DOMINANCE CONDITIONS FOR UTILITY FUNCTIONS WITH MULTIVARIATE RISK AVERSION

by

Marco Scarsini

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Department of Statistics
Stanford University
Stanford, California
Dominance Conditions for Utility Functions
With Multivariate Risk Aversion*

Marco Scarsini†

Abstract

Stochastic dominance conditions are given for \( n \)-variate utility functions, when \( k \)-variate risk aversion is assumed for \( k = 1, 2, \ldots, n \). These conditions are expressed through a comparison of distribution functions, as in the well known univariate case, and through a comparison of random variables defined on the same probability space.

Running head: Multivariate risk aversion

Key words: Stochastic dominance, multivariate risk aversion.

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† Department of Statistics, Stanford University, Stanford, CA 94305, on leave from Istituto di Matematica Finanziaria, Università di Parma, Via Kennedy 6, 43100 Parma, Italy.
1. Introduction

Stochastic dominance has been extensively studied in the univariate case, that is when the
decision maker has to choose between random variables (generally, random amounts of money).
Conditions of stochastic dominance are usually related to risk aversion. In the multivariate
case few attempts have been performed of giving general conditions of stochastic dominance,
but they generally do not refer to risk aversion (Levhari, Parous and Peleg (1975), Levy and
Paroush (1974a), (1974b), Huang, Kira and Vertinsky (1978), Huang, Vertinsky and Ziemba
(1978), or they impose the condition that the utility functions represent the same preference
ordering (Levy and Levy (1984)). A remarkable exception is Russel and Seo (1978), where
multivariate stochastic dominance rules for the class of all risk averters are provided. Scarsini
(1985) gives stochastic dominance conditions for the bivariate case when bivariate risk aversion
is assumed. In this note we examine the general n-variate case and provide some stochastic
dominance condition under the assumption of multivariate risk aversion.

2. Multivariate Risk Aversion

Richard (1975) defined bivariate risk aversion for decision makers acting according to the
expected utility principle. The definition of bivariate risk aversion is obtained through
comparison of lotteries in the following way. Consider the lotteries $L_I$ and $L_{II}$

$$L_I = \begin{cases} (x_1, x_2) & (y_1, y_2) \\ 0.5 & 0.5 \end{cases} \quad L_{II} = \begin{cases} (x_1, y_2) & (y_1, x_2) \\ 0.5 & 0.5 \end{cases}$$

If a decision maker (DM) prefers $L_{II}$ to $L_I$ for all $x_1 \leq y_1$, $x_2 \leq y_2$, then he is said to be
bivariate risk averse. It is easy to see that this is tantamount to saying that his bivariate
utility function $u$ satisfies the following

$$u(x_1, x_2) + u(y_1, y_2) \leq u(x_1, y_2) + u(y_1, x_2)$$

(1)

for all $x_1 \leq y_1$, $x_2 \leq y_2$, i.e.,

$$\forall_{s=x_1}^{y_1} \forall_{t=x_2}^{y_2} u(s, t) \leq 0 \quad \text{for} \quad x_1 \leq y_1, \quad x_2 \leq y_2,$$
where $\Delta^b_{s=a} u(s, t) = u(b, t) - u(a, t)$.

Bivariate risk aversion (2-RA) has nothing to do with what is generally known as “risk aversion”, even if both of them are defined through comparison of lotteries. A DM is risk averse if and only if his utility function is concave. Condition (1) neither implies nor is implied by concavity of $u$ (not even component-wise). It is not difficult to verify that (1) does not even imply that $u$ be increasing.

2-RA characterizes the behavior of the DM w.r.t. combinations of goods two-by-two. The lotteries that he has to compare are “marginally” equivalent: both of them offer $z_1$ or $z_2$ with even probabilities, and $y_1$ or $y_2$ with even probabilities. The difference between $L_I$ and $L_{II}$ lies just in the pairing of quantities. Under $L_I$ the possible outcomes are either a lot of item 1 and a lot of item 2, or a little of item 1 and a little of item 2. Under $L_{II}$ the possible outcomes are either a little of item 1 and a lot of item 2, or vice versa. The bivariate risk averse DM will prefer $L_{II}$ to $L_I$.

