PEAKEDNESS IN MULTIVARIATE DISTRIBUTIONS

BY

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Peakedness in Multivariate Distributions

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ABSTRACT

Peakedness provides one of the principal descriptive indices of a distribution. A definition for the univariate case was developed by Birnbaum (1948) and generalized to the multivariate case by Sherman (1955). Of course, other multivariate definitions can be developed. In this paper we study and extend some closure properties of peakedness. In particular, two multivariate generalizations are given of a majorization inequality of Proschan (1965). The results are shown to apply to the family of elliptically contoured distributions, and to have implications in estimation and for tests of hypotheses.
1. INTRODUCTION

Peakedness is a descriptive index of a distribution that provides an indication of concentration. Although the notion is intuitive, a formal definition for the univariate case was first provided by Birnbaum (1948).

**Definition 1.1.** A (univariate) random variable $Y$ is said to be more peaked about $\mu$ than a random variable $X$ about $\nu$ if

$$\text{P}[|Y-\mu| < t] \geq \text{P}[|X-\nu| < t] \quad \text{for all } t.$$  \hspace{1cm} (1.1)

When $\mu = \nu = 0$ we say that $Y$ is more peaked than $X$, and write

$$\text{P} \quad Y \geq X.$$ \hspace{1cm} (1.2)

Even in the univariate case, various questions arise almost immediately. If $Y_1 \geq X_1$ and $Y_2 \geq X_2$, does it follow that $Y_1 + Y_2 \geq X_1 + X_2$? Suppose $\{Y_m\}$ and $\{X_m\}$ are sequences with $\lim \text{P}(Y_m) = Y$, $\lim \text{P}(X_m) = X$. If $Y_m \geq X_m$, $m = 1,2,\ldots$ does it follow that $Y \geq X$? The answer to both questions is in the affirmative, as we shall see.

Following a different direction an extension of the univariate case to multivariate random variables can be considered. This was first discussed by Sherman (1955).

**Definition 1.2.** An $n$-dimensional random variable $Y$ is said to be more peaked than $X$ if they have densities and if
holds for all $A \in \mathbb{C}_n$, the class of compact, convex, symmetric (about the origin) sets in $\mathbb{R}^n$.

In the present paper we obtain some results on closure properties (Section 2), further study the multivariate case (Section 3), and discuss some applications (Section 4).

2. CLOSURE PROPERTIES

A closure property for the sum of random vectors was obtained by Sherman (1955) as follows. Let $\mathcal{Q}$ denote the closed convex cone generated by the indicator functions of all compact, convex, symmetric (about the origin) sets in the norm that is the maximum of the uniform and $L_1$ norms.

**Proposition 2.1** (Sherman, 1955). Let $X_1, X_2, Y_1, Y_2$ be independent continuous random variables with densities $f_1, f_2, g_1, g_2$, respectively. If (a) $f_1, f_2, g_1, g_2$ are in $\mathcal{Q}$, and (b) $Y_i \geq X_i, i = 1, 2$, then

\[
P_{Y_1 + Y_2 \geq X_1 + X_2}.
\]

Although this proposition provides a sufficient condition for (2.1) to hold, condition (a) is not easy to verify. However, the following sufficient condition for (a) to hold is quite straightforward to verify, and indeed, it is well known that many densities have this property.

**Proposition 2.2.** If $f : \mathbb{R}^n \rightarrow [0, \infty)$ is a continuous density function
that is bounded, symmetric (about the origin) and log-concave, then \( f \) is in \( G \).

