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BY

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TECHNICAL REPORT NO. 211

MAY 1985

PREPARED UNDER THE AUSPICES OF
NATIONAL SCIENCE FOUNDATION
DMS 84-11411
INGRAM OLKIN, PROJECT DIRECTOR

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
MULTIVARIATE NEW BETTER THAN USED DISTRIBUTIONS: A SURVEY

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April 1985

(1)Supported in part by the National Science Foundation at Stanford University, and in part by the Natural Sciences and Engineering Research Council Canada.

(2)Supported by the Air Force Office of Scientific Research, U.S.A.F. under Grant AFOSR-84-0205. Reproduction in whole or in part is permitted for any purpose of the United States Government.
Abstract

Various multivariate NBU (new better than used) notions are surveyed in this paper. The relationships among them are studied. Some preservation properties of these notions under different kinds of operations are listed and open questions are indicated. Multivariate NBU distributions with exponential marginals are discussed. Finally some examples are given.

AMS subject classification (1979): 60K10

Key words and phrases: Homogeneous and subhomogeneous functions, coherent life functions, NBU distributions, exponential distributions.
1. Introduction.

A random variable $T$ or its distribution function $F$ is said to be new better than used (NBU) if $P(T > 0) = 1$ and if

$$P(T > s + t | T > s) < P(T > t) \text{ for all } s, t > 0.$$ 

This property was found to be important in the study of replacement policies (Marshall and Proschan, 1972) and has received a considerable amount of attention in the literature of reliability theory.

Because dependent components are often encountered in the context of reliability theory multivariate extensions of the NBU property are of interest. There is no one extension that is clearly the "right one" and various possibilities have been proposed by Buchanan and Singpurwala (1977), Marshall and Shaked (1979, 1982, 1985), Arjas (1981a,b), Block and Savits (1981), El-Neweihi (1981), Ghosh and Ebrahimi (1981), Ohl and Nishida (1982), El-Neweihi, Proschan and Sethuraman (1983) and by van den Berg and Kesten (1984). The purpose of this paper is to review some of these multivariate extensions and to see something of how they relate to each other.

In the following, "increasing" is used to mean "nondecreasing" and "decreasing" is used to mean "nonincreasing". Functions considered in this paper are tacitly assumed to be measurable.

2. Multivariate NBU distributions based on classes of functions.

The multivariate NBU conditions of Marshall and Shaked (1985) were motivated by the following fact (Marshall and Shaked, 1982, Corollary 3.6):

(i) $g(T_1, \ldots, T_n)$ is NBU whenever $T_1, \ldots, T_n$ are independent NBU random variables provided (ii) $g$ is a nonnegative increasing function defined on
\( R^n_+ = (0, \infty)^n \) that is subhomogeneous, i.e.,

\[(2.1) \quad g_{\alpha}(x) \leq g(\alpha x) \quad \text{for all} \quad \alpha \in R^n_+ \quad \text{and all} \quad \alpha \in [0, 1]. \]

More recently, van den Berg and Kesten (1984) have shown that (i) holds whenever (iii) \( g \) is a nonnegative increasing function defined on \( R^n_+ \) such that

\[ g^{-1}(a+b, \infty) \subseteq g^{-1}(a, \infty) + g^{-1}(b, \infty) \quad \text{for all} \quad a, b > 0, \]

where \( g^{-1}A = \{x : g(x) \in A\} \) and \( g^{-1}A + g^{-1}B = \{x+y : x \in g^{-1}A, y \in g^{-1}B\} \). Moreover, they show that (ii) implies (iii). Various classes \( \mathcal{C} \) of such functions \( g \) are described below, and for any such class, two kinds of multivariate distributions arise.

2.1 Definition. A random vector \( \mathbf{T} = (T_1, \ldots, T_n) \) or its distribution is said to be \( \mathcal{C} \)-generated from NBU random variables if for some positive integer \( k \) there exist independent NBU random variables \( X_1, \ldots, X_k \) and functions \( g_1, \ldots, g_n \) in \( \mathcal{C} \) defined on \( R^n_+ \) such that

\[ T_i = g_i(X_1, \ldots, X_k), \quad i = 1, \ldots, n. \]

A random vector \( \mathbf{T} = (T_1, \ldots, T_n) \) or its distribution is said to be in the \( \mathcal{C} \)-closure of the NBU random variables if \( g(T_1, \ldots, T_n) \) has an NBU distribution for all \( g \in \mathcal{C} \) defined on \( R^n_+ \).

Some classes \( \mathcal{C} \) of particular interest are

\[ \mathcal{C}_1 = \{g: \quad \text{for some positive integer} \quad n, \quad g \quad \text{is defined on} \quad R^n_+ \quad \text{and} \quad \} \]
\[ g(x_1, \ldots, x_n) = \min_{i \in J} x_i \text{ for some } J \subseteq \{1, \ldots, n\}, \]

\[ \mathcal{E}_2 = \{g : \text{for some positive integer } n, g \text{ is defined on } \mathbb{R}_{+}^{n} \text{ and } \]

\[ g(x_1, \ldots, x_n) = \sum_{i \in J} x_i \text{ for some } J \subseteq \{1, \ldots, n\}, \]

\[ \mathcal{E}_3 = \{g : g \text{ is a coherent life function}, \text{ where "coherent life functions" is in the sense of Esary and Marshall (1970); such functions are allowed to have irrelevant components.} \]

\[ \mathcal{E}_4 = \{g : \text{for some positive integer } n, g \text{ is defined on } \mathbb{R}_{+}^{n} \text{ and } \]

\[ g \text{ is nonnegative, increasing and homogeneous}, \text{ where homogeneous function is one for which (2.1) holds with equality.} \]

\[ \mathcal{E}_5 = \{g : \text{for some positive integer } n, g \text{ is defined on } \mathbb{R}_{+}^{n} \text{ and } \]

\[ g \text{ is nonnegative, increasing and subhomogeneous}, \}

\[ \mathcal{E}_6 = \{g : g \text{ satisfies (iii)}, \}

\[ \tilde{\mathcal{E}}_1 = \{g : g(x) = \psi(a_1 x_1, a_2 x_2, \ldots) \text{ for some } \psi \in \mathcal{E}_1 \text{ and } a_1, a_2, \ldots \in (0, \infty)\}, \]

\[ i = 1, 2, 3. \]