Our aim is to generalize this idea to the general $n$-dimensional case, where the DM has to choose between different random bundles of $n$ nonhomogeneous goods. It is still assumed that he decides according to the expected utility criterion. $n$-variate risk aversion ($n$-RA) will be defined again by comparing different lotteries. In particular we shall compare two lotteries $L_A$ and $L_B$ (say), having the following form. The possible outcomes of $L_A$ are all the $n$-tuples $(z_1, \cdots, z_n)$, where $z_i = x_i$ or $z_i = y_i$ ($i = 1, \cdots, n$), containing either zero or an even number of $y$'s and the remaining $x$'s, all of these with equal probability ($2^{-n}$). The possible outcomes of $L_B$ are all the $n$-tuples $(z_1, \cdots, z_n)$ containing an odd number of $y$'s and the remaining $x$'s, all of these with equal probability ($2^{-n}$). So, any $n$-tuple $(z_1, \cdots, z_n)$ is either in $L_A$ or in $L_B$, but not in both. Lotteries $L_A$ and $L_B$ are marginally equivalent: if any variable with index $i$ is deleted in the outcomes of the two lotteries, then the two remaining lotteries (with $(n - 1)$-dimensional outcomes) are identical.

If the DM prefers $L_B$ to $L_A$, for all $x_i \leq y_i$ ($i = 1, \cdots, n$), then he is said to be $n$-variate risk averse. Lottery $L_A$ contains among its possible outcomes the worst one $(x_1, \cdots, x_n)$. This
is the reason for calling preference of \( L_B \) over \( L_A \) n-RA: the (n-variate) risk averse DM is the one who wants to avoid the worst possibility and is ready to pay a positive price for doing that.

A necessary and sufficient condition for n-RA is that the DM has an n-variate utility function \( u \) satisfying

\[
(-1)^n \sum_{s=x} y u(s) \leq 0 \quad \text{for} \quad x \leq y
\]

where

\[
\sum_{s=x} y u(s) = \sum_{s_1=x_1} y_1 \cdots \sum_{s_n=x_n} y_n u(s_1, \ldots, s_n).
\]

Obviously, (2) holds when \( u \) is absolutely continuous with

\[
(-1)^n \frac{\partial^n u}{\partial s_1 \cdots \partial s_n} \leq 0 \quad \text{a.s.}
\]

n-RA concerns only the attitude of the DM towards combination of \( n \) different items. It does not imply and it is not implied by any other k-RA, with \( k < n \). A DM can be simultaneously n-variate risk averse and \((n - 1)\)-variate risk seeking w.r.t. all the \( n \) possible \((n - 1)\)-tuplets of goods in the bundle \((z_1, \ldots, z_n)\). Furthermore, k-RA w.r.t. a certain set of items does not imply k-RA w.r.t. any other set, provided the two sets do not coincide. Again, n-RA has nothing to see with usual risk aversion. There is no implication (in any sense) between (2) and concavity of the utility function.

From now on, we shall be interested in DM's showing k-RA for all \( k \)-tuplets of goods, and for all \( k = 1, \ldots, n \). From what we said before, the n-variate utility functions of these DM's will satisfy the following condition:

\[
(-1)^k \sum_{s_{i_1}=s_{i_1}} y_{i_1} \cdots \sum_{s_{i_k}=s_{i_k}} y_{i_k} u(\cdots, s_{i_1}, \ldots, s_{i_k}, \cdots) \leq 0
\]

for all \( k \leq n \), for all \( x_{ij} \leq y_{ij} \), \( j = 1, \ldots, k \), and for all values of the other arguments of the utility function whose index is other than \( i_j \), \( j = 1, \ldots, k \). Call \( U_n \) the class of such utility functions.
3. Main Results

Before giving stochastic dominance conditions for utility functions in $U_n$, we report a well known theorem concerning stochastic dominance for $n$-variate increasing utility functions, when no assumption is made about any kind of risk aversion.