**Proof.** If suffices to show that there exists an array of compact, convex, and symmetric sets \( \{A_{mr}\} \) for \( r = 1, 2, \ldots, m = 1, 2, \ldots \), and a sequence of functions \( f_m(x) \) of the form \( f_m(x) = \sum_{r=1}^{\infty} I_{A_{mr}}(x) \), such that

\[
\limsup_{m \to \infty} \sup_x \left| f_m(x) - f(x) \right| \to 0, \tag{2.2}
\]

\[
\lim_{m \to \infty} \int \left| f_m(x) - f(x) \right| dx \to 0. \tag{2.3}
\]

Define

\[
A_{mr} = \{x \mid x \in \mathbb{R}^n, f(x) \geq r/m\},
\]

then \( A_{mr} \in G_n \) for every \( m \) and \( r \) (convexity follows because all log-concave density functions are unimodal), and \( A_{mr} \supseteq A_{m,r+1} \) for every \( r \) and \( m \). Now denote \( B_{mr} = A_{mr} \cap \overline{A}_{m,r+1} \) and define \( f(x) = r/m \) for \( x \in B_{mr} \), that is,

\[
f_m(x) = \begin{cases} 
\sum_{r=1}^{\infty} I_{A_{mr}}(x) & \text{for } x \in A_{mr}, \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( \left| f_m(x) - f(x) \right| \leq 1/m \) holds for all \( x \in \mathbb{R}^n \). Consequently, \( f_m(x) \) converges to \( f(x) \) in the uniform norm and (2.2) holds.

To show that (2.3) also holds, note that there exists a convex function
Q(x) of x with Q(x) = Q(-x) for which f(x) is of the form exp(-Q(x)). It follows that there exists a large $M_0$ and an $K > 0$ such that

$$||x|| > \frac{1}{K} \log m \Rightarrow Q(x) \geq K||x||$$

for all $m \geq M_0$. Thus for $m \geq M_0$ the volume (vol) of $A_{ml}$ satisfies

$$\text{vol} \{x|f(x) > \frac{1}{m}\} = \text{vol} \{x|Q(x) < \log m\} \leq \text{vol} \{x|K^*||x|| \leq \log m\} \leq \text{vol} \{x|\max_{1 \leq i \leq n}|x_i| \leq K^{-1}\log m\} = (2 \log m)^n / K^n.$$ 

Consequently we have

$$\int |f_m(x) - f(x)| dx \leq \int_{A_{ml}} \frac{1}{m} dx + \int_{A_{ml}} |f_m(x) - f(x)| dx \leq \frac{1}{m} \left[ \frac{2}{K} \right] \log \frac{1}{m} + \int_{A_{ml}} |f_m(x) - f(x)| dx \equiv I_1 + I_2 \quad \text{(say)}.$$ 

Clearly $I_2 \to 0$ as $m \to \infty$. Also, since (by l'Hospital's rule)

$$\lim_{m \to \infty} (1/m) \log(1/m)^n = 0, I_1 \to 0,$$

which completes the proof. $\square$

An application of Propositions 2.1 and 2.2 is the fact that: If the densities of $X_1, \ldots, X_N, Y_1, \ldots, Y_N$ are bounded, symmetric (about the origin), and log-concave, $X_1, \ldots, X_N$ are independent, $Y_1, \ldots, Y_N$ are independent, then $Y_m \geq X_m (m=1, \ldots, N)$ implies that

$$(2.4) \quad \sum_{m}^{N} c_m Y_m \geq \sum_{m}^{P} c_m X_m$$

for all real numbers $c_1, \ldots, c_N$. 

Two special cases follow: (a) When \( N=2, c_1=c_2=1 \) and \( X_2=Y_2=U \), (2.4) reduces to \( Y + U \geq X + U \); (b) when \( X_1, \ldots, X_N \) are i.i.d., \( Y_1, \ldots, Y_N \) are i.i.d., and \( Y_i \geq X_i \), (2.4) yields \( \frac{1}{N} \sum_1^N Y_i \geq \frac{1}{N} \sum_1^N X_i \).

