MATRICES VERSIONS OF THE CAUCHY
AND KANTOROVICH INEQUALITIES

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ABSTRACT

A version of Cauchy's inequality is obtained which relates two matrices by an inequality in the sense of the Loewner ordering. In that ordering a symmetric idempotent matrix is dominated by the identity matrix and this fact yields a simple proof.

A consequence of the matrix Cauchy inequality leads to a matrix version of the Kantorovich inequality, again in the sense of Loewner.

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1. Introduction.

The Cauchy inequality for (row) vectors \( x, y \in \mathbb{R}^n \) asserts that

\[(xx')yy' \geq (xy')^2,\]

with equality if and only if \( y = \alpha x \) for some scalar \( \alpha \).

If \( A \) is a positive definite Hermitian matrix and \( A^{1/2} \) is its unique positive definite Hermitian square root, the choice

\[ x = uA^{1/2}, \quad y = uA^{-1/2} \]

in (1) yields

\[(2) \quad uA^{-1}u' \geq (uAu')^{-1}\]

for all \( u \in \mathbb{R}^n \) satisfying \( uu' = 1 \).

If the characteristic roots, \( \alpha_1, \ldots, \alpha_n \), of \( A \) satisfy

\[ 0 < m \leq \alpha_i \leq M, \quad i = 1, \ldots, n, \]

the Kantorovich inequality provides a reversal to (2):

\[(3) \quad uA^{-1}u' \leq \frac{(m+M)^2}{mM}(uAu')^{-1}\]

where the constant is the square of the ratio of the arithmetic to geometric mean of \( m \) and \( M \).

Many derivations of (1) and (3) have been devised. We obtain a matrix version of (1) and (3) that incidentally provides yet another proof of Cauchy's inequality.

For Hermitian matrices \( B, C \) the notation \( B > C \) (\( B \geq C \)) means that \( B - C \) is positive definite (semidefinite); this is a Loewner ordering for Hermitian matrices.

A Cauchy Inequality for Matrices.

If \( X \) is a \( k \times n \) matrix and \( Y \) is an \( \ell \times n \) matrix of rank \( \ell \),

\[(4) \quad XX^* \geq XY^*(YY^*)^{-1}YX^*\]

is a matrix version of (1), and yields (1) when \( k = \ell = 1 \).
Inequality (4) follows quite simply by noting that $Y^*(YY^*)^{-1}Y$ is idempotent and hence has characteristic roots equal to 0 or 1. Consequently,

\[ I \geq Y^*(YY^*)^{-1}Y. \]

Inequality (4) now follows by pre- and post-multiplication by $X$ and $X^*$ respectively.

Take $k = \ell$, let $A > 0$ and let $U$ be a $k \times n$ matrix satisfying $UU^* = I$. The substitution of

\[ X = UA^{1/2}, \quad Y = UA^{-1/2} \]

in (4) yields

\[ UA^{-1}U^* \geq (UAU^*)^{-1}, \]

which is a matrix version of (2).

If $A$ and $A^{-1}$ are partitioned conformably as

\[ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} A_{11}^{-1} & A_{12} \\ A_{21} & A_{22}^{-1} \end{pmatrix}, \]

then the choice $U = (I_k \ 0)$ in (6) yields

\[ A_{11}^{11} \geq A_{11}^{-1}, \]

a result of Chollet (1982). An alternative proof of (8) is given by Marcus (1982). His proof is similar to the following alternative proof of (4).

If $A$ is partitioned as in (7), then $A \geq 0$ and $A_{11} > 0$ if and only if

\[ A_{11} > 0, \quad A_{22} \geq A_{21}A_{11}^{-1}A_{12}. \]

This equivalence is an immediate consequence of

\[ \begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}. \]

If

\[ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} Y \\ X \end{pmatrix}(Y^*X^*) = \begin{pmatrix} YY^* & YX^* \\ YX^* & XX^* \end{pmatrix}, \]

where $X$ is a $k \times n$ matrix and $Y$ is an $\ell \times n$ matrix of rank $\ell$, then (9) yields (4).
3. A Kantorovich Inequality for Positive Definite Matrices.

For \( z \in [mM], m > 0 \), the convexity of \( z^{-1} \) implies that

\[
(11) \quad z^{-1} \leq \frac{m + M}{mM} - \frac{z}{mM}.
\]

(See Marshall and Olkin, 1964, p. 509). If \( A \) has the representation \( A = \Gamma D_\alpha \Gamma^* \), where \( \Gamma \) is unitary and \( D_\alpha = \text{diag}(\alpha_1, \ldots, \alpha_n) \), and if \( 0 < m \leq \alpha_i \leq M, \quad i = 1, \ldots, n \), then from (11)

\[
(12) \quad D_\alpha^{-1} \leq \frac{m+I}{mM} - \frac{D_\alpha}{mM},
\]

pre- and post-multiplication of (12) by \( \Gamma \) and \( \Gamma^* \), respectively, yields

\[
(13) \quad A^{-1} \leq \frac{m+M}{mM} I - \frac{A}{mM}.
\]

Consequently, if \( U \) is a \( p \times n \) matrix satisfying \( UU^* = I \), then

\[
(14) \quad UA^{-1}U^* \leq \frac{m+M}{mM} I - \frac{UAU^*}{mM} \leq \frac{(m+M)^2}{4mM} (UAU^*)^{-1}.
\]

The first inequality in (14) follows directly from (13) by pre- and post- multiplication by \( U \) and \( U^* \), respectively, and provides a stronger result than Kantorovich's inequality. The second inequality in (14) is obtained by completing the square.

If \( A \) is a Hermitian matrix satisfying \( 0 < mI \leq A \leq MI \), then the choice \( U = (I, 0) \) in (14) yields an inequality complementary to (8):

\[
A^{11} \leq \frac{(m+M)^2}{4mM} A^{-1}_{11}.
\]

References