Theorem 1. Let $X, Y$ be $n$-dimensional random vectors. The following three conditions are equivalent

a) $Pr\{X \in A\} \leq Pr\{Y \in A\}$ for any decreasing set $A$ (a set is decreasing if its indicator function is nonincreasing).

b) $Eu(X) \geq Eu(Y)$ for any increasing function $u$ such that the expectations exist.

c) There exist $X', Y', X' \sim X, Y' \sim Y$ such that

$$\mathbf{X'} \geq \mathbf{Y'} \quad \text{a.s.}$$

(the symbol $\sim$ means "distributed as").

Bi-implication a) $\iff$ b) has been proved by Lehmann (1955) and Levhari, Paroush and Peleg (1975). Bi-implication a) $\iff$ c) relies on some results proved in Strassen (1965). See also Brumelle and Vickson (1975), and Hansen, Holt and Peled (1978). Kamae, Krengel and O'Brien (1977) proved a version of Theorem 1 for random variables with values in a partially ordered polish space.

Let $X$ and $Y$ be two $n$-dimensional random vectors distributed according to the distribution functions $F$ and $G$, respectively. Consider the following conditions:

i) $F(x) \leq G(x)$ for all $x \in \mathbb{R}^n$.

ii) $Eu(X) \geq Eu(Y)$ for all right continuous $u \in U_n$ for which expectations exist.

iii) For all $\alpha \in \mathbb{R}^n_+$, there exist $T_\alpha, S_\alpha$ such that

$$T_\alpha \sim \max(\alpha_1 X_1, \ldots, \alpha_n X_n), \quad S_\alpha \sim \max(\alpha_1 Y_1, \ldots, \alpha_n Y_n) \quad \text{and} \quad T_\alpha \geq S_\alpha \quad \text{a.s.}$$

Theorem 2. Conditions i), ii) and iii) are equivalent.
Rüschendorf (1980) has proven a result similar to the bi-implication $i) \iff ii)$ for \( \Delta \)-monotone functions defined on the unit hypercube. The reason for proving the whole theorem anyway is twofold. First we think that our proof is a little simpler than Rüschendorf’s one. Then, \( \Delta \)-monotone functions are not the same as functions in \( U_n \), and cannot be obtained from them by means of simple algebraic manipulations.

**Proof of Theorem 2.** ii) implies i). Trivial, since

\[
F(x) = \int I_{[0,x]}(t)dF(t) = EI_{[0,x]}(X)
\]

\[
G(x) = \int I_{[0,x]}(t)dG(t) = EI_{[0,x]}(Y)
\]

where \( I_A(\cdot) \) is the indicator function of the set \( A \), and

\[-I_{[0,x]} \in U_n\]

i) implies ii). Let

\[
u(x) = -\int_{t=x}^{\infty} d\mu(t)\quad (3)
\]

where \( \mu \) is a positive measure on \( IR_+^n \). Then \( \nu \) is right continuous and \( \nu \in U_n \). Let i) hold. We have

\[
\int_{IR_+^n} -F(x)d\mu(x) \geq \int_{IR_+^n} -G(x)d\mu(x).
\]

Since \( F(x) = \int_{t=0}^{x} dF(t) \), \( G(x) = \int_{t=0}^{x} dG(t) \), we can apply Fubini’s Theorem and obtain

\[-\int_{IR_+^n} \int_{x=t}^{\infty} d\mu(x)dF(t) \geq -\int_{IR_+^n} \int_{x=t}^{\infty} d\mu(x)dG(t)\]

that is,

\[
\int_{IR_+^n} \nu(t)dF(t) \geq \int_{IR_+^n} \nu(t)dG(t)
\]

for any \( \nu \) of type (3).

