TESTING FOR EXPONENTIAL AND MARSHALL-OLKIN DISTRIBUTIONS

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Abstract

Statistical tests for testing composite goodness-of-fit hypotheses to the exponential and bivariate Marshall-Olkin exponential distributions are proposed and investigated. The tests are based on the empirical moment generating function and are constructed to be consistent against all alternative distributions. A small sample simulation study indicates that the proposed tests have very good power properties.

Key Words: exponential distribution, goodness-of-fit tests, Marshall-Olkin distribution, multivariate empirical moment generating function.
1. **Introduction**

Exponential distributions play a central role in the modelling and analysis of stochastic processes in which events occur as a series of random shocks. Such processes occur both in survival analysis and reliability theory; the essential feature of the processes is simply that component failure is caused by a random shock rather than by wear. The need to model more complicated multicomponent systems has led to the introduction of a variety of multivariate exponential distributions; see for example Morgenstern (1956), Gumbel (1960), Freund (1961), Moran (1967), Marshall and Olkin (1967a,b), Downton (1970), Hawkes (1972), Paulson (1973), Block and Basu (1974) and Proschan and Sullo (1974). The most important of these distributions seem to be the Marshall-Olkin and the Block-Basu distributions. The latter distribution is the absolutely continuous part of the Marshall-Olkin distribution but does not have exponential marginals. The Marshall-Olkin distribution has exponential marginals, a lack of memory property and may be derived from both fatal and non-fatal shock models. In the bivariate case, it has recently been derived from the unique natural bivariate Bernoulli distribution (Marshall and Olkin (1985)). It has been applied in reliability theory (Harris (1968)) where it may be used to model the time to failure of paired components such as aircraft engines or the time to the registration of events on adjacent Geiger counters and may be used in survival analysis to model the time to failure of paired organs such as eyes, lungs, kidneys etc. (see the third example in section 5). A further application of the model has been to competing risks problems (Moeschberger and David (1971)) but this application will not be pursued in the present paper. The estimation of the parameters of the Marshall-Olkin distribution has been investigated by Arnold (1968) but there are no tests available for testing for goodness-of-fit for the distribution. In contrast, in the
univariate case, there is a daunting literature on the subject of testing for exponentiality; for a recent survey, see the review paper of Doksum and Yandell (1984).

Our purpose in this paper is two-fold: firstly, to develop a new test for univariate exponentiality and secondly to extend the test to test for the Marshall-Olkin distribution. The test we propose is based on the empirical moment generating function. In the present context, this approach leads to a powerful test which is consistent against all alternatives and which extends in a straightforward way to the Marshall-Olkin problem. It would also appear to be a useful approach towards deriving tests for some of the other multivariate exponential distributions though we will not consider these problems in this paper.

Specifically, let \( X_1, \ldots, X_n \) be independent and identically distributed random \( d \)-vectors, \( d \geq 1 \), with common distribution function \( F \). If \( d = 1 \), the hypothesis of exponentiality is
\[
H_1: F(x) = 1 - \exp(-x/\lambda), \quad x \geq 0, \text{ for some } \lambda > 0.
\]

Under \( H_1 \) the moment generating function is
\[
\psi(t) = E \exp(tX) = (1 - \lambda t)^{-1}, \quad t < 1/\lambda,
\]
and the behaviour of \( \psi(t) \) on any interval \([-a,b]\), \( a > 0 \), \( 0 < b < 1/\lambda \), completely specifies \( H_1 \). The test we recommend is to reject \( H_1 \) at the \( \alpha \) level if

\[
M_n = \sup_{-0.45 \leq t \leq 0.45} \left| n^{1/2} \sum_{j=1}^{n} \{ \exp(tX_j/S_n) - (1 - t)^{-1} \} \right| \geq c_\alpha,
\]

where \( S_n = \{(n - 1)^{-1} \sum_{j=1}^{n} (X_j - \bar{X}_n)^2\}^{1/2} \) and \( c_\alpha \) is the \( \alpha \) critical point tabulated in table 1.

For simplicity in treating the multivariate problem, we restrict attention to \( d = 2 \) though the extension of our results to general \( d \geq 1 \) is
straightforward. The bivariate Marshall-Olkin distribution is defined by

\[ P(X > x) = P(X_1 > x_1, X_2 > x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)\}, \ x_1, x_2 > 0, \]

where \( \lambda_1, \lambda_2 > 0 \) and \( \lambda_{12} > 0 \), which has both an absolutely continuous and a singular component. The corresponding moment generating function is much easier to work with. Marshall and Olkin (1967a,b) showed that

\[
(1.2) \quad \Psi(t) = \Psi(t_1, t_2) = \frac{1 - \delta_1 t_1/(\lambda_1 + \lambda_{12}) - \delta_2 t_2/(\lambda_2 + \lambda_{12}) + \delta_3 t_1 t_2/(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}{\{1 - \delta_1 t_1/(\lambda_1 + \lambda_{12}) - \delta_2 t_2/(\lambda_2 + \lambda_{12})\} \{1 - t_1/(\lambda_1 + \lambda_{12})\} \{1 - t_2/(\lambda_2 + \lambda_{12})\}}
\]

for any \( t = (t_1, t_2) \) such that \( t_1 + t_2 < \lambda_1 + \lambda_2 + \lambda_{12} \), where

\[
\delta_1 = (\lambda_1 + \lambda_{12})/(\lambda_1 + \lambda_2 + \lambda_{12}), \ \delta_2 = (\lambda_2 + \lambda_{12})/(\lambda_1 + \lambda_2 + \lambda_{12}), \text{ and}
\]

\[
\delta_3 = \lambda_{12}/(\lambda_1 + \lambda_2 + \lambda_{12}).
\]

