_2) \), with respect to the quadratic loss function is given in equations (4.1.9), (4.1.10), as \( \bar{\lambda}_0 = 3.616, \bar{\lambda}_1 = 5.701, \) and \( \bar{\lambda}_2 = 4.927 \). The marginal posterior variances of \( \Lambda_0, \Lambda_1, \Lambda_2 \) are computed from (4.1.13) and (4.1.14); the result is \( \text{Var}(\Lambda_0 | d) = 3.07, \text{Var}(\Lambda_1 | d) = 4.596, \) and \( \text{Var}(\Lambda_2 | d) = 5.234 \).

Using (4.1.8), the marginal posterior density of \( \Lambda_1 \) is computed. The resulting form is shown in Figure 1. We include in Figure 1 a gamma approximation of the marginal posterior density of \( \Lambda_1 \). Posterior densities of \( \Lambda_0 \) and \( \Lambda_2 \) can be computed similarly using (4.1.7) and (4.1.8).

Figure 1
Credibility intervals for the parameters can be established by using the corresponding approximated marginal densities.

7. Conclusions

Although the likelihood function for the \((k+1)\)-parameter MVE parallel system is a rather complicated one, Bayesian inferences are developed not only for obtaining closed form point estimators, but also for constructing credibility intervals of the individual parameters. This complexity of the likelihood function is due to the non-avoidable adoption of a complicated dominating reference measure that used to derive a density, because the \((k+1)\)-parameter MVE distribution is not absolutely continuous with respect to the usual Lebesgue measure.

The dominating measures approach and the close approximations developed in this work encourage the statistician to apply the Bayesian approach to the estimation problem even for non-absolutely continuous distributions like the MVE, which has singular parts. It is also useful to notice the importance of these Bayesian optimal estimators, since it is difficult to establish whether or not an M.V.U.E. exists. This is because the induced family of distributions of the minimal sufficient statistics for the \((k+1)\)-parameter MVE family is not complete.
REFERENCES