**Proposition 2.3.** Let \( \{X_m\}_{m=1}^{\infty} \) and \( \{Y_m\}_{m=1}^{\infty} \) be two sequences of \( n \)-dimensional random variables. If \( \mathcal{L}(X_m) \overset{d}{\rightarrow} \mathcal{L}(X) \), \( \mathcal{L}(Y_m) \overset{d}{\rightarrow} \mathcal{L}(Y) \) and \( Y_m \geq X_m, m = 1, 2, \ldots \), then \( Y \geq X \).

**Proof.** For any given \( A \in \mathcal{G}_n \) let \( I_A(x) \) be its indicator function, and let \( F_m, F, G_m, G \) be the distribution of \( X_m, X, Y_m \) and \( Y, \) respectively. Then

\[
P[X \in A] = \int I_A(x) dF(x) = \lim_{m \to \infty} \int I_A(x) dF_m(x) \\
\leq \lim_{m \to \infty} \int I_A(x) dG_m(x) = P[Y \in A].
\]

The equalities are a consequence of the Helly-Bray Lemma, and the inequality follows by hypothesis. \( \square \)

Two other properties that are related to the above deal with marginal distributions and linearly transformed variables.

**Proposition 2.4.** If \( X = (\hat{X}, \hat{X}) \) and \( Y = (\hat{Y}, \hat{Y}) \) are \( n \)-dimensional random variables partitioned conformably with \( \hat{X} \) and \( \hat{Y} \) \( q \)-dimensional, and if \( Y \geq X \), then \( \hat{Y} \geq \hat{X} \).

**Proof.** For any given \( A \in \mathcal{G}_q \) (in \( \mathbb{R}^q \)) define

\[
A_m = A \cap \{(x_{q+1}, \ldots, x_n) \mid x_i \leq m, q+1 \leq i \leq n \}.
\]

Then \( A_m \in \mathcal{G}_n \) for all \( m \) and
\[
P(X \in A) = \lim_{m \to \infty} P(X \in A_m) \leq \lim_{m \to \infty} P(Y \in A_m) = P(Y \in A).
\]

The equalities follow from the monotone convergence theorem, the inequality follows by hypothesis.

**Proposition 2.5.** \( Y \preceq X \) holds iff \( CY \preceq CX \) holds for all \( k \times n \) matrices \( C \), \( k \leq n \).

**Proof.** The "if" part follows immediately by choosing \( C \) to be the identity matrix. The proof for the only if part will be given in three steps:

(a) First consider the case that \( C \) is \( n \times n \) and is nonsingular. Then, for any given \( A \in \mathbb{G}_n \), \( CX \in A \) iff \( x \in C^{-1}A \) and \( C^{-1}A \in \mathbb{G}_n \). Thus

\[
P(CX \in A) = P(X \in C^{-1}A) \leq P(Y \in C^{-1}A) = P(CY \in A).
\]

(b) If \( C \) is any \( n \times n \) matrix (singular or nonsingular), then \( C \) can be decomposed as

\[
C = E_1 \ldots E_r DF_1 \ldots F_s
\]

where \( D = \text{diag}(I_q, 0) \), the \( E_i \)'s and \( F_j \)'s are elementary transformations, and \( I_q \) is the identity matrix of order \( q \). That is, the \( E_i \)'s and \( F_j \)'s can be obtained by either multiplying a row of the identity matrix by a scalar, or by interchanging two rows, or by adding a scalar multiple of one row to another. Each of such transformations is nonsingular, thus we have, by (a), \( E_i Y \geq E_i X \) and \( F_j Y \geq F_j X \) for all \( i, j \). Furthermore, since \( DX (DY) \) and \( X \) (and \( Y \)) are identically distributed, by Proposition 2.4 we have \( DY \geq DX \).

Consequently, \( CY \geq CX \) holds for all \( n \times n \) matrices \( C \).
(c) If $C$ is $k \times n$, $k < n$, we consider the $n \times n$ matrix $C^p = \text{diag}(C, I_{n-k})$, and apply (b) to yield $C^p Y \succeq C^p X$, then apply Proposition 2.4 to obtain $p \ Y \succeq C X$.

\[ \square \]

Remark 2.6. After obtaining Proposition 2.5 we now see that Proposition 2.4 is a special case (by choosing $C = \text{diag}(I_q, 0)$ in Proposition 2.5). Of course, the proof of Proposition 2.5 depends on Proposition 2.4.