Any right continuous function \( \nu \in U_n \) can be written as

\[
u(x) = -\left[ \int_{t=x}^{\infty} d\mu(t) + \sum_i \int_{t^{(i)}=x^{(i)}}^{\infty} d\mu_i(t) + \sum_i \sum_j \int_{t^{(i,j)}=x^{(i,j)}}^{\infty} d\mu_{i,j}(t) + \cdots + c \right]\quad (4)
\]
where $\mu$ is a positive measure on $\mathbb{R}_+^n$, $\mu_i$ is a positive measure on $\mathbb{R}_+^{n-1}$, $\mu_{ij}$ is a positive measure on $\mathbb{R}_+^{n-2}$, etc., and

$$t^{(i)} = (t_1, t_2, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n) \in \mathbb{R}_+^{n-1}$$

$$t^{(ij)} = (t_1, t_2, \cdots, t_{i-1}, t_{i+1}, \cdots, t_{j-1}t_{j+1}, \cdots, t_n) \in \mathbb{R}_+^{n-2}$$

e.tc. (see von Neumann (1950)). In order to prove the implication we have only to repeat the procedure used for arbitrary functions of type (3), for each addend in (4). Recall that if

$$F(x) \leq G(x) \quad \text{for all} \quad x \in \mathbb{R}_+^n,$$

then obviously

$$F^*(y) \leq G^*(y) \quad \text{for all} \quad y \in \mathbb{R}_+^k, \quad k < n$$

where $F^*, G^*$ are any corresponding $k$-dimensional marginals of $F$ and $G$.

iii) implies (i). By Theorem 1, if

$$T_\alpha \geq S_\alpha \geq 0 \quad \text{a.s., for all} \quad \alpha \in \mathbb{R}_+^n,$$

then

$$Pr\{T_\alpha \leq x\} \leq Pr\{S_\alpha \leq x\} \quad \text{for all} \quad x \in \mathbb{R}_+ \quad \text{for all} \quad \alpha \in \mathbb{R}_+^n,$$

i.e.,

$$Pr\{\max(\alpha_1X_1, \cdots, \alpha_nX_n) \leq x\} \leq Pr\{\max(\alpha_1Y_1, \cdots, \alpha_nY_n) \leq x\}$$

$$Pr\{\alpha_1X_1 \leq x, \cdots, \alpha_nX_n \leq x\} \leq Pr\{\alpha_1Y_1 \leq x, \cdots, \alpha_nY_n \leq x\}$$

$$Pr\{X_1 \leq x/\alpha_1, \cdots, X_n \leq x/\alpha_n\} \leq Pr\{Y_1 \leq x/\alpha_1, \cdots, Y_n \leq x/\alpha_n\}$$

$$Pr\{X \leq t\} \leq Pr\{Y \leq t\} \quad \text{for all} \quad t \in \mathbb{R}_+^n.$$

i) implies ii). By reversing the procedure, if

$$Pr\{X \leq t\} \leq Pr\{Y \leq t\} \quad \text{for all} \quad t \in \mathbb{R}_+^n,$$

then

$$Pr\{\max(\alpha_1X_1, \cdots, \alpha_nX_n) \leq x\} \leq Pr\{\max(\alpha_1Y_1, \cdots, \alpha_nY_n) \leq x\}$$
for all \( x \in \mathbb{R}_+ \), for all \( \alpha \in \mathbb{R}^n_+ \). Therefore, by Theorem 1 there exist \( T_\alpha, S_\alpha \) such that

\[
T_\alpha \sim \max(\alpha_1 X_1, \cdots, \alpha_n X_n)
\]
\[
S_\alpha \sim \max(\alpha_1 Y_1, \cdots, \alpha_n Y_n)
\]
\[
T_\alpha \geq S_\alpha \quad \text{a.s.} \quad \text{Q.E.D.}
\]

Theorem 1 and Theorem 2 are somewhat analogous in that three conditions are shown to be equivalent. The first condition concerns inequalities of probabilities of a class of sets. The second one concerns inequalities of expected values of a class of utility functions. The third one concerns (almost sure) inequalities between random variables defined on the same probability space.

