FUNCTIONAL EQUATIONS FOR
MULTIVARIATE EXPONENTIAL DISTRIBUTIONS

by

Albert W. Marshall and Ingram Olkin

Technical Report No. 272
July 1990

Prepared Under the Auspices
of
National Science Foundation Grant
DMS 87-08083
Ingram Olkin, Project Director

Department of Statistics
Stanford University
Stanford, California
FUNCTIONAL EQUATIONS FOR
MULTIVARIATE EXPONENTIAL DISTRIBUTIONS

by

Albert W. Marshall and Ingram Olkin
University of British Columbia and Western Washington University; Stanford University

Technical Report No. 272

Prepared Under the Auspices
of
National Science Foundation Grant
DMS 87-08083
Ingram Olkin, Project Director

Department of Statistics
Stanford University
Stanford, California
FUNCTIONAL EQUATIONS FOR
MULTIVARIATE EXPONENTIAL DISTRIBUTIONS

by

Albert W. Marshall
University of British Columbia and Western Washington University

and

Ingram Olkin
Stanford University

ABSTRACT

A large number of characterizations of univariate exponential distributions are known; these often lead to a functional equation, relatively few of which have been extended to the multivariate case. This paper is an exposition of multivariate extensions and relatives of the functional equation which stems from the following characterization: let $W_k$ be the minimum of $k$ independent copies of $X$. Then (i) $kW_k$ has the same distribution as $X$ for $k = 1, 2, \ldots$ if and only if (ii) $X$ has an exponential distribution.

Key words: characterization of distributions, minima of exponential variables.

AMS Classification: 62H05, 62E10, 39B60.

---

1 Supported in part by the National Science Foundation and by the Natural Sciences and Engineering Research Council of Canada.

2 Supported in part by the National Science Foundation.
1. Introduction.

A large number of characterizations of univariate exponential distributions are known; these often lead to a functional equation, relatively few of which have been extended to the multivariate case. This paper is an exposition of multivariate extensions and relatives of the functional equation which stems from the following characterization: Let $W_k$ be the minimum of $k$ independent copies of $X$. Then

(i) $kW_k$ has the same distribution as $X$ for $k = 1, 2, \ldots$ if and only if

(ii) $X$ has an exponential distribution.

This characterization was first stated explicitly in these terms by Desu (1971), but it was implicit already in the work of Fisher and Tippett (1928). A similar result, but with Condition (i) weakened, is an immediate consequence of a theorem of Sethuraman (1965). This and other modifications of Desu’s result are discussed in Section 3.

Desu’s Condition (i) can be written in terms of the survival function $\bar{F}$ of $X$ as the functional equation

\begin{equation}
[\bar{F}(x/k)]^k = \bar{F}(x), \quad x \geq 0, k = 1, 2, \ldots.
\end{equation}

This is a classical functional equation, perhaps best recognized as such by taking logarithms; see Aczél (1966, p. 31). Moreover, it is a direct consequence of the more standard functional equation for exponential distributions,

\begin{equation}
\bar{F}(x + t) = \bar{F}(x)\bar{F}(t), \quad x, t \geq 0.
\end{equation}

Although (1.2) is formally stronger than (1.1), the two equations admit the same family of survival functions as solutions. As discussed below, this convenient fact does not extend to higher dimensions.

A natural multivariate extension of Desu’s characterization takes $X$ to be a random vector, and $W_k$ to be the component-wise minimum of $k$ independent copies of $X$. This extension of Condition (i) is equivalent to the functional equation

\begin{equation}
[\bar{F}(\frac{x}{k})]^k = \bar{F}(x), \quad x \geq 0, k = 1, 2, \ldots,
\end{equation}
which in turn is a natural multivariate version of (1.1).