Below, the class \( \mathcal{E}_1 \)-generated from NBU random variables is denoted by \( \tilde{G}_1 \) and the \( \mathcal{E}_1 \)-closure of the NBU random variables is denoted by \( \tilde{C}_1 \), \( i = 1, \ldots, 6 \). Similarly define \( \tilde{G}_i \) and \( \tilde{C}_i \) in case \( \mathcal{E}_1 \) is replaced by \( \tilde{E}_1 \).

Various properties of these classes are obtained by Marshall and Shaked (1982, 1985); in particular they show that

\[ \tilde{G}_1 \subseteq \tilde{G}_3 \subseteq \tilde{G}_4 \subseteq \tilde{G}_5 \subseteq \tilde{G}_6 = \tilde{C}_4 \subseteq \tilde{C}_3 \subseteq \tilde{C}_1 \subseteq \tilde{C}_2 \subseteq \tilde{C}_2. \]

Some of the above classes and implications are mentioned by Block and Savits (1981), El-Neweihi, Proschan and Sethuraman (1983) and by El-Neweihi
The classes $G_6$ and $C_6$ are not well understood. In particular it is not known whether $G_6 \subset C_6$.

3. Some Additional NBU Conditions.

Several additional multivariate NBU conditions based on analogies with univariate conditions have been proposed by various authors. Some interrelationships among such conditions are studied in this section.

Let $\mathbb{T}$ be a nonnegative random vector, let $F$ be its distribution function and let $\overline{F}(t) = P(\mathbb{T} > t)$ denote its survival function. The condition

$$\overline{F}(\mathbb{S}+\mathbb{T}) < \overline{F}(\mathbb{S})\overline{F}(\mathbb{T}) \text{ for all } \mathbb{S} > 0, \mathbb{T} > 0$$

was first mentioned by Buchanan and Singpurwalla (1977). A related but stronger condition, proposed by van den Berg and Kesten (1984), makes use of the notion of an upper set, i.e., a set $U$ with the property that $u \in U$ and $u < v$ together imply $v \in U$. If $\mu_\mathbb{T}(A) = P(\mathbb{T} \in A)$ for each Borel set $A$, the condition of van den Berg and Kesten (1984) is that

$$\mu_\mathbb{T}(U+V) < \mu_\mathbb{T}(U)\mu_\mathbb{T}(V) \text{ for all measurable upper sets } U, V \subset \mathbb{R}^n$$

(here $A + B = \{s + t : s \in A, t \in B\}$); they call $\mathbb{T}$ "strongly NBU".

Of course (3.1) is just the restriction of (3.2) to upper sets $U$ and $V$ of the form $\{x : x > s\}$ and $\{x : x > t\}$. Denote by (3.1) the class of distributions satisfying condition (3.1). Then

$$\text{(3.2) } \subset \text{(3.1).}$$
From the arguments of van den Berg and Kesten (1984) it follows that

\[(3.2) \mathcal{C} C_6.\]

Condition (3.1) implies that

\[\overline{F}(\mathbf{t}) < \prod_{j=1}^{n} \overline{F}(0, \ldots, 0, t_j, 0, \ldots, 0) \text{ for all } \mathbf{t} = (t_1, \ldots, t_n),\]

where \(t_j\) is the \(j^{th}\) coordinate of \((0, \ldots, 0, t_j, 0, \ldots, 0)\). This inequality in turn implies that \(\text{Cov}(T_i, T_j) < 0\) for \(i \neq j\). It follows that many distributions of interest cannot satisfy (3.1) or (3.2). To overcome this difficulty, Marshall and Shaked (1979) introduced the condition

\[(3.3) \overline{F}(s + \mathbf{t}) < \overline{F}(s) \overline{F}(\mathbf{t}) \text{ whenever } s > 0, \mathbf{t} > 0 \text{ are similarly ordered,}\]

i.e., \((s_i - s_j)(t_i - t_j) > 0\) for all \(i, j = 1, \ldots, n\).

Another condition,

\[(3.4') \overline{F}(se^{+}t) < \overline{F}(se) \overline{F}(te) \text{ for all } s > 0, t > 0 \text{ (e = (1, \ldots, 1))}\]

was introduced by Buchanan and Singpurwalla (1977), who also considered the condition

\[(3.5') \overline{F}(se^{+}t_e) < \overline{F}(se) \overline{F}(te) \text{ for all } s, t, e > 0.\]

Condition (3.4') does not imply that the lower dimensional marginals of \(F\) satisfy (3.4') [see, e.g., Ghurye and Marshall (1984)]. It is easy to see that the same is true also for condition (3.5'). Thus it may be appropriate to consider the modified conditions

\[(3.4) \text{ Condition (3.4') together with this condition imposed upon all marginal distributions.}\]
(3.5) Condition (3.51') together with this condition imposed upon all marginal distributions.

Conditions (3.1), (3.3), (3.4), (3.5) are discussed by Ghosh and Ebrahimi (1981).

