POSITIVE DEPENDENCE OF A CLASS OF MULTIVARIATE EXPONENTIAL DISTRIBUTIONS

by

Ingram Olkin and Y. L. Tong

Technical Report No. 267
December 1989

Prepared Under the Auspices
of
National Science Foundation
DMS 87-08083
Ingram Olkin, Project Director

Department of Statistics
Stanford University
Stanford, California
POSITIVE DEPENDENCE OF A CLASS OF MULTIVARIATE EXPONENTIAL DISTRIBUTIONS

by

Ingram Olkin\(^1\) and Y. L. Tong\(^2\)
Stanford University and Georgia Institute of Technology

ABSTRACT

The positive dependence of a subclass of multivariate exponential distributions is examined. This class is characterized by an index vector, which is used as an ordering to yield degrees of positive dependence. The results presented have a direct implication on the reliability function of a system and the survival probability function of a shock model.

Key words: System reliability, component systems, shock models, majorization, Schur-convex functions.

\(^1\) Research supported in part by the National Science Foundation Grant DMS-8708083.
\(^2\) Research supported in part by the National Science Foundation Grant DMS-8801327, A01 and by the U. S. Air Force Office of Scientific Research.
POSITIVE DEPENDENCE OF A CLASS OF MULTIVARIATE EXPONENTIAL DISTRIBUTIONS

I. Olkin and Y. L. Tong

1. Introduction.

Multivariate exponential distributions play an important role in reliability theory, shock models, and other related applications. (We use the name multivariate exponential distribution to mean a joint distribution with univariate exponential marginal distributions.) In the early stages of development, attention was focused on alternative forms of bivariate exponential distributions, and noted works include Freund (1961), Marshall and Olkin (1967a, 1967b), Downton (1970), Block and Basu (1974), Friday (1976). Some of these bivariate exponential distributions can be generalized to the multivariate case; versions of multivariate exponential distributions were introduced by Marshall and Olkin (1967a), Arnold (1968), Proschan and Sullo (1974), and Basu and Block (1975). The problem of parameter estimation was studied by Proschan and Sullo (1976), and a rank test for independence of multivariate exponential variables was proposed by Al-Saadi and Young (1982). For a comprehensive review of the literature, see Johnson and Kotz (1972, Chapter 41), Friday (1976), Block (1985), and Marshall and Olkin (1985).

In this paper we study a subclass of multivariate exponential distributions of Marshall and Olkin (1967a), and present some analytical results concerning their positive dependence properties. This class of distributions can be characterized by an index vector, and has applications in reliability and shock models (Section 2). In Section 3 we present results concerning the positive dependence of the random variables via a partial ordering of the index vectors and a partial ordering of the parameter vectors. These results have a direct implication on the reliability function of a system and the survival probability function of a shock model. Finally, in Section 4 we briefly discuss some related applications.

2. A Class of Multivariate Exponential Distributions with a Common Univariate Marginal Distribution.

Let $U_1, \ldots, U_n$, $V_1, \ldots, V_n$ and $W$ be independent univariate exponential random
variables with $EU_i = \lambda_1^{-1}$, $EV_i = \lambda_2^{-1}$ \((i = 1, \ldots, n)\) and $EW = \lambda_0^{-1}$. Let \(k = (k_1, \ldots, k_n)\) be a vector of nonnegative integers with

$$
\sum_{s=1}^{n} k_s = n, \quad k_1 \geq \cdots \geq k_r \geq 1 \text{ and } k_{r+1} = \cdots = k_n = 0,
$$

for some \(r \leq n\). In the results given below the monotonicity of the \(k_i\)'s in (2.1) is not essential, as we shall see; this condition is made primarily for notational convenience.