At least four different multivariate extensions of (1.2) have been discussed in the literature. The most obvious extension

\begin{equation}
\bar{F}(x + z) = \bar{F}(x)\bar{F}(z), \quad x, z \geq 0,
\end{equation}

was found by Marshall and Olkin (1967) to have as solutions only cases with marginals independent (See also, Aczél, 1966, p. 348). This equation can be weakened so as to admit more interesting solutions as follows:

\begin{equation}
\bar{F}(x + z) = \bar{F}(x)\bar{F}(z) \quad \text{for all} \quad x, z \geq 0 \quad \text{that are similarly ordered.}
\end{equation}

In fact, the survival functions satisfying (1.5) are the multivariate exponential distributions of Marshall and Olkin (1967); see Marshall and Shaked (1979, p. 355). Still weaker is the functional equation

\begin{equation}
\bar{F}(x + t\varepsilon) = \bar{F}(x)\bar{F}(t\varepsilon) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad x \geq 0, \quad \varepsilon = (1,\ldots,1),
\end{equation}

which was used by Marshall and Olkin (1967) together with constraints on the marginals to obtain a multivariate exponential distribution. Additional solutions of (1.6) were given by Ghurye and Marshall (1984).

An extension of (1.2) unrelated to (1.6) but weaker than (1.5) is

\begin{equation}
\bar{F}((\alpha + 1)x) = \bar{F}(\alpha x)\bar{F}(x) \quad \text{for all} \quad \alpha > 0, x \geq 0.
\end{equation}

This functional equation was considered by Pickands (1976) and is discussed further in Section 2.

Sets of survival function solutions of the functional are related in Figure 1.

\begin{align*}
\subset (1.6) \\
(1.4) \subset (1.5) \\
\subset (1.7) = (1.3)
\end{align*}

**Figure 1. Survival function solutions.**

No additional inclusions hold.

The remainder of the paper is devoted to (1.3) and some modifications of it.

2. Multivariate distributions with exponential scaled minima.
Any function homogeneous on rays emanating from the origin will satisfy (1.3), but the only solutions which are survival functions are of interest here, and they are not so easily identified. However, it is shown below in Section 3 that the survival functions which satisfy (1.3) are just those with “exponential scaled minima.”

The distribution \( F \) of \((X_1, \ldots, X_n)\) is said to have exponential scaled minima if \( \min_{1 \leq i \leq n}(X_i/a_i) \) has an exponential distribution whenever \( 0 \leq a_i < \infty \) for all \( i = 1, \ldots, n \) and \( a_i > 0 \) for some \( i = 1, \ldots, n \). (Here and in what follows, the convention \( 1/0 = \infty \) is used.) Distributions of this kind were introduced by Esary and Marshall (1974). According to the representation of Pickands (1976, 1981), \( F \) has exponential scaled minima if and only if the survival function \( \bar{F} \) of \( F \) can be written in the form

\[
\bar{F}(\bar{x}) = \exp \left\{ - \int_{\mathcal{S}} \left[ \min_{1 \leq i \leq n} q_i x_i \right] d\nu(\bar{g}) \right\},
\]

where \( \nu \) is a finite measure on \( \mathcal{S} = \left\{ \bar{g} : q_i \geq 0, \ i = 1, \ldots, n \text{ and } \sum_{i=1}^{n} q_i = 1 \right\} \).

By taking all but one of the \( a_i \) to be 0, it is easy to see that distributions with exponential scaled minima have exponential univariate marginals; this fact is also immediate from (2.1). Examples of distributions with exponential scaled minima include the multivariate exponential distribution of Marshall and Olkin (1967).

In addition to satisfying (1.3), distributions with exponential scaled minima constitute the solutions of (1.7). To see this, fix \( \bar{z} \geq 0, \ \bar{z} \neq \bar{0} \) and let \( Z = \min(X_i/x_i) \); then \( P(Z > \alpha) = \bar{F}(\alpha \bar{z}) \). The condition that \( F \) have exponential scaled minima can be written as the condition that the survival function of \( Z \) satisfies (1.2) for all \( \bar{z} \geq \bar{0} \) \( \bar{z} \neq \bar{0} \). Thus, \( F \) has exponential scaled minima if and only if

\[
\bar{F}((\alpha + \beta) \bar{z}) = \bar{F}(\alpha \bar{z}) \bar{F}(\beta \bar{z}) \text{ for all } \alpha, \beta \geq 0, \ \bar{z} \geq \bar{0}, \ \bar{z} \neq \bar{0}.
\]

Replacement of \( \alpha \) by \( \alpha \beta \) and \( \bar{z} \) by \( \bar{z}/\beta \) yields (1.7).

