A NOTE ON THE CHI-SQUAREDNESS OF QUADRATIC FORMS

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A representation is given, in terms of generalized inverses of the covariance structure of
$y$, of all quadratic forms $y'Qy$ in a normal variable $y$ which are (non-central) chi-squared
in distribution.

1. Introduction. Khatri (1962, 1963) shows that for $y \sim N(\mu, V)$ the quadratic form
$y'Qy$, $Q$ symmetric, follows a $\chi^2_r(\lambda)$ distribution if and only if

$$ VQVQV = VQV \tag{1.1} $$
$$ VQVQ\mu = VQ\mu \tag{1.2} $$
$$ \mu'QVQ\mu = \mu'Q\mu \tag{1.3} $$

in which case the degrees of freedom and the non-centrality parameter are respectively given
by

$$ r = \text{rank} (VQV) = \text{trace} (QV), \text{ and } \lambda = \mu'Q\mu. $$

In the following we give a representation of all $Q$ to satisfy (1.1), in terms of generalized inverses
of $V$, and using this representation we give equivalent conditions for (1.2) to hold in addition
to (1.1), and for (1.3) to hold in addition to (1.1) and (1.2).

For earlier work on the problem considered here and related problems see Mitra (1968),
Bhimasankaram and Majumdar (1980) and Baksalary, Hanke and Kala (1980).

2. Chi-squaredness of a quadratic form. Let $C(X)$ denote the column space of a
matrix $X$. A $g_1$-inverse $X^-$ of a matrix $X$ satisfies $XX^-X = X$ and $g_2$-inverse $X^*$ of a matrix
$X$ satisfies $X^*XX^* = X^*$.

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Lemma: Let $V$ be a symmetric and nonnegative definite matrix. Then a necessary and sufficient condition for a symmetric matrix $Q$ to satisfy $VQVQV = VQV$ is that $Q$ can be written as

$$Q = V^{-} - V^*$$

(2.1)

where $V^{-}$ is a symmetric $g_1$-inverse of $V$ and $V^*$ is a symmetric nonnegative definite $g_2$-inverse of $V$ such that $C(V^*) \subset C(V)$.

Proof: Let

$$V = P \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} P'$$

(2.2)

be the (complete) singular value decomposition of $V$, i.e. $P$ is an orthogonal matrix of the same order as $V$ and $\Delta$ is a diagonal matrix containing the non-zero eigenvalues of $V$.

Further let

$$Q^* = P'QP = \begin{bmatrix} Q_1 & Q_2 \\ Q'_2 & Q_3 \end{bmatrix}$$

(2.3)

be conformably partitioned. Then we have with (2.2), (2.3):

$$VQVQV = VQV$$
$$\Leftrightarrow \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q'_2 & Q_3 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q'_2 & Q_3 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} =$$
$$= \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q'_2 & Q_3 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}$$
$$\Leftrightarrow Q_1\Delta Q_1 = Q_1$$
$$\Leftrightarrow \Delta^{1/2}Q_1\Delta Q_1\Delta^{1/2} = \Delta^{1/2}Q_1\Delta^{1/2}$$
$$\Leftrightarrow \Delta^{1/2}Q_1\Delta^{1/2} \text{ idempotent}$$
$$\Leftrightarrow Q^* \text{ can be written as } Q^* = \begin{bmatrix} \Delta^{-1/2}Q_1\Delta^{-1/2} & Q_2 \\ Q'_2 & Q_3 \end{bmatrix}$$

where $Q_2$ arbitrary, $Q_3$ arbitrary symmetric and $Q_1$ an arbitrary symmetric idempotent matrix.

This is equivalent to saying that $Q$ can be written as

$$Q = P \begin{bmatrix} \Delta^{-1} & Q_2 \\ Q'_2 & Q_3 \end{bmatrix} P' - P \begin{bmatrix} \Delta^{-1/2}(I - Q_1)\Delta^{-1/2} & 0 \\ 0 & 0 \end{bmatrix} P'$$
$$= V^{-} - V^*.$$
hand, it is easy to show that any matrix $Q$ of the form (2.1) satisfies (1.1). This completes the proof.