Clearly

\[(3.1) \subset (3.3) \subset (3.4) \subset (3.5) = C_1, (3.3) \subset \tilde{C}_1 \subset C_1.\]

Although the above ideas are not clearly related to those of Section 2, some connections can be found. Block and Savits (1980) approximate upper sets by unions of sets of the form \(\{x: x > \bar{a}\}\), and in this way show that a nonnegative homogeneous function \(g\) can be approximated by functions \(h\) of the form

\[h(x_1, \ldots, x_n) = \max_{1 \leq j \leq m} \min_{1 \leq i \leq n} (x_i / \bar{w}_j^{(i)}),\]

where \(\bar{w}_1^{(1)}, \ldots, \bar{w}_n^{(n)} \in \mathbb{R}_+^m\). Equivalently,

\[h(x_1, \ldots, x_n) = \tau(x_1 / \bar{w}_1^{(1)}, \ldots, x_n / \bar{w}_n^{(m)})\]

where \(\tau\) is a coherent life function of order \(mn\) and \(\bar{w}_1^{(1)}, \ldots, \bar{w}_n^{(n)} \in \mathbb{R}_+^m\); here the notation \(a / \bar{w}_1^{(i)}, \ldots, a / \bar{w}_m^{(i)}\) and \(1/0 = \infty\) is used.

Consequently condition \(C_4\) can be written in the form

\[(3.6) \quad P[\tau(T_1 / \bar{w}_1^{(1)}, \ldots, T_n / \bar{w}_n^{(n)}) > a + b] < aP[\tau(T_1 / \bar{w}_1^{(1)}, \ldots, T_n / \bar{w}_n^{(n)}) > b] \quad \text{for all coherent life functions of order } mn, \]

\[\bar{w}_1^{(1)}, \ldots, \bar{w}_n^{(n)} \in \mathbb{R}_+^m, \quad m = 1, 2, \ldots, \text{ and } a, b > 0.\]
Condition (3.6) suggests additional conditions of the form

\[(*) \quad P(\tau(T_1/u(1) + y(1)), \ldots, T_n/(u(n) + y(n))) > 1)\]
\[< P(\tau(T_1/u(1), \ldots, T_n/y(n)) > 1) P(\tau(T_1/y(1), \ldots, T_n/y(n)) > 1)\]

for all coherent life functions \( \tau \) of order \( mn \).

In particular (3.6) together with (3.3) suggests

(3.7) inequality (*) holds whenever \( u^{(j)} \) and \( y^{(j)} \) are similarly ordered,
\[ j = 1, \ldots, n; \quad m = 1, 2, \ldots, \]

(3.8) inequality (*) holds whenever the \( mn \)-dimensional vectors
\[ (u^{(1)}, \ldots, u^{(n)}) \quad \text{and} \quad (y^{(1)}, \ldots, y^{(n)}) \]
are similarly ordered, \( m = 1, 2, \ldots \).

The ordering condition of (3.8) implies that of (3.7). Moreover, if
\( u \) and \( y \) are similarly ordered vectors and \( \tau(t) = \min_{1 \leq j \leq n} t_j \), then (3.3) follows from (3.8) with \( m = 1 \). It is also clear that (3.8) implies (3.6).

Thus

\[(3.7) \quad C(3.8) \quad C(3.3) \quad \text{and} \quad (3.8) \quad C(3.6) = C_4 \quad .\]

From quite another viewpoint El-Neweihi (1981) introduced the condition

(3.9) \[ T \geq \min[\bar{Y}/a, \bar{Z}/(1-a)] \quad \text{for all} \quad a \in (0, 1), \quad \text{where} \quad \bar{Y} \quad \text{and} \quad \bar{Z} \]
are independent and distributed as \( T \).
here $\mathbb{V}^{x} \triangleright \mathbb{V}$ means that $E_\phi(\mathbb{V}) < E_\phi(\mathbb{V})$ for all functions $\phi: \mathbb{R}^m \rightarrow \mathbb{R}$ that are increasing in each argument and for which the expectations exist. El-Neweihi (1981) showed that

$$G_5 \subset (3.9) \subset C_5 (-C_4).$$

A nonnegative function $h$ defined on $\mathbb{R}_+^k$ is said to be log subadditive if

$$h((\alpha + \beta)z) < h(\alpha z)h(\beta z) \quad \text{for all} \quad z \in \mathbb{R}_+^k, \alpha, \beta > 0.$$

The random vector $\tilde{z}$ is multivariate NBU in the sense of Savits (1983) if

$$E_h(z, \tilde{z}) \text{ is log subadditive in } z \quad \text{for every function } h \text{ defined on } \mathbb{R}_+^m \times \mathbb{R}_+^n \text{ (for some } m) \text{ which is log subadditive and is such that } h(z, \tilde{z}) \text{ is increasing in } \tilde{z} \quad \text{for all fixed } z.$$

According to Savits (1983),

$$(3.9) \subset (3.10) \subset C_4.$$

In the following summary "\subset" is replaced by "\rightarrow" for typographical reasons.

\begin{align*}
(3.7) & \rightarrow (3.8) \\
(3.2) & \rightarrow (3.1) \\
(3.3) & \rightarrow (3.4) \\
G_1 & \rightarrow G_3 \\
G_5 & \rightarrow (3.9) \rightarrow (3.10) \rightarrow C_5 \leftrightarrow C_4 \leftrightarrow \tilde{z} \rightarrow C_1 \rightarrow G_1 \rightarrow (3.5)
\end{align*}
All these implications have been shown above with the one exception that $G_3 \mathbf{c}$ (3.8), which can be proved with the aid of the following lemma.