3. The Characterizations.

Suppose that \( X_1, X_2, \ldots \) is a sequence of independent copies of the random variable \( X \) and let \( W_k = \min(X_1, \ldots, X_k), \ k = 1, 2, \ldots \).

3.1.a. Theorem. (Fisher and Tippett, 1928; Desu, 1971). (i) \( kW_k \) and \( X \) have the same nondegenerate distribution for \( k = 1, 2, \ldots \) if and only if (ii) \( X \) has an exponential distribution.

A direct proof of this theorem easily follows from (1.1).
The Proof of Fisher and Tippett. In their pioneer paper on extreme value theory Fisher and Tippett (1928) consider a functional equation, which when transformed from maxima to minima, takes the form

\[ \bar{F}^k(x) = \bar{F}(a_k x + b). \]

Of course (3.1) is of this form. Fisher and Tippett show that when \( \bar{F}(0) = 1 \), then \( b = 0 \), \( a_k = k^\alpha \), and \( \bar{F}(x) = e^{-\lambda x^\alpha} \); Theorem 3.1.a deals with the case \( \alpha = 1 \).

Now suppose that \( \bar{X}_1, \bar{X}_2, \ldots \) is a sequence of independent copies of the random vector \( \bar{X} = (X_1, \ldots, X_n) \). Let

\[ W_{ki} = \min(X_{1i}, \ldots, X_{ki}), \quad c = 1, 2, \ldots, k = 1, 2, \ldots. \]

The following theorem is due to Pickands (1976) and is quoted by Galambos and Kotz (1978, p. 119). A somewhat simplified proof is offered here.

3.1.b. Theorem. (Pickands, 1976). (i) \( k \bar{W}_k \) and \( \bar{X}_k \) have the same distribution with nondegenerate marginals for \( k = 1, 2, \ldots \) if and only if (ii) \( \bar{X} \) has a distribution with exponential scaled minima.

Proof. Condition (i) can be written in the form of a functional equation for the survival function \( \bar{F} \) of \( X \), as follows:

\[ \left[ \bar{F}(t/k) \right]^k = \bar{F}(t), \quad t \in \mathbb{R}^n, \quad k = 1, 2, \ldots. \quad (3.2) \]

Let \( t \in \mathbb{R}^n_+ \), \( t > 0 \), and let \( a_i = t_i/t \). It follows from (3.2) that

\[ \left[ \bar{F} \left( \frac{a_1 t}{k}, \ldots, \frac{a_n t}{k} \right) \right]^k = \bar{F}(a_1 t, \ldots, a_n t), \quad t > 0, \quad k = 1, 2, \ldots, \quad a \in \mathbb{R}^n_+. \quad (3.3) \]

Suppose that \( a \in \mathbb{R}^n_+ \), \( a \neq 0 \), and let

\[ \bar{G}(t) = \bar{F}(a_1 t, \ldots, a_n t) = P\{\min X_i/a_i > t\}. \quad (3.4) \]

In terms of \( \bar{G} \), (3.3) becomes

\[ \bar{G}(t) = \left[ \bar{G}(t/k) \right]^k, \quad k = 1, 2, \ldots, \quad t > 0. \quad (3.5) \]

Since the marginals of \( F \) are non-degenerate, \( G \) is not degenerate at 0. Consequently, if (i) holds it follows that

\[ \bar{G}(t) = e^{-\lambda t}, \quad t \geq 0 \text{ for some } \lambda > 0, \quad (3.6) \]
and thus (ii) holds.

If (ii) holds,  \( \varrho \in \mathcal{R}_+^n \) and \( \varrho \neq \bar{\varrho} \), then \( \bar{G} \) defined in (3.4) satisfies (3.6), i.e. (3.3) is satisfied. With \( t_i = a_i t \), this yields (3.2) which is equivalent to (i). ||

Several modifications of Theorem 3.1.a have been obtained, and these are listed below, together with multivariate analogs.