Choosing $C = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}$ in Proposition 2.5 yields the following fact:

Fact 2.7. For $n = 2q$ let $X = (\hat{X}, \hat{X})$, $Y = (\hat{Y}, \hat{Y})$ be such that $\hat{X}, \hat{X}, \hat{Y}, \hat{Y}$ are q-dimensional random variables. If $Y \succeq X$, then $p \ Y + \hat{Y} \succeq \hat{X} + \hat{X}$.

Remark 2.8. Assume that $p \ Y \succeq \hat{X}$, $\hat{Y} \succeq \hat{X}$, $\hat{X}$ and $\hat{X}$ are independent, $\hat{Y}$ and $\hat{Y}$ are independent, does this imply $p \ (\hat{Y}, \hat{Y}) \succeq (\hat{X}, \hat{X})$? Without any additional conditions on their densities the answer to this question is negative. To see this, consider $n = 2$, $q = 1$, and let the densities of $\hat{X}, \hat{X}, \hat{Y}, \hat{Y}$ be as given in the example in Sherman (1955). If $p \ (\hat{Y}, \hat{Y}) \succeq (\hat{X}, \hat{X})$ were true, then Fact 2.7 would imply $p \ (\hat{Y} + \hat{Y}) \succeq (\hat{X} + \hat{X})$ which is known to be false.

3. SCHUR PROPERTIES

The questions posed in this section deal with comparisons of linear combinations for which the weights are ordered by majorization. (For definitions and properties of majorization see Marshall and Olkin, 1979.) A result of Proschan (1965) deals with univariate random variables.

Theorem 3.1 (Proschan, 1965). If $Z_1, \ldots, Z_N$ are independent identically
distributed univariate random variables with common log-concave density function \( f(z) \) symmetric about the origin, and if \( a = (a_1, \ldots, a_N) \) and \( b = (b_1, \ldots, b_N) \) are two real nonnegative vectors for which \( a \succ b \), then

\[
\sum_{i=1}^{N} b_i Z_i \overset{P}{\geq} \sum_{i=1}^{N} a_i Z_i.
\]

Univariate densities that are log-concave are said to be strongly unimodal (Ibragimov, 1956) or Pólya frequency functions of order 2 (Karlin, 1968, p.12 or Barlow and Proschan, 1975, p.76). It was shown by Proschan (1965) that Theorem 3.1 fails if the strong unimodality condition is replaced by the weaker condition: \( f(z) \) is unimodal in the sense that \( \{z : f(z) \geq u\} \) is an interval for all \( u \). The Cauchy distribution provides a counterexample.

Theorem 3.1 has a generalization to the multivariate case. The present proof is different from Proschan's proof for the univariate case and depends on an application at the result of Anderson (1955) and Sherman (1955). It has the virtue of removing the nonnegativity condition of the weights.

**Theorem 3.2.** If \( Z_1, \ldots, Z_N \) are independent, identically distributed \( n \)-dimensional random variables with common log-concave density function \( f(z) \) which is symmetric (about the origin), and if \( (a_1, \ldots, a_N) \succ (b_1, \ldots, b_N) \), then

\[
\sum_{i=1}^{N} b_i Z_i \overset{P}{\geq} \sum_{i=1}^{N} a_i Z_i.
\]

The univariate version of Theorem 3.1 requires that \( Z_1, \ldots, Z_N \) be independent. In the following extension, \( Z_1, \ldots, Z_N \) are permitted to be
dependent provided other conditions are satisfied.