For given \(\sim k\) let \(\sim X = \sim X(k) = (X_1, \ldots, X_n)\) be an \(n\)-dimensional multivariate exponential random vector defined by

$$
\begin{align*}
X_j &= \min(U_j, V_1, W), \quad j = 1, \ldots, k_1, \\
X_j &= \min(U_j, V_2, W), \quad j = k_1, \ldots, k_1 + k_2, \\
& \quad \vdots \\
X_j &= \min(U_j, V_r, W), \quad j = \sum_{i=1}^{r-1} k_i + 1, \ldots, n.
\end{align*}
$$

That is, each of the \(X_j\)'s depends on a different \(U_j\) and a common \(W\), the first \(k_1\) depend on \(V_1\), the next \(k_2\) depend on \(V_2\), etc. It is clear that the univariate marginal distributions of the \(X_j\)'s are exponential with a mean \((\lambda_1 + \lambda_2 + \lambda_0)^{-1}\), and that the joint distribution of the \(X_i\)'s is not exchangeable unless \(k = (n, 0, \ldots, 0)\) or \(k = (1, 1, \ldots, 1)\). Furthermore, it seems intuitively clear that the components of \(\sim X(n, 0, \ldots, 0)\) are more positively dependent than that of \(\sim X(1, 1, \ldots, 1)\) because the former depends on the same variable \(V_1\), and an analytical proof of such a result can be found in Shaked and Tong (1985). Thus a question of interest is how the "degree" of positive dependence of the components of \(\sim X(k)\) affects the probability functions

$$
F_{\sim k, \sim \lambda}(t) \equiv P_{\sim k, \sim \lambda} \left[ \bigcup_{i=1}^{n} \{X_i \leq t\} \right], \quad \bar{F}_{\sim k, \sim \lambda}(t) = P_{\sim k, \sim \lambda} \left[ \bigcap_{i=1}^{n} \{X_i > t\} \right]. \quad (2.4)
$$

A related question is how these probability functions depend on \(\lambda_1, \lambda_2, \) and \(\lambda_0\) when both \(\sim k\) and \(\lambda = \lambda_0 + \lambda_1 + \lambda_2\) are kept fixed. This problem has an important application in reliability theory and shock models as illustrated below:

**Example 2.1. System Reliability.** Suppose that three types of components labeled \(A\), \(B\), and \(C\) are connected in a parallel-series fashion such that \(k_s\) of the type \(A\) components
are connected to component $B_s$ ($s = 1, \ldots, r$); all of the type $B$ components are connected to a type $C$ component in series as shown in Figure 1. If the life length distributions of the components $A_1, B, C$ are exponential with means $\lambda_1^{-1}, \lambda_2^{-1},$ and $\lambda_0^{-1},$ respectively, then the system reliability is simply

$$R_{k, \lambda}(t) \equiv P_{k, \lambda} \{\text{lifelength of system} > t\} = 1 - F_{k, \lambda}(t).$$ \hspace{1cm} (2.5)

Note that when $k = (1, 1, \ldots, 1)$ (that is, $k_1 = \cdots = k_n = 1$), then the system in Figure 1 is equivalent to that in Figure 2 where the lifelengths of components $A_1', \ldots, A_n'$ have independent exponential distributions with a common mean $(\lambda_1 + \lambda_2)^{-1}$.

**Figure 1 Here**

A parallel-series system

**Figure 2 Here**

An equivalent system

**Example 2.2. Shock Models.** In a fatal shock model suppose that the components of an $n$-component system die after receiving a fatal shock from one of several sources, and that independent Poisson processes govern the occurrence of shocks. If (i) the waiting times for components of type $A, B, C$ governed by shocks are independent Poisson variables with means $\lambda_1 t, \lambda_2 t, \lambda_0 t$, respectively, (ii) a separate type $A$ shock applies to each of the $n$ components, (iii) a shock to a type $B$ component applies to the first $k_1$ components, another shock to a type $B$ component applies to the next $k_2$ components, etc., and (iv) a shock to type $C$ applies to all the $n$ components, then the survival probability function of concern is

$$F_{k, \lambda}(x_1, \ldots, x_n) = P_{k, \lambda} \left[ \bigcap_{i=1}^{n} \{X_i > x_i\} \right]$$

$$= \exp \left[ -\lambda_1 \sum_{i=1}^{n} x_i - \lambda_2 \sum_{i=1}^{r} \max (x_{k_{i-1}+1}, \ldots, x_{k_i}) - \lambda_0 \max_{1 \leq i \leq n} x_i \right].$$ \hspace{1cm} (2.6)