3.1 Lemma. If $T_1, \ldots, T_n$ are independent NBU random variables, then (3.7) holds.

Proof. According to the minimal path representation for $\tau$ (Esary and Marshall, 1970),

$$\tau(T_1/\omega^{(1)}, \ldots, T_n/\omega^{(n)}) = \max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_\alpha} [T_j/\omega^{(j)}]$$

where $A_\alpha \mathbf{c} \{(j, \ell): j = 1, \ldots, n, \ell = 1, \ldots, m\}, \alpha = 1, \ldots, K,$ are the minimal path sets of $\tau$. Let

$$X_j(z) = 1 \text{ if } T_j/z > 1$$
$$= 0 \text{ otherwise}$$

Then $\tau(T_1/\omega^{(1)}, \ldots, T_n/\omega^{(n)}) > 1$ if and only if $\max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_\alpha} X_j(\omega^{(j)}) = 1$. Since $T_1, \ldots, T_n$ are independent, the joint distribution of $X_j(\omega^{(j)}), j = 1, \ldots, n, \ell = 1, \ldots, m$ is determined by

$$(F_j(\omega^{(j)}), 1 \leq j \leq n, 1 \leq \ell \leq m) \equiv$$
$$(F_1(\omega^{(1)}), \ldots, F_n(\omega^{(n)}), F_2(\omega^{(2)}), \ldots, F_m(\omega^{(m)}), ..., F_n(\omega^{(n)}), ..., F_m(\omega^{(m)}))$$

It follows that there exists a function $h: [0, 1]^mn \rightarrow [0, 1]$ such that

$$P(\tau(T_1/\omega^{(1)}, \ldots, T_n/\omega^{(n)}) > 1) = P(\max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_\alpha} X_j(\omega^{(j)}) = 1)$$

$$= h((F_j(\omega^{(j)}), 1 \leq j \leq n, 1 \leq \ell \leq m)).$$

Because

$$\phi(z_1^{(1)}, \ldots, z_m^{(1)}, z_1^{(2)}, \ldots, z_m^{(2)}, \ldots, z_1^{(n)}, \ldots, z_m^{(n)}) = \max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_\alpha} z^{(j)}.$$
is an increasing function, \( h \) is increasing. Since \( F_1, \ldots, F_n \) are NBU

\[
(3.11) \quad h((\bar{F}_j(u_{\ell}^{(j)} + v_{\ell}^{(j)})), 1 \leq j \leq n, 1 \leq \ell \leq m) \leq h((\bar{F}_j(u_{\ell}^{(j)})) \bar{F}_j(v_{\ell}^{(j)})), 1 \leq j \leq n, 1 \leq \ell \leq m)
\]

Let \( \bar{S} = (S_1, \ldots, S_n) \) be a random vector independent of and distributed as \( \bar{T} \), and let \( Y_j(z) = 1 \) if \( S_j/z > 1 \), \( Y_j(z) = 0 \) otherwise. Notice that \((z_1, \ldots, z_n), (X_j(z_1), \ldots, X_j(z_n))\) and \((Y_j(z_1), \ldots, Y_j(z_n))\) are similarly ordered. It follows that if \( \bar{u}^{(j)} \) and \( \bar{v}^{(j)} \) are similarly ordered, then

\[
(X_j(u_1^{(j)}), \ldots, X_j(u_n^{(j)})), \quad (Y_j(v_1^{(j)}), \ldots, Y_j(v_n^{(j)}))
\]

and

\[
(X_j(u_1^{(j)})Y_j(v_1^{(j)}), \ldots, X_j(u_n^{(j)})Y_j(v_n^{(j)}))
\]

are all similarly ordered with an ordering that depends only on the common ordering of \( \bar{u}^{(j)} \) and \( \bar{v}^{(j)} \). Also, \( (X_j(u_1^{(j)}), \ldots, X_j(u_n^{(j)})) \) and \( (Y_j(v_1^{(j)}), \ldots, Y_j(v_n^{(j)})) \) are independent, \( j = 1, \ldots, n \). This insures that \( X_j(u_\ell^{(j)})Y_j(v_\ell^{(j)}), \ell = 1, \ldots, n \) have the dependency structure required to yield

\[
(3.12) \quad P\{ \max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_{\alpha}} X_j(u_\ell^{(j)})Y_j(v_\ell^{(j)}) = 1 \}.
\]

Since

\[
\max_{1 \leq \alpha \leq K} \min_{(h, \ell) \in A_{\alpha}} X_j(u_\ell^{(j)})Y_j(v_\ell^{(j)}) \leq \max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_{\alpha}} X_j(u_\ell^{(j)}),
\]

\[
\max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_{\alpha}} X_j(u_\ell^{(j)})Y_j(v_\ell^{(j)}) \leq \max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_{\alpha}} Y_j(v_\ell^{(j)})
\]

it follows that
\[
\max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_{\alpha}} X_j(u^{(j)}_{\ell}) Y_j(v^{(j)}_{\ell}) \\
< \max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_{\alpha}} X_j(u^{(j)}_{\ell}) \max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_{\alpha}} Y_j(v^{(j)}_{\ell})
\]

Thus

\[
P\{ \max_{1 \leq \alpha \leq K} \min_{(j, \ell) \in A_{\alpha}} X_j(u^{(j)}_{\ell}) Y_j(v^{(j)}_{\ell}) = 1 \} \\
= h(\bar{F}_j(u^{(j)}_{\ell}), 1 \leq j \leq n, 1 \leq \ell \leq m) h(\bar{F}_j(v^{(j)}_{\ell}), 1 \leq j \leq n, 1 \leq \ell \leq m).
\]

By combining (3.11), (3.12) and (3.13), (3.7) is obtained.

3.2 **Proposition.** \( G_3 \subset (3.8). \)

**Proof.** Assume that \( (T_1, \ldots, T_n) \) have a distribution in \( G_3 \), i.e.,

\[
T_j = \tau_j(X_1, \ldots, X_k)
\]

where \( \tau_j \) is a coherent life function of order \( k \), and \( X_1, \ldots, X_k \) are independent NBUE random variables. For any coherent life function of order \( kn \) and nonnegative vectors \( u^{(1)}, \ldots, u^{(n)} \) of dimension \( m \),

\[
\tau(T_1/u^{(1)}, \ldots, T_n/u^{(n)}) = \tau(\tau_1(X)/u^{(1)}, \ldots, \tau_n(X)/u^{(n)})
\]

can be written in the form

\[
\tau^* \left( \frac{X_1}{u^{(1)}}, \ldots, \frac{X_k}{u^{(n)}} \right)
\]

for some coherent life function \( \tau^* \) of order \( kn \). If \( (u^{(1)}, \ldots, u^{(n)}) \) and \( (v^{(1)}, \ldots, v^{(n)}) \) are similarly ordered then (3.8) follows from Lemma 3.1
applied to $\tau^*$. \\

4. **Preservation properties.**

There are a number of desirable properties that a family $\mathcal{F}$ of multivariate NBU distributions may or may not satisfy, and many of these properties are known to be satisfied by the various families discussed in Sections 2 and 3. Such results are summarized in this section.