3.2. **First modification.** (Sethuraman, 1965; Arnold, 1971). In Theorem 3.1.a, the condition (i) that \( kW_k \) and \( X \) have the same distribution, \( k = 1, 2, \ldots \) can be replaced by the condition that \( kW_k \) and \( X \) have the same distribution for two values \( m_1 \) and \( m_2 \) of \( k \) such that \( \log m_1 / \log m_2 \) is irrational. This result is given explicitly by Arnold, and it is an immediate consequence of Theorem 2 of Sethuraman (1965).

The same modification can be made in (i) of Theorem 3.1.b, because that change is reflected only as a similar change in (3.5), where the one dimensional result applies.

3.3 **Second Modification.** Gupta (1973) shows that in Theorem 3.1.a, the condition (i) can be replaced by the condition that \( kW_k \) and \( X \) have the same non-degenerate distribution for some one value of \( k > 1 \), and their common distribution \( F \) is such that

\[
\lim_{x \downarrow 0} \frac{F(x)}{x} \quad \text{exists.}
\]

(3.7)

Condition (i) of Theorem 3.1.b can be similarly modified, where (3.7) is replaced by

\[
\lim_{t \downarrow 0} \bar{F}(\varrho t) \quad \text{exists for all } \varrho \in \mathcal{R}_+^n.
\]

3.4 **Third modification.** (Shimizu, 1979). Let \( X_1, X_2, \ldots \) be a sequence of independent copies of \( X \), let \( \alpha_i > 0, \ i = 1, 2, \ldots, k, \sum \alpha_i = 1 \) and suppose that \( \log \alpha_i / \log \alpha_j \) is irrational for some pair \( i, j \). Then \( Z = \min(X_1/\alpha_1, \ldots, X_k/\alpha_k) \) and \( X \) have the same non-degenerate distribution if and only if \( X \) has an exponential distribution.

Now retain the hypotheses on \( \alpha_1, \ldots, \alpha_k \) and suppose that \( X_1, X_2, \ldots \) is a sequence of independent copies of \( \bar{X} = (X_1, \ldots, X_n) \). Let

\[
Z_i = \min(X_{i1}/\alpha_1, \ldots, X_{ik}/\alpha_k), \quad i = 1, \ldots, n.
\]

Then \( Z \) has the same distribution as \( \bar{X} \) if and only if \( \bar{X} \) has a distribution with exponential scaled minima.

Again this multivariate extension follows from the univariate case using the same approach as in the proof of Theorem 3.1.b.

As is indicated in the proof of Theorem 3.1.b, $\vec{X}$ has a distribution $F$ with exponential scaled minima if and only if

$$
\left[F \left( \frac{x_1 t}{k}, \ldots, \frac{x_n t}{k} \right) \right]^k = \bar{F}(x_1 t, \ldots, x_n t)
$$

for all $t > 0$, $\vec{x} \in \mathbb{R}^n_+$, $k = 1, 2, \ldots$. As indicated in Section 2, another equivalent condition is (1.7).

Suppose that (4.1), or alternatively (1.7), is required to hold only for vectors $\vec{x}$ with all nonzero components equal. Then in both cases there are additional solutions, and in fact it is easy to see that the class of all solutions is the set of distributions $F$ with exponential minima. These distributions are just those for which the corresponding random vector $\vec{X}$ satisfies $\min_{i \in A} X_i$ has an exponential distribution for all $A \subset \{1, 2, \ldots, n\}$, $A \neq \emptyset$. This class of distributions was introduced and studied by Esary and Marshall (1974).

Suppose that (4.1), or alternatively (1.7), is further weakened so that it is required to hold only for vectors with one nonzero component. Then the class of solutions is just the class with exponential marginals.

There appears to be no way of strengthening (4.1) short of replacing the power by a product so as to limit the possible solutions to a smaller family with exponential scaled minima such as the family of multivariate exponential distributions introduced by Marshall and Olkin (1967).
References