where $k_0 \equiv 1$. When $x_1 = \cdots = x_n = t$, (2.6) reduces to

$$F_{k, \lambda}(t) = \exp \left[ -n \lambda_1 t - r \lambda_2 t - \lambda_0 t \right].$$ \hspace{1cm} (2.7)
It should be noted that a larger class of multivariate exponential distributions can be defined using a similar formulation. For example, for \( n \geq 2 \) and \( m \geq 2 \) let \( \{ U_{ij} \} \) be independent exponential variables with \( EU_{ij} = \lambda_j^{-1} \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \). Let \( D = (d_{ij}) \) be an \( n \times m \) matrix of positive integers with \( 1 \leq d_{ij} \leq n \) for all \( i, j \). Define an \( n \)-dimensional random vector \( \vec{X}(D) = (X_1(D), \ldots, X_n(D)) \) by
\[
X_i(D) = \min_{1 \leq j \leq m} U_{d_{ij}, j}, \quad i = 1, \ldots, n.
\]
Then clearly the class of random variables defined in (2.2) is a special case with \( m = 3 \) and \( d_{i1} = i \) for \( i = 1, \ldots, n \), \( d_{i2} = \cdots = d_{k_1, 2} = 1 \), \( d_{k_1 + 1, 2} = \cdots = d_{k_1 + k_2, 2} = 2, \ldots, d_{\sum_{i=1}^{r-1} k_i + 1, 2} = \cdots = d_{n, 2} = r \), and \( d_{i3} = 1 \) for \( i = 1, \ldots, n \). The physical interpretation of the random vector \( \vec{X}(D) \) in reliability theory and shock models is similar to the interpretation in the examples discussed. Since we have not obtained useful results for this larger class of random variables (and distributions), we confine our attention to the (smaller) class of random variables \( \vec{X}(\vec{k}) \) defined in (2.2).


In this section we present analytic results concerning partial orderings of the positive dependence of the random variables \( X_1(\vec{k}), \ldots, X_n(\vec{k}) \). Specifically, we show that when the random variables are more positively dependent in a fashion to be defined, then they tend to hang together more, in the sense that both \( F_{\vec{k}, \lambda}(t) \) and \( \tilde{F}_{\vec{k}, \lambda}(t) \) defined in (2.4) become larger.

The key question is how to characterize the degree of positive dependence of \( \vec{X}(\vec{k}) \) (or more precisely, the components of \( \vec{X}(\vec{k}) \)) via \( \vec{k} \) and \( \vec{\lambda} \). In the following we show that, for fixed \( \vec{\lambda} \), \( X(\vec{k}) \) is more positively dependent than \( X(\vec{k}') \) in this sense when \( \vec{k} \) majorizes \( \vec{k}' \). [For a comprehensive discussion of majorization theory see Marshall and Olkin (1979).] This is clear in the extreme cases when \( \vec{k} = (n, 0, \ldots, 0) \) and \( \vec{k}' = (1, 1, \ldots, 1) \) as noted in Section 2.

For \( \vec{k} \) fixed, we consider a “decreasing transformation” of the parameter vector \( \vec{\lambda} \).

**Definition 3.1.** Let \( \vec{\lambda} = (\lambda_1, \lambda_2, \lambda_0) \) and \( \vec{\lambda}^* = (\lambda_1^*, \lambda_2^*, \lambda_0^*) \) denote two vectors of parameters. The vector \( \vec{\lambda}^* \) is said to be a decreasing transformation of \( \vec{\lambda} \) (denoted \( \vec{\lambda} \succ \vec{\lambda}^* \)) if either (i) \( \lambda_0^* = \lambda_0 \) and \( \lambda_1^* - \lambda_1 = 2 - \lambda_2 > 0 \) or (ii) \( \lambda_1^* = \lambda_1 \) and \( \lambda_2^* - \lambda_2 = \lambda_0 - \lambda_0^* > 0 \).
Note that if $\lambda^* \succ \lambda$, then $\sum_{j=1}^3 \lambda_j = \sum_{j=1}^3 \lambda^*_j$ (thus the marginal distributions remain unchanged) and the random variables are more positively dependent under $\lambda^*$. The following example will serve to illustrate the ideas.