Since the Marshall-Olkin family is not closed under linear transformations (Marshall and Olkin (1967a, §3.3)), we are unable to propose a test for the full composite hypothesis that the distribution of \( X \) is Marshall-Olkin with some \( \lambda_1, \lambda_2 > 0 \) and \( \lambda_{12} > 0 \). However, we are able to test the following hypothesis:

\[ H_2: \text{The distribution of } X = (X_1, X_2) \text{ is Marshall-Olkin with some} \]

\[ \lambda_1, \lambda_2 > 0, \lambda_{12} > 0 \text{ with given ratios } \delta_1, \delta_2 \text{ and } \delta_3. \]

This hypothesis may be of interest in a number of practical situations since the \( \delta \)'s can be interpreted in terms of the intensities of the underlying shock processes. Two cases of primary interest are

a) \( \delta_1 = \delta_2 = 2/3, \delta_3 = 1/3 \) where the single shocks to the two components and the double shocks occur at the same unknown intensity, and

b) \( \delta_1 = \delta_2 = 3/5, \delta_3 = 1/5 \) where the rate of double shocks is half of the common, unknown rate of single shocks.

Note that if \( \delta_3 = 0 \), then \( X_1 \) and \( X_2 \) are independent and there are no double shocks and in this case there is no need to specify \( \delta_1 \) and \( \delta_2 \). For given \( 0 < \delta_1, \delta_2, \delta_3 < 1 \), set
\[ \psi_\delta(t_1, t_2) = \frac{1 - \delta_1 t_1 - \delta_2 t_2 + \delta_3 t_1 t_2}{(1 - \delta_1 t_1 - \delta_2 t_2)(1 - t_1)(1 - t_2)}, \quad \delta_1 t_1 + \delta_2 t_2 < 1. \]

Then our test of \( H_2 \) is to reject \( H_2 \) at the \( \alpha \) level if

\[ M_\delta = \sup_{-b \leq t_1, t_2 \leq b} \left| n^{-1/2} \sum_{j=1}^n \left[ \exp\{t_1 x_j/S_{n1} + t_2 x_j/S_{n2}\} - \psi_\delta(t_1, t_2) \right] \right| \geq c_{\alpha \delta}, \tag{1.3} \]

where \( 0 < b < \frac{1}{2} (\delta_1 + \delta_2)^{-1}, S_{nk} = (n - 1)^{-1} \sum_{j=1}^n (x_{jk} - \bar{x}_{nk})^2 \), \( k = 1, 2 \), and \( c_{\alpha \delta} \) is the \( \alpha \) critical point tabulated in table 1. For \( H_{2a} \), we take \( b = 0.35 \) and for \( H_{2b} \), \( b = 0.40 \).

In section 2 we develop the theoretical aspects of the tests. In fact we investigate two families of tests, one of which contains the tests described above. The selection of particular tests depends largely on finite sample properties and these are investigated in sections 3 and 4. In particular, in section 4, the finite sample power of various tests against selected alternatives is investigated by means of a simulation study. Finally, the test are applied to real data sets in section 5.

2. Asymptotic Theory

Suppose that we observe \( n \) independent random vectors

\[ X_j = (X_{j1}, \ldots, X_{jd}), \quad 1 \leq d, \quad 1 \leq j \leq n, \]

with common distribution function \( F(x), \quad x = (x_1, \ldots, x_d). \) For each \( x, y \in \mathbb{R}^d \), let \( \langle x, y \rangle = \sum_{k=1}^d x_k y_k \) denote the inner product in \( \mathbb{R}^d \). Then suppose that the moment generating function

\[ \psi(t) = E \exp(\langle t, x \rangle) = \int_{\mathbb{R}^d} \exp(\langle t, x \rangle) \, dF(x), \]

exists in a non-degenerate \( d \)-dimensional subset \( I \) of \( \mathbb{R}^d \), necessarily containing the origin. The empirical moment generating function is defined by

\[ \psi_n(t) = n^{-1} \sum_{j=1}^n \exp(\langle t, X_j \rangle) = \int_{\mathbb{R}^d} \exp(\langle t, x \rangle) \, dF_n(x), \]

where \( F_n \) is the empirical distribution function of \( X_1, \ldots, X_n \). It is straightforward to prove theorems which hold for each fixed \( t \in I \) but tests based on a fixed value of \( t \) are not consistent because the value of \( \psi(t) \)
at a fixed $t$ does not determine the underlying distribution $F$. However, the behaviour of $\psi(t)$ for $t$ in a non-degenerate finite subset of $I$ containing the origin as an inner point characterises the Marshall-Olkin and exponential distributions. Thus we develop results for $\psi_n(t)$ which hold uniformly in $t$ on suitable sets. In the present problem, the moment generating function plays the same role as the characteristic function in the problem of testing for normality (see Csörgő (1985)), but here the moment generating function yields a more usable characterisation property and is much simpler to work with than the characteristic function. The proofs of all results in this section are given in the appendix.