Properties of interest include:

(4.1) If $F \in \mathcal{F}$, then the one dimensional marginal of $F$ are NBU,

(4.2) If $F \in \mathcal{F}$ is of dimension $n > 1$, then the $n-1$ dimensional marginals of $F$ are in $\mathcal{F}$,

(4.3) If $(T_1, \ldots, T_n)$ has a distribution in $\mathcal{F}$, then for any permutation $\pi$, $T_{\pi(1)}, \ldots, T_{\pi(n)}$ has a distribution in $\mathcal{F}$,

(4.4) $\mathcal{F}$ is closed under limits in distribution,

(4.5) If $T_1, \ldots, T_n$ are independent univariate NBU random variables, then $\sim = (T_1, \ldots, T_n)$ has a distribution in $\mathcal{F}$,

(4.6) If $\sim^{(1)}$ and $\sim^{(2)}$ have distributions in $\mathcal{F}$ and are independent, then $(\sim^{(1)}, \sim^{(2)})$ has a distribution in $\mathcal{F}$,

(4.7) If $T$ is a univariate NBU random variable, then $\sim = (T, \ldots, T)$ has a distribution in $\mathcal{F}$,

(4.8) If $\sim^{(1)}$ and $\sim^{(2)}$ have distributions in $\mathcal{F}$ of the same dimension and are independent, then $\sim^{(1)} + \sim^{(2)}$ has a distribution in $\mathcal{F}$,

(4.8') If $\sim = (T_1, \ldots, T_n)$ has a distribution in $\mathcal{F}$ and if $U_i = \sum_{j \in S_i} T_j$ for some $S_i \subseteq \{1, \ldots, n\}$, $i = 1, \ldots, k$, then $(U_1, \ldots, U_k)$ has a distribution in $\mathcal{F}$,

(4.9) If $\sim^{(1)}$ and $\sim^{(2)}$ have distributions in $\mathcal{F}$ of the same dimension $n$ and are independent, then
\[ T^{(1)} \wedge T^{(2)} = (\min(T^{(1)}_1, T^{(2)}_1), \ldots, \min(T^{(1)}_n, T^{(2)}_n)) \] has a distribution in \( \mathcal{F} \).

(4.9') If \( T = (T_1, \ldots, T_n) \) has a distribution in \( \mathcal{F} \) and \( U_j = \min_{j \in S_i} T_j \) for some \( S_i \subset \{1, \ldots, n\}, i = 1, \ldots, k \), then \( (U_1, \ldots, U_k) \) has a distribution in \( \mathcal{F} \).

(4.10) If \( \tau_1, \ldots, \tau_n \) are coherent life functions of order \( n \) and \( T \) has an \( n \)-dimensional distribution in \( \mathcal{F} \), then the distribution of \( (\tau_1(T), \ldots, \tau_n(T)) \) is in \( \mathcal{F} \).

(4.11) If \( T = (T_1, \ldots, T_n) \) has a distribution in \( \mathcal{F} \), then for all \( a_1, \ldots, a_n > 0 \), \( (a_1 T_1, \ldots, a_n T_n) \) has a distribution in \( \mathcal{F} \).

Notice that (4.6) implies (4.5), and whenever (4.6) is satisfied, then (4.8') implies (4.8) and (4.9') implies (4.9).

All of the families of NBU distributions discussed in Section 2 and 3 satisfy (4.1), (4.2), (4.3) and (4.4). Whether or not (3.7) satisfies (4.5) is not known, but all of the remaining families satisfy (4.5). In Table 4.1, further properties are given where known, together with some references.

Numbers in brackets [ ] refer to references in the bibliography; numbers in braces \{ \} refer to examples in this paper. Properties without reference are easy to obtain or are given by Marshall and Shaked (1985).

4.1 Example. Following Section 7 and Esary and Marshall (1979), one can see that if \( (T_1, T_2) \in \mathcal{G}_1 \), then either \( T_1 \) and \( T_2 \) are independent or the distribution of \( (T_1, T_2) \) has positive mass on at least one ray \( t_1 = \alpha t_2 \). As noted in Subsection 4.5 of Block and Savits (1982), the distribution of \( T^{(1)}(1) + T^{(2)}(2) \) can be absolutely continuous with dependent components even though \( T^{(1)}(1) \) and \( T^{(2)}(2) \) have distribution in \( \mathcal{G}_1 \). Then, the distribution of \( T^{(1)}(1) \) and \( T^{(2)}(2) \) cannot be in \( \mathcal{G}_1 \).
| $G_1$ | Yes | Yes | No(4.1) | No  | Yes | Yes | No(6.9) | No(6.6) |
| $G_2$ | Yes | Yes | Yes     | Yes | ?   | ?   | No(4.2) | No(6.6) |
| $G_3$ | Yes | Yes | Yes     | Yes | ?   | ?   | No(4.2) | Yes     |
| $G_4$ | Yes | Yes | Yes     | Yes | Yes | Yes | Yes     | Yes     |
| $G_5$ | Yes | Yes | Yes     | Yes | Yes | Yes | Yes     | Yes     |
| C_4   | Yes | Yes | Yes     | Yes | Yes | Yes | Yes     | Yes     |
| C_3   | Yes | Yes | Yes     | Yes | Yes | Yes | Yes     | Yes     |
| C_2   | Yes | Yes | Yes     | Yes | ?   | No(6.10) | No(6.10) | No(6.10) |
| C_1   | Yes | Yes | Yes     | Yes | ?   | Yes | Yes     | No(6.8) | Yes     |
| $G_1$ = (3.5) | Yes | Yes | ?   | Yes | Yes | Yes | No(6.8) | No(6.7) |
| (3.1) | Yes | No(6.11) | ?   | No(6.11) | Yes | No(6.11) | No(6.8) | Yes     |
| (3.3) | Yes | Yes | ?   | Yes | Yes | Yes | No(6.8) | No(6.6) |
| (3.4) | Yes | Yes | ?   | Yes | Yes | Yes | No(6.8) | No(6.6) |