**Example 3.2.** Let $n = 4$, $k = (2, 2, 0, 0)$ and $\lambda = (2, 5, 6)^t \succ (4, 3, 6) = \lambda^*$. Then the distribution of the lifelength of the system in Figure 1 under $\lambda$ (under $\lambda^*$) is identical to that in Figure 3 (Figure 4):

![Figure 3 Here](image)

![Figure 4 Here](image)

Here the lifelengths of the components $A_{ii'}$, $B_i$, and $C$ are independent exponential variables with means $\lambda_1^{-1}$, $\lambda_2^{-1}$ and $\lambda_0^{-1}$, respectively. In Figure 3, $X_1$ and $X_2$ involve the same component $A_{12}$, and $X_3$ and $X_4$ involve the same component $A_{32}$, so that it seems intuitively clear that the random variables are more positively dependent.

In the following we state the main results via these two partial orderings of positive dependence. In the first, $\lambda$ is fixed and we compare two $k$ vectors. In the second, $\lambda$ is fixed and we compare two $\lambda$ vectors.

**Theorem 3.3.** Let $n$, $\lambda$ and $t$ be arbitrary but fixed, and let $k$, $k'$ be two vectors satisfying (2.1). If $k \succ k'$, then

$$\tilde{F}_{k, \lambda}(t) \geq \tilde{F}_{k', \lambda}(t).$$

(3.1)

**Theorem 3.4.** Let $n$, $k$ and $t$ be arbitrary but fixed, and let

$$\lambda = (\lambda_1, \lambda_2, \lambda_0) \text{ and } \lambda' = (\lambda', \lambda_2, \lambda_0')$$

(3.2)

be two parameter vectors. If

$$\lambda' \geq \lambda_1, \quad \lambda'_1 + \lambda'_2 \geq \lambda_1 + \lambda_2 \quad \text{and} \quad \lambda'_1 + \lambda'_2 + \lambda'_0 = \lambda_1 + \lambda_2 + \lambda_0$$

(3.3)

hold, then

$$\tilde{F}_{k, \lambda}(x_1, \ldots, x_n) > \tilde{F}_{k, \lambda'}(x_1, \ldots, x_n) \text{ for all } (x_1, \ldots, x_n) \in \mathbb{R}_+^n.$$

5
Consequently we have

$$\widetilde{F}_{\mathbf{k}, \chi}(t) \succ \widetilde{F}_{\mathbf{k'}, \chi}(t) \text{ for all } t \in (0, \infty).$$  \hfill (3.4)

The proofs of these results depend on the following fact (whose proof is immediate):

**Fact 3.5.** If $\chi$ and $\chi'$ satisfy (3.3), then there exists a finite number of vectors $\chi^{(1)}, \ldots, \chi^{(N)}$ with positive components such that

$$\chi = \chi^{(1)} > \chi^{(2)} > \cdots > \chi^{(N-1)} > \chi^{(N)} = \chi'.$$

**Proofs of Theorems 3.3 and 3.4.** Theorem 3.3 follows immediately from (2.7) and the fact: If

$$\mathbf{k} = (k_1, \ldots, k_r, k_{r+1}, \ldots, k_n) \succ (k'_1, \ldots, k'_{r'}, k'_{r'+1}, \ldots, k'_n) = \mathbf{k'},$$

where $k_1 \geq \cdots \geq k_r \geq 1$, $k_{r+1} = \cdots = k_n = 0$, $k'_1 \geq \cdots \geq k'_{r'} \geq 1$, $k'_{r'+1} = \cdots = k'_n = 0$, then $r \leq r'$. Theorem 3.4 follows from (2.6) and Fact 3.5. ||

To investigate the function $F_{\mathbf{k}, \chi}(t)$, and hence the reliability function, note that

$$R_{\mathbf{k}, \chi}(t) = 1 - F_{\mathbf{k}, \chi}(t) = \bar{G}_0(t) \left[ 1 - \prod_{s=1}^{r} \left( 1 - \bar{G}_2(t) \left[ 1 - G_1^{k_s}(t) \right] \right) \right],$$  \hfill (3.5)

where

$$\bar{G}_j(t) = e^{-\lambda_j t}, \quad G_j(t) = 1 - \bar{G}_j(t), \quad \text{for } j = 0, 1, 2.$$