Motivated by the paper by Quandt and Ramsey (1978) which introduced the empirical moment generating function, Csörgő (1982) investigated the properties of $\psi_n(\cdot)$ for the case $d = 1$. The following result extends the corresponding univariate theorem.

**Theorem 1** Let $J$ be any compact subset of $I$ such that $2J \subseteq I$. Then

$$M_n(\cdot) = n^{1/2} \{ \psi_n(\cdot) - \psi(\cdot) \}$$

conveys weakly to a Gaussian process with mean zero and covariance

$$\psi(s + t) - \psi(s)\psi(t), \ s, t \in J.$$  

Here and in the sequel, weak convergence is understood to take place on the space of continuous $d$-variate functions defined on $J$; see the appendix for details. We note that the almost sure properties derived by Csörgő (1982) generalise a straightforward way.

Now set $d = 2$ and suppose that $X_j = (X_{j1}, X_{j2})$ has a covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix},$$

The limit process in Theorem 1 depends on the unknown parameters of the underlying distribution and hence cannot be used to test composite
hypotheses. For the Marshall-Olkin (and the univariate exponential) distributions, the limit process depends on \( \mathbb{E}X_{j1} = \sigma_1 = (\lambda_1 + \lambda_{12})^{-1} \) and \( \mathbb{E}X_{j2} = \sigma_2 = (\lambda_2 + \lambda_{12})^{-1} \). We consider standardising by the sample mean or the sample standard deviation. Specifically, put
\[
\psi_n(t_1, t_2) = \psi_n(t_1/X_{n1}, t_2/X_{n2}) = n^{-\frac{1}{2}} \sum_{j=1}^{n} \exp\{t_1 X_{j1} / X_{n1} + t_2 X_{j2} / X_{n2}\}
\]
and
\[
\psi_n(t_1, t_2) = \psi_n(t_1/S_{n1}, t_2/S_{n2}) = n^{-\frac{1}{2}} \sum_{j=1}^{n} \exp\{t_1 X_{j1} / S_{n1} + t_2 X_{j2} / S_{n2}\},
\]
where \( X_{nk} = n^{-\frac{1}{2}} \sum_{j=1}^{n} X_{jk} \) and \( S_{nk}^2 = (n - 1)^{-1} \sum_{j=1}^{n} (X_{jk} - X_{nk})^2, \quad k = 1, 2 \).

The second of the above standardisations is called "studentising" in the empirical characteristic function context (Murota and Takeuchi (1981)). The next two theorems give the effect of the standardisation.

**THEOREM 2** Suppose that \( \mathbb{E}X_{j1} = \mathbb{E}X_{j2} = 1 \) and that the covariance matrix \( \Sigma \) is finite. Then if \( J \) is any compact subset of \( I \) such that \( 2J \subseteq I \),
\[
n^{1/2}\{\psi_n(\cdot) - \psi(\cdot)\}
\]
converges weakly to a Gaussian process with mean zero and covariance
\[
\psi(s + t) - \psi(s)\psi(t) - t_1 \psi_1(t)\{\psi_1(s) - \psi(s)\} - s_1 \psi_1(s)\{\psi_1(t) - \psi(t)\}
\]
\[
- t_2 \psi_2(t)\{\psi_2(s) - \psi(s)\} - s_2 \psi_2(s)\{\psi_2(t) - \psi(t)\}
\]
\[
+ \lambda_{11} s_1 t_1 \psi_1(s)\psi_1(t) + \lambda_{22} s_2 t_2 \psi_2(s)\psi_2(t) + \lambda_{12} s_1 t_2 \psi_1(s)\psi_2(t) + s_2 t_1 \psi_2(s)\psi_1(t)\},
\]
\[
s = (s_1, s_2), \quad t = (t_1, t_2) \in J, \quad \psi_k(t) = \partial \psi(t) / \partial t_k, \quad k = 1, 2.
\]

**THEOREM 3** Suppose that in addition to the conditions of Theorem 2, \( \lambda_{11} = \lambda_{22} = 1 \) and \( \mathbb{E}X_{j1}^4 \) and \( \mathbb{E}X_{j2}^4 \) are finite. Then
\[
n^{1/2}\{\psi_n(\cdot) - \psi(\cdot)\}
\]
converges weakly to a Gaussian process with mean zero and covariance.
\[
\psi(s + t) - \psi(s)\psi(t) - \frac{1}{2} s_1 \psi_1(s)(\psi_1(t) - 2\psi_1(t)) - \frac{1}{2} s_2 \psi_2(s)(\psi_2(t) - 2\psi_2(t)) \\
- \frac{1}{2} t_1 \psi_1(t)(\psi_1(s) - 2\psi_1(s)) - \frac{1}{2} t_2 \psi_2(t)(\psi_2(t) - 2\psi_2(s)) \\
+ \frac{1}{4} t_1 s_1 \psi_1(t)\psi_1(s)(E(X_{j1} - 1)^4 - 1) + \frac{1}{4} t_2 s_2 \psi_2(t)\psi_2(s)(E(X_{j2} - 1)^4 - 1) \\
+ \frac{1}{4} [s_1 t_2 \psi_1(s)\psi_2(t) + s_2 t_1 \psi_1(t)\psi_2(s)]E(X_{j1} - 1)^2(X_{j2} - 1)^2 - 1), 
\]

\(s = (s_1, s_2), t = (t_1, t_2) \in J\), where \(\psi_k(t) = \partial \psi(t)/\partial t_k\) and \(\psi_{kl}(t) = \partial^2 \psi(t)/\partial t_k \partial t_l\), \(k, l = 1, 2\).