Table 4.1
4.2 Example. According to a result of Shaked (1983), if $X$ and $Y$ are independent NBU random variables such that $X + Y$ has an exponential distribution then for one of the random variables say $Y$, $P(Y = 0) = 1$. This means that Subsection 4.4 of Block and Savits (1982) applies to show that neither $G_2$ nor $\tilde{G}_2$ can satisfy (4.10).

5. Multivariate NBU Distributions with Exponential Marginals.

The results of Shaked (1983) can be used to identify all distributions with exponential marginals in some of the $G_i$ classes. In particular, it follows from Theorem 3.1 of Shaked (1983) that the class $G_1(\text{exp})$ consisting of the multivariate exponential distributions of Marshall and Olkin (1967) includes all distributions with exponential marginals in $G_1$ or $G_3$. Similarly, $\tilde{G}_1(\text{exp})$ includes all distributions with exponential marginals in $\tilde{G}_1$ or $G_3$ where $\tilde{G}_1(\text{exp})$ is the class of joint distributions of random variables $T_1, \ldots, T_n$ having a representation $T_j = \min(a_{j1}X_1, \ldots, a_{jk}X_k)$, $j = 1, \ldots, n$ where $X_1, \ldots, X_k$ are independent and exponentially distributed, $a_{ji} > 0$ and $\Sigma_{i=1}^n a_{ji} > 0$.

From Theorem 3.2 of Shaked (1983) it follows that $(T_1, \ldots, T_n)$ has a distribution with exponential marginals in $G_2$ if and only if for all $i \neq j$, $T_i = T_j$ with probability one or $T_i$ and $T_j$ are independent. Similarly the distribution of $(T_1, \ldots, T_n)$ has exponential marginals and is in $\tilde{G}_2$ if and only if for all $i \neq j$, $T_i = a_{ij}T_j$ for some $a_{ij} > 0$ or $T_i$ and $T_j$ are independent.

The identification of all distributions with exponential marginals in $G_i$ or $\tilde{G}_i$, $i = 1, \ldots, 6$ has not been accomplished. Notice that for these classes, another question arises: What is the class of possible joint distributions for $T_1, \ldots, T_n$ if $g(T_1, \ldots, T_n)$ is exponentially distributed.
for all $g \in \mathcal{C}_1$ or $\tilde{\mathcal{C}}_1$. For $i = 1$, $\mathcal{C}_1(\exp)$ and $\tilde{\mathcal{C}}_1(\exp)$ are the answers to this question, where $\mathcal{C}_1(\exp)$ [respectively, $\tilde{\mathcal{C}}_1(\exp)$] is the class satisfying condition (d) [respectively, condition (c)] of Esary and Marshall (1974). But for $i \neq 1$, the answers do not include the case of independence, as already noted in Figure 4.1 of Marshall and Shaked (1985). For $\mathcal{C}_3$, this question has been discussed by Esary and Marshall (1979, p. 363) in connection with $\mathcal{C}_3(\text{IHRA})$, their Condition B. The answer includes all distributions with exponential marginals for which there exists a permutation $\pi$ such that

$$P(T_{\pi(1)} < \cdots < T_{\pi(n)}) = 1.$$ 

For $\tilde{\mathcal{C}}_3$ and $\tilde{\mathcal{C}}_4$, even more stringent conditions follow from Theorem 2.6 of Block and Savits (1979): $T_j = c_j T_1$ for some $c_j > 0$, $j = 1, \ldots, n$ and $T_1$ is exponentially distributed.

6. Some Examples and Counterexamples.

Let $T_i$ be a random variable with survival function $\bar{F}_i$, $i = 1, 2$, and denote the joint survival function of $T_1, T_2$ by $\bar{F}$. The well-known Hoeffding-Fréchet bounds are

$$\bar{L}(t_1, t_2) < \bar{F}(t_1, t_2) < \bar{U}(t_1, t_2), \quad -\infty < t_1, t_2 < \infty,$$

where

$$\bar{L}(t_1, t_2) = \max[0, \bar{F}_1(t_1) + \bar{F}_2(t_2) - 1],$$

$$\bar{U}(t_1, t_2) = \min[\bar{F}_1(t_1), \bar{F}_2(t_2)], \quad -\infty < t_1, t_2 < \infty.$$ 

Both $\bar{L}$ and $\bar{U}$ are survival functions with marginals $\bar{F}_1$ and $\bar{F}_2$. In case $F_1$ and $F_2$ are NBU, it is of interest to determine which of the classes of Sections 2 and 3 contain $\bar{L}$ and $\bar{U}$. The following results are not complete, but do provide some examples of interest.
6.1 Example. If $F_1$ and $F_2$ are NBU, then $L$ satisfies (3.1).