Theorem 3.3. Suppose \( Z_1, \ldots, Z_N \) are exchangeable univariate random variables with joint density function \( \phi(z) \), and \( (a_1, \ldots, a_N) \) and \( (b_1, \ldots, b_N) \) are two real vectors. If the conditional density \( g \) of

\[
c(Z_1 + Z_2) + \sum_{i=3}^{N} a_i Z_i \mid Z_1 - Z_2 = v
\]

is symmetric (about the origin), and is unimodal for all fixed \( c, v \), and \( a_3, \ldots, a_n \), then \( a > b \) implies that

\[
\sum_{i=1}^{N} b_i Z_i \overset{D}{\geq} \sum_{i=1}^{N} a_i Z_i .
\]

To prove Theorems 3.2 and 3.3 we first observe two lemmas.

Lemma 3.4. Assume that \( f(z) \) satisfies the conditions in Theorem 3.2 (for \( n \geq 1 \)). For \( c \neq 0 \) let \( g(v_1 | v_2) \) denote the conditional density of \( V_1 = c(Z_1 + Z_2) \) given \( V_2 = (Z_1 - Z_2) = v_2 \). Then, for every fixed \( v_2 \) such that \( g \) exists, \( g \) is symmetric (about the origin) and is a log-concave function of \( v_1 \).

Proof. The joint density of \( V_1, V_2 \) is

\[
h(v_1, v_2) = f\left(\frac{v_1 + cv_2}{2c}\right) f\left(\frac{v_1 - cv_2}{2c}\right) |J| ,
\]

where \( |J| \) is the Jacobian of the linear transformation (a positive real number), and the conditional density of \( V_1 \) given \( V_2 = v_2 \) is of the form
\[ g(v_1|v_2) = h(v_1,v_2)/q(v_2) \]

where \( q(v_2) = \int h(v_1,v_2) dv_1 \). Thus \( g(v_1|v_2) = g(-v_1|v_2) \) follows immediately from \( h(-v_1,v_2) = h(v_1,v_2) \) for all fixed \( v_2 \). To show the log-concavity property, first note that for all \( v_1,v_1^* \) one has, for all \( \alpha \in [0,1] \),

\[
\begin{align*}
\log h(\alpha v_1 + (1-\alpha) v_1^*, v_2) \\
= \log f((2c)^{-1}[\alpha(v_1+cv_2) + (1-\alpha)(v_1^*-cv_2)]) \\
+ \log f((2c)^{-1}[\alpha(v_1-cv_2) + (1-\alpha)(v_1^*-cv_2)]) \\
+ \alpha \log |J| + (1-\alpha) \log |J| \\
\geq \alpha \log h(v_1,v_2) + (1-\alpha) \log h(v_1^*,v_2).
\end{align*}
\]

Thus one has

\[
\begin{align*}
\log g(\alpha v_1 + (1-\alpha) v_1^*|v_2) \\
\geq \alpha[\log h(v_1,v_2) - \log q(v_2)] + (1-\alpha)[\log(h(v_1^*,v_2) - \log q(v_2)] \\
= \alpha \log g(v_1|v_2) + (1-\alpha) \log g(v_1^*|v_2)
\end{align*}
\]

as to be shown.

Note that, by choosing \( n = 1 \) in Lemma 3.4, the conditions in Theorem 3.3 is weaker than that in Theorem 3.1.

**Lemma 3.5.** Under the assumptions stated in Theorem 3.2 the conditional density \( g \) of \( c(Z_1^* + Z_2) + \sum_{i=3}^{N} a_i Z_i \) given \( Z_1 - Z_2 = v_2 \) is symmetric.
(about the origin) and log-concave for all \( c, v_2 \) and \( a_i \) (i=1, ..., N) such that \( g \) exists.