The important point is that the Marshall-Olkin distributions satisfy the conditions of theorems 2 and 3. In particular, under the hypothesis \(H_2\), the processes

\[M_{nk}(\cdot) = n^{1/2}[\psi_{nk}(\cdot) - \psi_{0}(\cdot)], \quad k = 1, 2\]

with \(I = \{t = (t_1, t_2): \delta t_1 + \delta t_2 < 1\}\) converge weakly to the limiting mean-zero Gaussian processes \(M_k(\cdot)\) with covariance functions \(\mu_k(s, t)\), \(k = 1, 2\), given by Theorems 2 and 3, respectively, with the function \(\psi\) and its derivatives replaced by \(\psi_{0}\) and its derivatives and \(\sigma_1 = \sigma_2 = 1, \sigma_{12} = \delta_3, \ E(X_{j1} - 1)^4 = E(X_{j2} - 1)^4 = 9\) and \(E(X_{j1} - 1)^2(X_{j2} - 1)^2\)

\[= 2(\delta_3 + 1)^3 - 2(\delta_3 + 1)^2 + 1.\]

The limiting processes \(M_k(\cdot)\), \(k = 1, 2\), do not depend on parameters other than \(\delta_1, \delta_2\) and \(\delta_3\) fixed by \(H_2\). Results for \(d = 1\) are easily obtained by putting \(t_2 = s_2 = 0\) in the above results, so that, in particular, \(\psi_{0}(t) = (1 - t)^{-1}\). In this case, the covariances can be simplified greatly and we obtain that, under \(H_1\), the univariate processes \(M_{nk}(t) = n^{1/2}[\psi_{nk}(t) - (1 - t)^{-1}]\) converge weakly to \(M_k(t)\), \(k = 1, 2\), with covariance functions \(\mu_k(t, s)\) given by

\[\mu_1(t, s) = t^2 s^2/([1 - t - s](1 - t)(1 - s))\]

and

\[\mu_2(t, s) = (s^3 t^3 - 4 s^2 t^3 + 2 s t^3 + 8 s^2 t^2 - 4 s^3 t^2 - 3 s t^2 + st + 2 s^3 t - 3 s^2 t)/((1 - t - s)(1 - t)^3(1 - s)^3), \quad t + s < 1.\]
Theorems 2 and 3 provide guidance to the constraints which the set \( J \) must satisfy but do not specify any particular choice of \( J \). For simplicity, we set \( J = [-b, b] \times [-b, b] \), where \( 0 < b < \frac{1}{2} (\delta_1 + \delta_2)^{-1} \). These constraints ensure that, under the null hypothesis, the limiting processes have finite variances. Neither the theorems nor simple intuition indicate whether standardisation by \( S_n \) or \( \bar{X}_n \) is preferable. On the one hand, it is clear that scale transformations are required to standardise the Marshall-Olkin distribution and \( S_n \) is a common scale estimate. However, on the other hand, the sample mean is a simpler estimate and it also estimates the scale of the distribution so it is a plausible alternative to use. The theorems do imply that under the null hypothesis,

\[
\lim_{n \to \infty} P\{ \sup_{t \in J} n^{1/2}[\psi_{nk}(t) - \psi_\delta(t)] > y \} = P\{ \sup_{t \in J} |M_k(t)| > y \} = G_k(y), \quad k = 1, 2,
\]
say. Thus if we knew \( G_k \), \( k = 1, 2 \), we could derive asymptotic critical points for the tests. Unfortunately, \( G_k \), \( k = 1, 2 \), even though it is parameter-free, is not known. For the present problem we investigated the use of two upper bounds for \( G_k(y) \), \( k = 1, 2 \), to determine approximate, conservative critical points. The bound given by Fernique (1975, p. 51) gave critical points which were too large to be useful but the bound of Borell (1975), which is much simpler to use yielded usable results. Borell (1975) showed that

\[(2.1) \quad G_k(y) < 1 - \Phi((y - m_k)/\mu_k), \quad y > 0,
\]

where \( \mu_k = \sup_{t \in J} [M_k(t,t)]^{1/2} \), \( m_k \) is the median of the distribution of \( \sup_{t \in J} |M_k(t)| \), \( k = 1, 2 \), and \( \Phi \) is the standard normal distribution function. It is straightforward to calculate \( \mu_k \) from Theorems 2 and 3 (in fact, \( \mu_k^2 = \mu_k(b,b) \), \( k = 1, 2 \)) but \( m_k \) is not known. In the next section we estimate \( m_k \) in a simulation study and then use (2.1) to determine approximate asymptotic critical points for the tests.
Since the Marshall-Olkin distribution is characterised by its moment generating function just as the univariate exponential distributions, it follows that if the null hypothesis does not hold,

$$\sup_{t \in J} |n^{1/2}(\psi_{nk}(t) - \psi_{\delta}(t))| + \infty \text{ almost surely, } k = 1, 2,$$

both when $d = 1$ and $d = 2$. Thus the tests are consistent. We investigate the power of the tests by means of a simulation study described in section 4.