Proof. Trivially, if $L(s_1 + t_1, s_2 + t_2) = 0$, then

$$L(s_1 + t_1, s_2 + t_2) < L(s_1, s_2) L(t_1, t_2),$$

so assume $s_1, s_2, t_1, t_2$ are such that

$$L(s_1 + t_1, s_2 + t_2) = F_1(s_1 + t_1) + F_2(s_2 + t_2) - 1 > 0.$$ Then

$$F_1(s_1 + t_1) + F_2(s_2 + t_2) - 1 < F_1(s_1) F_1(t_1) + F_2(s_2) F_2(t_2) - 1$$

$$< F_1(s_1) F_1(t_1) + F_2(s_2) F_2(t_2) - 1 + [1-F_1(s_1)][1-F_2(t_2)]$$

$$+ [1-F_1(t_1)][1-F_2(s_2)] = [F_1(s_1) + F_2(s_2) - 1][F_1(t_1) + F_2(t_2) - 1].$$

6.2 Example. If $F_1$ and $F_2$ are NBU, then $L \in \mathcal{G}_3$.

Proof. Let $(S_1, S_2)$ have survival function $L$. For $a_1 > 0, a_2 > 0$, it is necessary to show that $\min(a_1 S_1, a_2 S_2)$ and $\max(a_1 S_1, a_2 S_2)$ are NBU. Here, it is possible to assume without loss of generality that $a_1 = a_2 = 1$.

To show that $\min(S_1, S_2)$ is NBU, it is necessary to show that

$$\tilde{G}(s + t) < \tilde{G}(s) \tilde{G}(t), \quad s, t > 0,$$

where $\tilde{G}(t) = L(t,t) = P\{\min(S_1, S_2) > t\}$. But this follows from Example 6.1.

To show that $\max(S_1, S_2)$ is NBU, it is necessary to show that

$$\tilde{H}(s + t) < \tilde{H}(s) \tilde{H}(t), \quad s, t > 0$$

where $\tilde{H}(t) = P\{\max(S_1, S_2) > t\}$. This is trivial if $\tilde{H}(s + t) = 1$, so assume that $s$ and $t$ are such that $\tilde{H}(s + t) = F_1(s+t) + F_2(s+t) < 1$. Then

$$F_1(s+t) + F_2(s+t) < F_1(s) F_1(t) + F_2(s) F_2(t) < [F_1(s) + F_2(s)][F_1(t) + F_2(t)].$$

6.3 Example. Observe that if $F_1$ and $F_2$ are NBU, then $L \notin \mathcal{G}_5$ because all distributions in $\mathcal{G}_5$ are associated in the sense of Esary, Proschan and Walkup (1967). If $F$ is associated, then $F(t_1,0) F(0,t_2) < F(t_1, t_2)$, contradicting Example 6.1 unless equality holds identically, which it does not.
for \( L \).

6.4 Example. The conditions that \( F_1 \) and \( F_2 \) are NBU do not imply that \( U \in C_1 \), hence they do not imply that \( U \) satisfies any of the multivariate NBU conditions discussed in this paper.

To see this, let \((T_1, T_2)\) have survival function \( \bar{U} \) and have NBU marginals. Then for some nonnegative increasing subadditive functions \( g_1 \) and \( g_2 \)

\[ (T_1, T_2) = (g_1(X), g_2(X)) \]

where \( X \) has an exponential distribution with \( \text{EX} = 1 \). Note that the right continuous inverses \( g_1^{-1} \) and \( g_2^{-1} \) of \( g_1 \) and \( g_2 \) are superadditive. If \( U \in C_1 \), then necessarily

\[ \max[g_1^{-1}(s+t), g_2^{-1}(s+t)] > \max[g_1^{-1}(s), g_2^{-1}(s)] + \max[g_1^{-1}(t), g_2^{-1}(t)], \ s, t > 0. \]

Fix \( a \) and \( b \) such that \( a < b < 2a \) and consider the case that

\[ g_1^{-1}(t) = mh_1 \text{ if } ma < t < (m+1)a, \ m = 0, 1, \ldots, \]

\[ g_2^{-1}(t) = mh_2 \text{ if } mb < t < (m+1)b, \ m = 0, 1, \ldots, \]

where \( 0 < 3h_1 < h_2 \). If \( a < s < b < t < 2a, s+t < 2b \), then the left side of (6.2) is \( h_2 \) and the right side of (6.2) is \( h_1 + h_2 \), so (6.2) fails.

Example 6.4 may be of particular interest when compared with the following:

6.5 Example. If \( F_1 \) and \( F_2 \) are IHRA, (increasing hazard rate average), then \( U \in G_5(\text{IHRA}) \), that is, there exist independent IHRA random variables \( X_1, \ldots, X_k \) and functions \( g_1, g_2 \) in \( G_5 \) such that \( T_i = g_i(X_1, \ldots, X_k), \ i = 1, 2 \), have the joint distribution \( U \).

To see this, observe that if \((T_1, T_2)\) has distribution \( U \) with IHRA marginals then (6.1) holds where \( X \) has an exponential distribution with
EX = 1, \( g_1, g_2 \) are nonnegative increasing, and \( g_1^{-1}, g_2^{-1} \) are starshaped.

6.6 Example. Let \( X \) have an exponential distribution with \( EX = 1 \). Then \((X,X)\) has the survival function

\[
\bar{F}(t_1,t_2) = \min[e^{-t_1}, e^{-t_2}] = e^{\max(t_1,t_2)}, \quad t_1, t_2 > 0.
\]

This distribution satisfies (3.3), (3.4) and belongs to most other classes discussed in this paper (not (3.1) or (3.2) because of its positive dependence).

On the other hand \((X,2X)\) has the survival function

\[
\bar{U}(t_1,t_2) = \min[e^{-t_1}, e^{-t_2/2}] = e^{\max(t_1,t_2/2)}, \quad t_1, t_2 > 0.
\]

Although this distribution clearly belongs to \( \tilde{G}_1 \), it violates (3.4) when \( s = 1, \ z = (0,2), \ s_2 + \z = (1,3) \). Thus

\[
\tilde{G}_1 \notin (3.4).
\]

Moreover, this example shows that the classes \( G_1, G_2, G_3 \), (3.3) and (3.4) fail to satisfy (4.11).