**Theorem 3.6.** Let $n$, $\chi$ and $t$ be arbitrary but fixed, and let $\mathbf{k}$, $\mathbf{k'}$ satisfy (2.1). If $\mathbf{k} \succ \mathbf{k'}$, then

$$R_{\mathbf{k}, \chi}(t) \leq R_{\mathbf{k'}, \chi}(t).$$  \hfill (3.6)

**Proof.** Although this theorem can be derived by applying a more general result in Tong (1989), here we give an independent proof using the specific expression in (3.5). Consider the function

$$h(k) = \log \prod_{s=1}^{r} \left\{ 1 - c_2 \left( 1 - c_1^{k_s} \right) \right\}$$

$$= \sum_{s=1}^{r} \log \left\{ 1 - c_2 \left( 1 - e^{k_s \log c_1} \right) \right\},$$

6
where \( c_1 = G_1(t) \), \( c_2 = \tilde{G}_2(t) \) are in \((0,1)\). The second derivative of \( \phi(u) = \log \left(1 - c_2 \left(1 - e^{\rho u}\right)\right) \) is
\[
\phi''(u) = c_2 \rho^2 (1 - c_2) e^{\rho u} \left(1 - c_2 \left(1 - e^{\rho u}\right)\right) > 0
\]
for all \( \rho = \log c_1 < 0 \) and \( u \geq 0 \). Consequently, the function \( h(k) \), and hence \( \prod_{s=1}^r \left(1 - \tilde{G}_2(t)^{k_s} \right) \), is a Schur-convex function of \( k \). (See Marshall and Olkin, 1979, p. 11.)

Thus \( R_{\tilde{k}, \lambda}(t) \) is a Schur-concave function. ||

Note that \( F_{\tilde{k}, \lambda}(t) = 1 - R_{\tilde{k}, \lambda}(t) \) is a Schur-convex function of \( \tilde{k} \).

This result asserts that when the true vector of concern is \( \tilde{k} \), then the components are more positively dependent; thus \( X_1, \ldots, X_n \) (the lifelengths of the components) tend to hang together more. As a consequence, \( F_{\tilde{k}, \lambda}(t) \) is larger than \( F_{\tilde{k}', \lambda}(t) \) and the corresponding reliability functions are also ordered, but in the reverse order.

The next result is an analog to Theorem 3.4.

**Theorem 3.7.** Let \( n, k, \) and \( t \) be arbitrary but fixed, and let \( \lambda, \lambda' \) be two possible parameter vectors. If \( \lambda, \lambda' \) satisfy (3.3), then
\[
R_{\tilde{k}, \lambda}(t) \leq R_{\tilde{k}', \lambda}(t).
\]

**Proof.** Using Fact 3.5 it suffices to show that (3.7) holds for all \( \lambda' > \lambda \).

Case (i): \( \lambda^*_{0} = \lambda_0 \) and \( \lambda_{1} - \lambda_1 = \lambda_2 - \lambda_2^* > 0 \). The result follows from the discussion in Example 3.1 of Tong (1989) as a special case.

Case (ii): \( \lambda_{1} = \lambda_1 \) and \( \lambda_{2}^* - \lambda_2 = \lambda_0 - \lambda_0^* = \delta > 0 \). It is clear that
\[
R_{\tilde{k}, \lambda}(t) = e^{-\lambda_0^* t} \left[1 - \prod_{s=1}^r \left(1 - z_s^* e^{-\lambda_{2s}^* t}\right)\right]
\]
\[
= e^{-\delta t} e^{-\lambda_0^* t} \left[1 - \prod_{s=1}^r \left(1 - z_s e^{\delta t} e^{-\lambda_{2s}^* t}\right)\right]
\]
\[
= \frac{1}{c} e^{-\lambda_0^* t} \left[1 - \prod_{s=1}^r \left(1 - c \omega z_s\right)\right],
\]

where \( z_s = z_s(t) = 1 - G_{1s}^{k_s}(t) \) (\( s = 1, \ldots, r \)) are arbitrary but fixed, \( z_s \in [0,1] \) and, for notational convenience,
\[
c \equiv e^{\delta t} > 1, \quad \omega \equiv e^{-\lambda_{2s}^* t} < 1.
\]
Thus it suffices to show that in this context

\[
\frac{1}{c} \left[ 1 - \sum_{j=1}^{r}(1 - cy_j) \right] \leq \left[ 1 - \sum_{j=1}^{r}(1 - y_j) \right],
\]

where, for simplicity, \( y_j \equiv \omega_{ij}; \ j = 1, \ldots, r. \)