3. Computation of test statistics and critical points

All the numerical calculations described in this paper were carried out on the DECsystem-20 computer at the University of Chicago. So far as possible, standard IMSL (1980) routines were used to ensure that the tests developed here are available with a minimum of additional programming and that the results are reproducible.

Notice that the functions $|n^{1/2}(\psi_{nk}(t) - \psi_{\delta}(t))|$, $k = 1, 2$, are "tied-down" at zero in the sense that they always equal zero at $t = 0$. For this reason, the test statistics were calculated by maximising $n(\psi_{nk}(t) - \psi_{\delta}(t))^2$, $k = 1, 2$, over each of the quadrants of $J$ and then taking the square root of the largest maximum. Each maximisation was carried out by the IMSL routine ZXMIN with the technique suggested in the manual (Vol. 3, p. Z-3) to incorporate the constraints imposed by the quadrants of $J$. To increase the accuracy of the routine, each evaluation of $\psi_{nk}$, $k = 1, 2$, was carried out in double precision. Typically, the procedure converged to within three significant figures in less than 500 function evaluations. In the event of non-convergence, the quadrant maximum was taken simply as $|n^{1/2}(\psi_{nk}(b) - \psi_{\delta}(b))|$, $k = 1, 2$. This choice was motivated by the discovery that under the null hypothesis, the maximum over $J$ is frequently (though not always) attained at $b$. 
The Marshall-Olkin distribution is simple to simulate although the calculation of tables of critical points is time consuming and expensive. Marshall and Olkin (1967a) showed that \((X,Y)\) has a Marshall-Olkin distribution if and only if

\[ X = \min(U,W), \quad Y = \min(V,W), \]

where \(U, V, W\) have exponential distributions with parameters \(\lambda_1, \lambda_2\) and \(\lambda_{12}\) respectively. Under the hypothesis \(H_2\), \(\lambda_1 = \lambda_2\) corresponds to the equal rate of single shocks to the two components and \(\lambda_{12}\) denotes the rate of double shocks. Alternatively, a convenient prescription in terms of the \(\delta's\) is \(\lambda_{12} = 1\) and \(\lambda_1 = \lambda_2 = (1 - \delta_1)/(2\delta_1 - 1)\). In our case, \(H_{2a}\) corresponds to \(\lambda_1 = \lambda_2 = \lambda_{12} = 1\) and \(H_{2b}\) corresponds to \(\lambda_1 = \lambda_2 = 2, \lambda_{12} = 1\). Similarly, it is straightforward to simulate critical points for a test of any other hypothesis of the form of \(H_2\).

In table 1, we present selected critical points for the tests of \(H_1\), \(H_{2a}\) and \(H_{2b}\). The critical points for each test are based on 500 simulations.

<table>
<thead>
<tr>
<th>SAMPLE SIZE n</th>
<th>Level</th>
<th>(H_1)</th>
<th>(H_{2a})</th>
<th>(H_{2b})</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>10%</td>
<td>1.11</td>
<td>2.3</td>
<td>3.8</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>1.45</td>
<td>2.8</td>
<td>4.3</td>
</tr>
<tr>
<td></td>
<td>1%</td>
<td>2.02</td>
<td>4.0</td>
<td>5.9</td>
</tr>
<tr>
<td>50</td>
<td>10%</td>
<td>1.05</td>
<td>2.4</td>
<td>4.7</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>1.23</td>
<td>2.6</td>
<td>5.9</td>
</tr>
<tr>
<td></td>
<td>1%</td>
<td>1.67</td>
<td>3.7</td>
<td>8.8</td>
</tr>
<tr>
<td>100</td>
<td>10%</td>
<td>1.00</td>
<td>2.4</td>
<td>5.9</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>1.16</td>
<td>2.8</td>
<td>6.9</td>
</tr>
<tr>
<td></td>
<td>1%</td>
<td>1.45</td>
<td>5.5</td>
<td>13.1</td>
</tr>
<tr>
<td>Borell bound</td>
<td>10%</td>
<td>2.67</td>
<td>6.87</td>
<td>13.16</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>3.31</td>
<td>8.55</td>
<td>16.36</td>
</tr>
<tr>
<td></td>
<td>1%</td>
<td>4.50</td>
<td>11.72</td>
<td>22.40</td>
</tr>
</tbody>
</table>

The test for \(H_1\) involves standardisation by \(S_n, b = 0.45\).
The test for \(H_{2a}\) involves standardisation by \(S_n, b = 0.35\).
The test for \(H_{2b}\) involves standardisation by \(S_n, b = 0.40\).
except for the critical points for the bivariate tests for $n = 100$ which were based on 350 simulations. The higher level critical points should be used with some caution, particularly in the bivariate case where many more replications are required to obtain stable estimates. Notice that the critical points derived from the Borell bound are very conservative. Of course we also simulated the critical points of the test statistics involving standardisation by $S_n$ for other choices of $b$ (the domain of supremum) as well as the critical points of the test statistics involving standardisation by $\bar{X}_n$. These are not reported here because of our findings on the power of the tests described in the next section.