**Proof.** If all \( a_i \)'s are zero then the lemma follows immediately from Lemma 3.4, thus we assume that at least one of the \( a_i \)'s is not zero. It is known that the convolution of two symmetric (about the origin) and log-concave densities is also symmetric and log-concave (the symmetry property can be easily checked, the log-concavity property can be found in Das Gupta (1973, Theorem 4.2). Thus the marginal density, or equivalently the conditional density of \( \gamma_3^{N} a_i Z_i \) given \( Z_1 - Z_2 = v_2 \) (by independence), is symmetric and log-concave. Consequently, if \( c = 0 \) then \( \gamma_3^{N} a_i Z_1 \) and \( Z_1 - Z_2 \) are independent, thus the Lemma follows immediately; if \( c \neq 0 \) then the conditional density \( g \) is the convolution of the conditional densities of \( c(Z_1 + Z_2) \) and that of \( \gamma_3^{N} a_i Z_i \), given \( Z_1 - Z_2 \), which is also symmetric and log-concave. □

**Proof of Theorem 3.2.** Let \( a \succ b \). Without loss of generality (see Marshall and Olkin (1979)) it can be assumed that

\[
(3.1) \quad a_1 > b_1 \geq b_2 > a_2, \quad \gamma_1^2 a_i = \gamma_1^2 b_i
\]

and \( b_i = a_i \) for \( i = 3, ..., N \). For any given \( A \in G_n \) write

\[
(3.2) \quad P[\gamma_1^N a_i Z_i \in A] = \int P[c(Z_1 + Z_2) + \gamma_3^N a_i Z_i \in A - \lambda v_2 | Z_1 - Z_2 = v_2] dH(v_2),
\]

where \( H \) is the c.d.f. of \( Z_1 - Z_2 \). Since \( g \) is symmetric and log-concave,
it is symmetric and unimodal. Thus by the main theorem in Anderson (1955) the conditional probability on the right-hand side of (3.2) is a non-increasing function of $\lambda$ for every fixed $v_2$. Consequently, the unconditional probability on the left-hand side of (3.2) is nonincreasing in $\lambda$.

Proof of Theorem 3.3. Again assume that (3.1) holds. Rewriting

$$\sum_{i=1}^{n} a_i Z_i = c(Z_{1}+Z_{2}) + \lambda(Z_{1}-Z_{2})$$

for $c = (a_1+a_2)/2$, $\lambda = (a_1-a_2)/2 > 0$, we have

$$P[|\sum_{i=1}^{n} a_i Z_i| \leq t]$$

$$= \int P[-t-\lambda v_2 \leq c(Z_{1}+Z_{2}) + \sum_{i=1}^{n} a_i Z_i \leq t-\lambda v_2 | Z_{1}-Z_{2} = v_2] dH(v_2),$$

where $H$ is the c.d.f. of $Z_{1}-Z_{2}$. Since the conditional density $g$ is symmetric and unimodal, the conditional probability is a nonincreasing function of $\lambda$ for every fixed $v_2$. Thus the unconditional probability is a nonincreasing function of $\lambda$.

4. APPLICATIONS

A number of applications of these results can be given. The present choices are illustrative of a variety of applications, and obviously are not exhaustive.
Applications 4.1. (Elliptically contoured distributions). If the density function of \( Z = (Z_1, \ldots, Z_N)' \) is permutation invariant and elliptically contoured, i.e., it is of the form \( \phi(z) = h(z'Tz) \) where \( h \) is nonincreasing, \( T = (\tau_{ij}) \) has elements

\[
\tau_{ij} = \begin{cases} 
\tau_1 & \text{for } i = j \\
\tau_1 \tau_2 & \text{for } i \neq j 
\end{cases}
\]

and is positive definite, then Theorem 3.3 applies. To see this, for \( a_1 > b_1 \geq b_2 > a_2 \) let \( c = (a_1 + a_2)/2, \lambda = (a_1 - a_2)/2, \)

\[
V_1 = c(Z_1 + Z_2), \quad V_2 = Z_1 - Z_2, \quad V_i = Z_i \quad \text{for } i = 3, \ldots, N;
\]