While the condition a) of Theorem 1 is not easily testable, condition i) of Theorem 2 involves only a comparison of two distribution functions, as in the usual univariate case.

4. Conclusions and Comparisons with Known Results

Mosler (1984) reports, without proof, a theorem which states bi-implication i) \( \Leftrightarrow \) ii), but only for continuously distributed r.v.'s and for a.e. \( n \)-times differentiable bounded utility functions. He provides important examples of utility functions in \( U_n \), in particular, the following utility independent functions:

\[
r(x) = \alpha + \beta \sum_{i=1}^{n} u_i(x_i)
\]

(5)

\[
u(x) = \alpha - \beta \prod_{i=1}^{n} (-u_i(x_i)) \quad u_i \leq 0
\]

(6)

where \( u_i : \mathbb{R}_+ \rightarrow \mathbb{R} \) is nondecreasing \( (i = 1, \cdots, n) \), \( \beta > 0 \).

For similar examples see also Levy and Paroush (1974b).

Of course (5) and (6) are not the only two possible forms of utility functions in \( U_n \). For examples, let

\[
v(x) = \max_{i=1,\cdots,n} u_i(x_i)
\]

where \( u_i : \mathbb{R}_+ \rightarrow \mathbb{R} \) is nondecreasing \( (i = 1, \cdots, n) \). Then \( v \in U_n \).
Let \( w : IR^n_+ \to IR \)

\[
w(x) = \beta - \beta \exp\{- \sum_{i=1}^{n} \lambda_i x_i - \sum_{i<j} \lambda_{ij} \max(x_i, x_j) - \sum_{i<j<k} \lambda_{ijk} \max(x_i, x_j, x_k) - \cdots - \lambda_{1,2,\ldots,n} \max(x_1, x_2, \ldots, x_n)\}
\]

where \( \beta \) and all \( \lambda \)'s are positive. Then \( w \in U_n \).

In general, let \( H : IR^n_+ \to IR \) be a survival function. Then any function \( s \), such that

\[
s(x) = \beta(1 - H(x)),
\]

with \( \beta > 0 \), is in \( U_n \).

As we said in the introduction, Russel and Seo (1978) have developed a technique, which utilizes epigraphs of functions, to provide dominance conditions for the class of all risk averters, i.e. for the class of all agents whose utility function is concave. Levhari, Paroush and Peleg (1975) have provided dominance rules for the class of quasi-concave utility functions, using standard approximation techniques. Our conditions are analytically closer to Levhari, Paroush and Peleg's rather than to Russel and Seo's, even if risk aversion is involved. In fact Russel and Seo's risk aversion is the usual one, related to concavity of the utility function, and does not take into account the dimension of the domain of the utility function. The n-RA, that we defined above, on the contrary, characterizes the attitude of the agent w.r.t. combinations of exactly \( n \) goods. The information about this attitude is obtained though comparison of lotteries that are marginally equivalent, namely, if the \( i \)th element is deleted in all the outcomes of the two lotteries, they become equal. A fortiori this is true if \( k \) out of \( n \) elements are deleted. The dominance conditions that we have obtained are rather simple, in that they involve only the comparison of two distribution functions, rather than the comparison of two measures on a large class of sets (for instance the increasing sets).

Levy and Levy (1984) have developed rules for dominance of a risky multivariate option over another for utility functions defined on the same preference ordering. They rely on some results of Kihlstrom and Mirman (1974) who define the multivariate utility function \( u_1 \) to be
more risk averse than the utility function $u_2$ if $u_1$ is an increasing convex transformation of $u_2$. This definition implies that comparison is allowed only for utility functions, which represent the same ordinal preferences. No such assumption is necessary for defining dominance w.r.t. $n$-RA. The utility functions in $U_n$ may have different indifference curves, namely they may represent different ordinal preferences.
References