4. Simulation results

A simulation study was carried out both to provide a basis for the recommendation of a particular member of the class of tests described in section 2 and to evaluate the relative small sample power of this and other tests against selected alternative distributions.

For the case $d = 1$, we considered three tests from each of the classes of tests defined by

$$\sup_{-b < t < b} \left\{ n^{1/2} \left( \psi_{nk}(t) - (1 - t)^{-1} \right) \right\}, \quad k = 1, 2.$$  

Recall that for $k = 1$, the observations are standardised by their sample mean while for $k = 2$, the observations are standardised by their sample standard deviation. The three tests from each class were evaluated for $b = 0.35, 0.40$ and $0.45$. Since $0 < b < 1/2$, these values reflect a compromise between choosing too small an interval and getting too close to $1/2$ which leads to numerical instability. For comparison, we compared these tests to three other tests for exponentiality. The first of these is the Kolmogorov-Smirnov statistic.
\[ T_{n1} = n^{1/2} \sup_{0 < y < 1} |D_n^{-1}(y) - y| \]

based on the scaled total time on test function

\[ D_n^{-1}(y) = \begin{cases} \left( nX_n \right)^{-1} \sum_{j=1}^{[ny]+1} (n+1-j)(X_n, j - X_n, j-1), & 0 < y < 1 \\ 1 & y = 1 \end{cases} \]

where \( 0 = X_n, 0 < X_n, 1 \leq \ldots \leq X_n, n \) is the ordered sample. The second test is also based on the total time on test transforms; let

\[ T_{n2} = \left| 12(n - 1)^{1/2} \left( \frac{V_n}{(n - 1)} - 0.5 \right) \right|, \]

where \( V_n = \sum_{j=1}^{n-1} j(n-j+1)(X_n, j - X_n, j-1)/ \sum_{j=1}^{n} X_n, j. \) Both of these tests and some of their optimality properties are described in Doksum and Yandell (1984, sections 5 and 6). The final test we consider is, in spirit, similar to our tests. It is the modified Kolmogorov–Smirnov statistic

\[ K_n = \sup_{0 < t < \infty} \left| F_n(t) - 1 + \exp(-t/X_n) \right|, \]

where \( F_n \) is the sample distribution function. This test is described in section 7 of Doksum and Yandell (1984).

The alternative distributions considered include the chi-squared (\( \chi^2 \)), gamma (\( \Gamma \)), Weibull (defined as on page 580 of Doksum and Yandell (1984) with \( \lambda = 1 \)) and the linear failure rate (LFR) distribution (defined on page 581 of Doksum and Yandell (1984) with \( \lambda = 1 \)). Each of these families of distributions includes the exponential distribution: \( \exp(1) = \chi^2_2 = \Gamma(1,1) = \text{Weibull}(1) = \text{LFR}(0) \). The Rayleigh distribution considered is \( \{\exp(1)\}^{1/2} \) and the Pareto (1) distribution is \( \exp(\exp(1)) \), i.e. like a log exponential.

The results of the simulations are presented in table 2. Each test is based on 10% critical points simulated from 500 exponential samples of size 50. Each empirical power is obtained from 500 samples of size 50 which means
that approximate 95% confidence intervals for the power are within 4.5 points on either side of the values given in Table 2. Within the class of tests k = 1, there is no sensitivity to the value of \( b \) in the chosen range but in the class k = 2, there is an increase in power as \( b \) increases. Except against the \( \Gamma(3,1) \) distribution, the class k = 2 performs better than k = 1. Moreover, the performance of the class k = 2 compares very favourably with that of the three other tests. (Of these, \( T_{n2} \) performs the worst.)

**TABLE 2**

PERCENTAGE POWER FOR 10% TESTS OF EXPONENTIALITY FOR \( n = 50 \).