6.7 Example. With \( a^2 > 3 \), the counterexample that Esary and Marshall (1979) use to show that their condition B does not imply their Condition A can be used to show that \( C_3 \notin \tilde{C}_1 \). This can be done by verifying that when \( a^2 > 3 \) then the hazard rate of \( \min(T_1, aT_2) \) at \( z = 0 \) is larger than the hazard rate at \( z = 1/2 \). The second sentence in page 370 of Esary and Marshall (1979) contains some misprints. It should read: "The derivative of \( -z^{-1}\log P[\min(T_1, aT_2) > z] \), for \( 1/2 < z < a/2(a-1) \) is nonnegative if and
only if
\[
\frac{2z}{2z-a} < \log \frac{a-2z}{2a}
\]
which is violated for a sufficiently large."

From this example it follows that \( C_1 \not\subseteq \text{C}_1 \). Thus condition (4.11) fails for the class \( C_1 \).

6.8 Example. Let \( T_1 \) and \( T_2 \) have the joint probabilities \( P(T_1 = t_1, T_2 = t_2) \) given by

<table>
<thead>
<tr>
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<th>2</th>
<th>3</th>
<th>5</th>
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</thead>
<tbody>
<tr>
<td>5</td>
<td>.15</td>
<td>.25</td>
<td>.19</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>.25</td>
</tr>
<tr>
<td>2</td>
<td>.01</td>
<td>0</td>
<td>.15</td>
</tr>
</tbody>
</table>

Since \( \max(T_1, T_2) \) is either 2 or 5, this random variable cannot be NBU so the distribution of \( (T_1, T_2) \) is not in \( C_3 \). On the other hand, tedious computations show that (3.1) holds. Thus (3.1) \( \not\subseteq C_3 \).

6.9 Example. Let \( X, Y \) and \( Z \) be independent random variables uniformly distributed on \([0,1]\) and let \( \tilde{T} = (T_1, T_2) = (\max(X,Z), \max(Y,Z)) \). Then \( \tilde{T} \) has the distribution \( F \in G_3 \) given by

\[
F_{\tilde{T}}(t_1, t_2) = m^2 M, \quad 0 \leq m \leq M \leq 1,
\]

where \( m = \min(t_1, t_2) \) and \( M = \max(t_1, t_2) \). If \( F \in G_1 \), then \( F_{\tilde{T}} \) also has the form

\[
F_{\tilde{T}}(t_1, t_2) = F_U(t_1)F_V(t_2)F_W(\max(t_1, t_2)), \quad t_1, t_2 > 0.
\]
Solution of this functional equation leads to a contradiction, so that $F \not\in G_1$. Thus $G_3 \not\in G_1$. This example also shows that Condition (4.10) fails for the class $G_1$.

6.10 Example. Suppose $T_1, T_2$ have the joint distribution which puts mass $\frac{1}{3}$ uniformly on the line segment $3t_1 + t_2 = 1$, $0 < t_1 < \frac{1}{3}$; mass $\frac{1}{3}$ uniformly in the rectangle $\frac{1}{3} < t_1 < \frac{2}{3}$, $0 < t_2 < 1$, and mass $\frac{1}{3}$ uniformly on the line segment $t_1 + \frac{1}{3} t_2 = 1$, $\frac{2}{3} < t_1 < 1$.

Then marginally $T_1$ and $T_2$ are uniformly distributed on $[0, 1]$, and $T_1 + T_2$ has the density $f_S$ given by

$$f_S(z) = z + \frac{1}{6}, \quad \frac{1}{3} < z < \frac{2}{3}$$
$$= \frac{5}{6}, \quad \frac{2}{3} < z < \frac{4}{3}$$
$$= (\frac{13}{6}) - z, \quad \frac{4}{3} < z < \frac{5}{3}$$
$$= 0, \quad \text{elsewhere.}$$
Since this density is log concave it is NBU, and consequently the distribution of \((T_1, T_2)\) is in \(C_2\).

The distribution of \(W = T_1 + \frac{1}{3} T_2\) puts mass \(1/3\) at the points \(1/3\) and 1; between \(1/3\) and 1, \(W\) has a density given by

\[
f_W(w) = 3w - 1, \quad \frac{1}{3} < w < \frac{2}{3}; \quad f_W(w) = 3 - 3w, \quad \frac{2}{3} < w < 1.
\]

It is easy to see that for small \(\varepsilon\), \(F_W(1/3+\varepsilon) \leq F_W(1/3) < F_W(2/3+\varepsilon)\) so that \(W = T_1 + \frac{1}{3} T_2\) is not NBU.

The random variable \(U = \min(T_1, T_2)\) has a density given by

\[
f_U(u) = 2, \quad 0 < u < 1/4; \quad f_U(u) = 2/3, \quad 1/4 < u < 1/3;
\]

\[
f_U(u) = 2(1-u), \quad 1/3 < u < \frac{2}{3}; \quad f_U(u) = 4/3, \quad 2/3 < u < 3/4,
\]

\[
f_U(u) = 0; \quad \text{elsewhere.}
\]

This distribution has hazard rate \(r\) such that \(2 = r(0) > r(1/4) = 4/3\) and consequently is not NBU.

This example shows that \(C_2 \not\subseteq C_2,\ C_2 \not\subseteq C_1\), and of course this means that \(C_2\) fails to satisfy (4.9'), (4.10'), and (4.11).

6.11 Example. It is well known that increasing functions of independent random variables are associated and hence have nonnegative covariances (Esary, Proschan and Walkup, 1967). As was already noted in Section 3, random vectors satisfying (3.1) or (3.2) must have nonpositively correlated components. Since vectors of independent NBU random variables belong to (3.1) and (3.2) it follows that conditions (3.1) and (3.2) can be violated by increasing
functions of random variables satisfying (3.1) or (3.2). The class \( G_6 \)
contains only associated random vectors whereas the class (3.2) has as members
some negatively correlated random vectors. Hence (3.2) \( \not\in G_6 \). Similarly the
positive dependence of members of \( G_1 \) and the negative dependence of members
of (3.1) implies \( G_1 \not\in (3.1) \) .
References