then the joint density of \( V = (V_1, \ldots, V_N)' \) is \( h(v'(\Lambda'T\Lambda)v)/|\Lambda| \). where \( \Lambda \) satisfies \( V = \Lambda^{-1}Z \). Partitioning the matrix \( \Lambda'T\Lambda \) into the form

\[
\Lambda'T\Lambda = \begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix},
\]

where \( C_{11} \) is a 2x2 matrix, an elementary calculation shows that

\[
C_{11} = \text{diag}(\tau_1(1+\tau_2)/2c^2, \tau_1(1-\tau_2)/2).
\]
Since the marginal density of a subset of elliptically contoured-distributed random variables is again elliptically contoured, and depends on $\Lambda' T \Lambda$ only through the submatrix $C_{ij}$ (see e.g. Kelker (1970)), the density of $(V_1, V_2)'$ is a nonincreasing function of

$$(v_1, v_2) C_{11}(v_1, v_2)' = \frac{1}{2} \tau_1 [(1+\tau_2) v_1^2/c^2 + (1-\tau_2) v_2^2] .$$

Consequently the conditional density of $V_1$ given $V_2 = v_2$ is a non-increasing function of $v_2^2$. Thus it is symmetric and unimodal, and the conditions of Theorem 3.3 are satisfied.

Not that, as a special case, if the joint distribution of $Z_1, ..., Z_N$ is multivariate normal with means 0, variances $\tau_1$ and correlations $\tau_2 \in (-1/(n-1), 1)$, then Theorem 3.3 applies.

**Application 4.2.** (i.i.d. multivariate normal variables.) Let $Z_1, ..., Z_N$ be n-dimensional i.i.d. multivariate normal variables with mean vector 0 and a positive definite matrix $\Sigma$. Then their density $f(z)$ is symmetric and log-concave; hence Theorem 3.2 applies.

**Application 4.3.** (A monotonicity property for the confidence probability.) As in Proschan (1965), Theorems 3.2 and 3.3 yield the following monotonicity property: Let $W_1, ..., W_N, ...$ be a sequence of i.i.d. n-dimensional random variables with mean vector $\mu$ and density $f(w)$. Let $A \subset \mathbb{R}^n$ be convex and symmetric (about the origin),

$$\overline{W}_N = \frac{1}{N} \sum_{m=1}^{N} W_m (N=1,2,...) ,$$

and let $Q + \overline{W}_N$ denote the confidence region for $\mu$ on the basis of $W_1, ..., W_N$. If $f(w)$ is symmetric about $\mu$ (i.e.,
If \( f(\mu+v) = f(\mu-v) \) for every \( v \) and is a log-concave function of \( w \), then the confidence probability \( P[\mu \in Q + \bar{w}_N] \) is a nondecreasing function of \( N \).

If \( A \) is compact, this follows immediately from Theorem 3.2 and from letting

\[
\begin{align*}
a_1 &= \ldots = a_N = \frac{1}{N}, \quad a_{N+1} = 0, \\
b_1 &= \ldots = b_N = b_{N+1} = \frac{1}{N+1}.
\end{align*}
\]

If \( A \) is not compact, then one considers a sequence of compact sets \( \{A_n\} \) which converges to \( A \) and applies Theorem 3.2 and the monotone convergence theorem.

**Application 4.4.** (Monotonicity of the type I error in certain multivariate hypothesis-testing problems). Consider the sequence of \( n \)-dimensional random variables \( \bar{W}_1, \ldots, \bar{W}_N, \ldots \) described in Application 4.3. In testing the hypotheses

\[
H_0: \mu = \mu_0 \quad \text{vs} \quad H_1: \mu \neq \mu_0,
\]

if \( H_0 \) is rejected iff \( \bar{W}_N - \mu_0 \) \( \notin A \) for some \( A \) which is symmetric (about the origin) and convex, by Theorem 3.2 the type I error is a non-increasing function of the sample size \( N \).
References