| DISTRIBUTION         | \( \chi_1^2 \) | \( \chi_3^2 \) | \( \Gamma(.5,1) \) | \( \Gamma(1.5,1) \) | \( \Gamma(3,1) \) | WEIBULL (.5) | WEIBULL (2) | LOGNORMAL | LN(0,1)! | U(0,1) | RAYLEIGH | LFR (.5) | LFR (1) | LFR (2) | PARETO (1) |
|----------------------|---------------|---------------|-------------------|-------------------|-----------------|--------------|-------------|-----------|----------|---------|---------|---------|--------|--------|---------|----------|
|                      | 0.35          | 0.40          | 0.45              | 0.35              | 0.40            | 0.45         | \( T_{n1} \) | \( T_{n2} \) | \( K_n \)            |
| \( \chi_1^2 \)      | 60            | 64            | 63                | *                 | *               | *            | 63          | 55        | 54       |
| \( \chi_3^2 \)      | 99            | 99            | *                 | *                 | *               | *            | *           | *         | *        |
| \( \Gamma(.5,1) \)  | 71            | 70            | 70                | 74                | 76              | 81           | 89          | 87        | 91       |
| \( \Gamma(1.5,1) \) | 36            | 39            | 39                | 51                | 54              | 59           | 56          | 56        | 51       |
| \( \Gamma(3,1) \)   | 99            | 99            | 99                | 49                | 55              | 63           | *           | *         | *        |
| WEIBULL (.5)        | 98            | 98            | 98                | *                 | *               | *            | *           | *         | 93       |
| WEIBULL (2)         | *             | *             | *                 | *                 | *               | *            | *           | *         | *        |
| LOGNORMAL           | 46            | 47            | 46                | 91                | 92              | 94           | 37          | 23        | 42       |
| LN(0,1)!            | 59            | 65            | 65                | *                 | *               | *            | 66          | 57        | 54       |
| U(0,1)              | *             | *             | *                 | *                 | *               | *            | *           | 99        | 97       |
| RAYLEIGH             | *             | *             | *                 | *                 | *               | *            | *           | *         | *        |
| LFR (.5)            | 27            | 32            | 31                | *                 | *               | *            | 37          | 28        | 28       |
| LFR (1)             | 51            | 56            | 55                | *                 | *               | *            | 62          | 53        | 52       |
| LFR (2)             | 73            | 76            | 76                | *                 | *               | *            | 79          | 73        | 69       |
| PARETO (1)          | 89            | 89            | 89                | *                 | *               | *            | *           | 63        | *        |

* Denotes 100% power.

The powers of the tests based on their asymptotic 10% critical points for samples of size 50 were also evaluated. For the first six tests, the Borell bound (see (2.1)) critical points based on the median simulated from 500
samples of size 50 were used. Asymptotically $T_{n1}$ has the Kolmogorov–Smirnov distribution and $T_{n2}$ has the $|N(0,1)|$ distribution so these points are tabulated. Critical points for $K_n$ were obtained from the approximate formula given by Doksum and Yandell (1984, p. 592). The results are not reproduced here but several conclusions can be drawn. For $k = 1$, the tests have almost no power. For $k = 2$, the power of the test with $b = 0.35$ decreases slightly and decreases further as $b$ increases. This may simply be due to the strong possibility that the Borell bounds become more and more conservative as $b$ increases. Interestingly, the power of $T_{n1}$ and $T_{n2}$ decreased only slightly but the power of $K_n$ decreased dramatically. This indicates that the approximation for the critical points of $K_n$ is very poor.

For the case $d = 2$, we carried out a limited simulation to evaluate the power of the test $M_{n0}$ of $H_{2a}$ against selected alternatives. As before, the 10% test for samples of size 50 was used and each empirical power is based on 500 simulations. The alternatives considered included the independent exponential or Marshall-Olkin with $\delta_3 = \lambda_{12} = 0$ distributions, the Marshall-Olkin distributions specified by $H_{2b}$, a bivariate Weibull distribution (the square root of each component of a random vector with the Marshall-Olkin distribution specified by $H_{2a}$) and a bivariate distribution obtained by taking the exponential rather than the square root function in the definition of the Weibull distribution.

The test of $H_{2a}$ has low power (14% and 11% respectively) against the first two alternatives considered. This result is not surprising since one would expect that as the dimension increases, larger samples would be required to be able to distinguish between similar distributions. However, the test has excellent power (100% and 83%) against the other two alternatives even though the sample size is relatively small.
5. **Examples**

Barlow and Campo (1975) gave the failure times for the right rear brakes of 107 D9G-66A Caterpillar tractors. (The data set is reproduced in Doksum and Yandell (1984).) The test statistic for the data yielded a value of $M_n = 3.64$ which is significant at the 1% level. This result is consistent with the results reported in Doksum and Yandell (1984) and with those of the other tests investigated in Section 4 except for $T_{n2} = 1.36$ which has a p-value of 0.174.

As a second example, we applied $M_n$ to the uncensored portion of the data set given by Hollander and Proschan (1979). These are the survival times of 90 stage IV prostate cancer patients treated with oestrogen in a Veterans Administration Cooperative Urological Research Group (1967) study. The data were originally analysed by Koziol and Green (1976); the data set given by Hollander and Proschan (1979) is an updated version of the original data. Koziol and Green (1976) tested the hypothesis that the survival time for deaths from cancer of the prostate has an exponential distribution with mean 100 months. We have ignored the censored portion of the data set and tested the composite exponential hypothesis. We obtained $M_n = 1.92$ which is significant at the 1% level and as a cross-check, $T_{n1} = 3.27$ (significant at the 1% level), $T_{n2} = 0.48$ (p-value 0.632) and $K_n = 2.03$ (significant at the 5% level). Doksum and Yandell (1984) review the conclusions of various other tests which incorporate the censored portion of the data and conclude that the data provides a borderline example and may or may not be exponentially distributed, though there is more evidence for a non-exponential distribution.