If $y \sim N(0, V)$ the lemma typifies all quadratic forms $y'Qy$ which are distributed chi-square since (1.2), (1.3) are trivially satisfied in this case. It is interesting to note the reduction of (2.1) in the cases of nonsingular $V$ and $V = I$ to well-known matrix types.

We note further that for (1.1) to hold it is sufficient that $V^*$ in (2.1) is any symmetric $g_2$-inverse of $V$. The nonnegative definiteness of $V^*$ and the column space condition $C(V^*) \subseteq C(V)$ can therefore be dropped.

Along the lines of the proof of the lemma it can be shown that a general (not necessarily symmetric) solution $Q$ to (1.1) can be written as $Q = V^- - V^*$, where $V^-$ and $V^*$ are arbitrary $g_1$- and $g_2$-inverses respectively of $V$.

This is still true when $V$ is any $m \times n$ matrix. Thus a general solution $X$ to the matrix equation $BXBXB = BXB$ can be written as $X = B^- - B^*$, i.e. as the difference of general solutions to the equations $BXB = B$ and $XBX = X$ respectively.

Finally, since the nonnegative definite solutions $Q$ to $VQVQV = VQV$ and $QVQVQ = QVQ$ coincide for nonnegative definite matrices $V$ (Bhamasankaram and Majumdar, 1980). We can conclude that any nonnegative definite solution to $QVQVQ = QVQ$ can be written as $Q = V^- - V^*$ where $V^-$ is a nonnegative definite $g_1$-inverse of $V$ and $V^*$ is a nonnegative definite $g_2$-inverse of $V$ with $C(V^*) \subseteq C(V)$.

Using the representation (2.1) we can now give equivalent conditions for (1.2) to hold in addition to (1.1) and for (1.3) to hold in addition to (1.1) and (1.2).

Corollary 1: If $Q$ is of the form (2.1) then

$$VQVQ\mu = VQ\mu \leftrightarrow V^*\mu = V^*VV^-\mu$$  \hspace{1cm} (2.4)

If (2.4) holds and $Q$ is of the form (2.1) then

$$\mu'QVQ\mu = \mu'Q\mu$$

$$\leftrightarrow \mu'VV^-\mu = \mu'V^-\mu$$  \hspace{1cm} (2.5)
Proof: With (2.1) we have

\[ VQ = VV^- - VV^* , \text{ and} \]
\[ VQVQ = VV^- - VV^*VV^- \]

which proves (2.4), and since

\[ QVQ = V^-VV^- - V^-VV^* - V^*VV^- + V^* , \]

(2.5) is proved using (2.4).

Summing up, with the lemma and the corollary we have proved that for \( y \sim N(\mu, V) \), the quadratic form \( y'Qy \) is distributed chi-squared if and only if

(i) \( Q \) can be written as \( Q = V^- - V^* \) and
(ii) \( V^*\mu = V^*VV^-\mu \) and
(iii) \( \mu'V^-VV^-\mu = \mu'V^-\mu \).

3. Independence of quadratic forms. Khatri (1962, 1963) showed that two quadratic forms \( y'Q_1y \) and \( y'Q_2y \) in a normal variable \( y \sim N(\mu, V) \) are independently distributed if and only if

\[ VQ_1VQ_2V = 0 \quad (3.1) \]
\[ VQ_1VQ_2\mu = 0 \quad (3.2) \]
\[ VQ_2VQ_1\mu = 0 \quad (3.3) \]
\[ \mu'Q_1VQ_2\mu = 0 \quad (3.4) \]

We note that conditions (3.2) through (3.4) are trivially satisfied when \( y \sim N(0, V) \) or when (3.1) holds and \( Q_1 \) and \( Q_2 \) are nonnegative definite.

Thus we obtain the following corollary.

Corollary 2: Let \( Q_1 = V_1^- - V_1^* \) and \( Q_2 = V_2^- - V_2^* \) with \( V_j^-, V_j^* \) as in (2.1), \( j = 1, 2 \). Further let \( y \sim N(0, V) \) and \( Q_1, Q_2 \) nonnegative definite.

Then the two quadratic forms \( y'Q_1y \) and \( y'Q_2y \) are independently distributed if and only if

\[ (I - VV_1^*)(I - VV_2^*)V = 0. \]
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