Inequality (3.9) clearly holds for \( r = 1 \) and 2 (for \( r = 1 \) it becomes an equality). Now suppose that it holds for \( r - 1 \). Then a simple argument yields

\[
\frac{1}{c} \left[ 1 - \prod_{j=1}^{r}(1 - cy_j) \right] = \frac{1}{c} \left[ 1 - \prod_{j=1}^{r-1}(1 - cy_j) \right] (1 - cy_r) + y_r
\]

\[
\leq \left[ 1 - \prod_{j=1}^{r-1}(1 - y_j) \right] (1 - cy_r) + y_r \leq \left[ 1 - \prod_{j=1}^{r-1}(1 - y_j) \right] (1 - y_r) + y_r
\]

\[
= 1 - \prod_{j=1}^{r}(1 - y_j),
\]

where the first inequality is based on the induction hypothesis, and the second inequality holds from the reliability context. ||

For the purpose of illustration of the applications of Theorems 3.6 and 3.7 consider the following specific example.

**Example 3.8.** For fixed \( n \) and \( k \) let \( \lambda \sim (3, 2, 8), \lambda' \sim (5, 2, 6). \) From the sequence

\[
\lambda^{(1)} \sim (3, 2, 8) > \lambda^{(2)} \sim (3, 3, 7) > \lambda^{(3)} \sim (5, 1, 7) > \lambda^{(4)} \sim (5, 2, 6),
\]

it follows that

\[
R_{\sim}^{k, \lambda^{(1)}}(t) \leq R_{\sim}^{k, \lambda^{(2)}}(t) \leq R_{\sim}^{k, \lambda^{(3)}}(t) \leq R_{\sim}^{k, \lambda^{(4)}}(t).
\]

In the special case that \( k \sim (n, 0, \ldots, 0) \) or \( k^{(2)} \sim (1, 1, \ldots, 1) \), it is clear from Figures 1 and 2 that

\[
R_{\sim}^{k^{(1)}}, \lambda^{(1)}(t) \leq R_{\sim}^{k^{(2)}, \lambda^{(1)}}(t) = R_{\sim}^{k^{(1)}, \lambda^{(4)}}(t).
\]
Thus in this special case, the effect on the reliability function when \( \tilde{k} \) is changed from \((n, 0, \ldots, 0)\) to \((1, 1, \ldots, 1)\) for fixed \( \tilde{\lambda} = (3, 2, 8) \) is identical to that when \( \tilde{\lambda} \) is changed from \((3, 2, 8)\) to \((5, 2, 6)\) when \( \tilde{k} \) is fixed to be \((n, 0, \ldots, 0)\).


The results given in Section 3 can be used to study a variety of applications as in the case of optimal allocation of components in a system or network as shown in Figure 1. In particular, when the number, \( r \), of components of type \( B \) is kept fixed, and the component lifelengths are exponentially distributed, the system reliability is maximized when the \( \tilde{k} \) vector is such that \(|k_i - k_{i'}| \leq 1\) for all \( 1 \leq i, i' \leq r \).

A similar application can be found in the shock model described in Example 2.2. In that application the probability function \( \tilde{F}_{\tilde{k}, \tilde{\lambda}}(\tilde{t}) \) is maximized when the vector \( \tilde{k} = (n - r + 1, 1, \ldots, 1, 0, \ldots, 0) \) for all fixed \( \tilde{\lambda} \) and \( \tilde{t} \).
Figure 1
A parallel-series system

Figure 2
An equivalent system

Figure 3
A parallel-series system

Figure 4
A parallel-series system
References