The final example involves data from a multicenter randomised, controlled clinical trial sponsored by the National Eye Institute to evaluate photocoagulation treatment for proliferative diabetic retinopathy (Diabetic...
Retinopathy Study Research Group, 1981). Between 1972 and 1975, 1758 patients with a best corrected visual acuity in each eye of 20/100 or better were enrolled in the study. For each patient, one eye was assigned to immediate photocoagulation treatment by one of three methods (argon laser, xenon arc or a combined treatment) while the other eye was assigned to follow up without treatment. Best corrected visual acuity was chosen as the primary response variable for evaluating treatments. In the study, "severe visual loss" was defined to have occurred if a best corrected visual acuity < 5/200 was measured at two consecutively completed 4 month follow-up visits and this was the primary endpoint for the evaluation of treatment effects. For the present purposes, we define a failure to have occurred at the first visit at which a best corrected visual acuity of 5/200 or worse was measured. The times to failure in days of the uncensored portion of the data for white male patients receiving argon laser treatment are given in Table 3. The data have been extracted from a computer tape prepared by the National Eye Institute.

**TABLE 3**

UNCENSORED FAILURE TIMES IN DAYS FOR EYES OF 71 WHITE MALES WITH DIABETIC RETINOPATHY.

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In the presented data, some ties within pairs occur (10 out of 71) so that the Marshall-Olkin distribution is certainly a plausible model for the data. The test statistic for the test of the null hypothesis $H_{2b}$ was $M_{n0} = 13.4$ for which the $p$-value is less than 0.01. (For $H_{2a}$, $M_{n0} = 9.38$ which is even more significant.) While we therefore reject $H_{2b}$ (and $H_{2a}$), it is of course still plausible that the data follows a Marshall-Olkin distribution arising from shock processes with a much lower rate of double shocks though this conjecture is weakened by the fact that the marginal distributions appear not to be exponential ($M_n = 3.83$ and $M_n = 3.72$ respectively).

APPENDIX

For any compact set $T$, let $C_0(T)$ be the Banach space of continuous $d$-variate functions defined on $T$, endowed with the supremum norm. The weak convergence results stated in section 2 and proved below are all understood in the context of this space; see Billingsley (1968 p. 54) for a general treatment.

Proof of Theorem 1: We have $EM_n(t) = 0$ and $EM_n(s)M_n(t) = \psi(s + t) - \psi(s)\psi(t)$ for any $s, t \in J$ and $n = 1, 2, \ldots$. Hence by the multidimensional central limit theorem the finite dimensional distributions of $M_n(\cdot)$ converge to those of the Gaussian process in the theorem. Moreover, by a one-term $d$-variate Taylor expansion, we see that

$$E(M_n(s) = M_n(t))^2 < E(\exp(\langle s, x \rangle) - \exp(\langle t, x \rangle))^2$$

$$= \sum_{k=1}^d \sum_{\ell=1}^d (s_k - t_k)(s_\ell - t_\ell) \frac{\partial^2 \psi}{\partial t_k \partial t_\ell} (2s + (\theta_k + \theta_\ell)(t - s))$$

$$< kd^3 |s - t|^2$$

where $0 < \theta_k, \theta_\ell < 1$ and
\[ K = \max_{1 \leq k, l \leq d} \sup_{t \in J} \frac{\partial^2 \psi}{\partial t_k \partial t_l^2} (2t) < \infty. \]

This inequality establishes that \( \{M_n(\cdot)\} \) is tight and hence the theorem. \( \square \)

**Proof of Theorem 2:** Notice that \( n^{1/2}(X_{nk}^2 - 1) = -n^{1/2}(X_{nk} - 1) + \varepsilon_{nk} \),
where \( \varepsilon_{nk} \overset{P}{\to} 0 \), as \( n \to \infty \), \( k = 1,2 \). Then

\[
n^{1/2}(\psi_{n1}(t) - \psi(t))
= n^{-1/2} \sum_{j=1}^n \{ \exp(\langle t, X_j \rangle) - \psi(t) + (t_1 X_{j1}(X_{n1}^2 - 1) + t_2 X_{j2}(X_{n2}^2 - 1)) \exp(\langle t, X_j \rangle) \} + R_n^{(1)}(t)
= n^{-1/2} \sum_{j=1}^n \{ \exp(\langle t, X_j \rangle) - \psi(t) - t_1 \psi_1(t) (X_{j1} - 1) - t_2 \psi_2(t) (X_{j2} - 1) \} + R_n^{(2)}(t),
\]

where the remainder terms \( R_n^{(l)}(t) \), \( l = 1,2 \), are uniformly negligible in probability. The result follows by Theorem 1 and a direct, somewhat lengthy covariance computation. \( \square \)

**Proof of Theorem 3:** First,

\[
n^{1/2}(S_{nk}^2 - 1) = -n^{1/2}(S_{nk} - 1) + \varepsilon_{nk}
= -n^{-1/2} \sum_{j=1}^n (X_{jk}^2 - 2X_{jk} + 1)/2 + \delta_{nk}
= -n^{-1/2} \sum_{j=1}^n (X_{jk}^2 - 2X_{jk} + 1)/2 + \delta_{nk},
\]

where \( \varepsilon_{nk}, \delta_{nk} \overset{P}{\to} 0 \), as \( n \to \infty \), \( k = 1,2 \). Then the proof is completed by the same argument as that used to prove Theorem 2. \( \square \)

**Acknowledgements:** We are grateful to Brian Yandell for sending us a copy of Doksum and Yandell (1984). The second named author is grateful to Charles Taylor for helpful conservations and to Paul Meier and Ted Karrison for their assistance in obtaining the Diabetic Retinopathy Study data.
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