REPEATED MEASUREMENT ARMA-PROCESSES II

by

S. H. C. du Toit

Technical Report No. 281
January 1991

Prepared Under the Auspices
of
National Science Foundation Grant
DMS 90-02502
Ingram Olkin, Project Director

Department of Statistics
Stanford University
Stanford, California
REPEATED MEASUREMENT ARMA-PROCESSES II

by

S. H. C. du Toit
University of Pretoria

Technical Report No. 281

Prepared Under the Auspices of
National Science Foundation Grant
DMS 90-02502
Ingram Olkin, Project Director

Department of Statistics
Stanford University
Stanford, California
REPEATED MEASUREMENT ARMA-PROCESSES II

by

S. H. C. du Toit
Department of Statistics
University of Pretoria

ABSTRACT

In Part I of the Technical Report a general procedure for the analysis of structured means and covariance matrices using maximum normal likelihood is described. Results of this procedure were implemented in the computer program AUFit, which requires a subroutine to evaluate the mean and covariance matrices and which normally uses numerical derivatives. Furthermore, closed form expressions for various types of ARMA models are given.

When a particular model is to be employed on a routine basis, it is worthwhile to simplify expressions for the likelihood function and derivatives and results to this end are given in Part II.

Since this report deals specifically with situations where the number of time points \(T\) are relatively small whereas the number of repetitions \(N\) are large, alternative graphical methods for the identification of ARMA processes are given.

This report also deals with how to handle additional concomitant variables and how to extend results to vector-variate repeated measurement time series models. Applications are given.
1. Introduction.

In Part I of the Technical Report a general model involving structured means and covariance matrices is proposed to describe possible changes in the response pattern over time (cf. Part I, (2.6)). It is assumed that the deviations from the model are generated by an ARMA($n,m$)-process (cf. Part I, (2.2)). A general procedure for the analysis of structured means and covariance matrices using maximum normal likelihood is described in Part I, Section 3. Results of this procedure were implemented in the computer program AUFIT (Du Toit and Browne, 1982 and 1989). This program requires a subroutine to evaluate the mean and covariance structure and optionally a subroutine to evaluate nonlinear constraints that may be imposed on the model parameters. Practical experience has shown that AUFIT works well with numerical derivatives, although the user may optionally provide Fortran code for the calculation of exact derivatives.

Furthermore, a closed form expression for the covariance matrix of a stationary ARMA($n,m$) process is derived in Part I, Section 4 (cf. (4.6)). By relaxing the usual assumptions for stationarity three additional ARMA models are proposed in Part I Section 5 and in Section 6 of Part I a number of applications are given.

In order to provide exact expressions for the elements of the gradient vector and approximate Hessian matrix, it is sufficient to obtain expressions for the derivatives of the mean and covariance matrices with respect to the unknown parameters (cf. Part I, Section 3, (3.7) and (3.10)). Expressions for the derivatives of the means will depend on the choice of response function. Therefore expressions for the evaluation of $\frac{\partial \Sigma(\gamma)}{\partial \eta}$ only, will be given in Section 2 of this report.

When a particular model is to be employed on a routine basis, it is worthwhile to simplify expressions for the likelihood function and derivatives. In Section 3 of this report, results are given for a few special cases.

Since this report deals specifically with situations where the number of time points ($T$) are relatively small whereas the number of repetitions ($N$) are large, alternative graphical methods for the identification of ARMA correlation structures are given in Section 4.

In Section 5 we consider situations of the following type. Each of a sample of $N$ subjects is measured $T$ times at regular intervals yielding measurements $y_1, y_2, \ldots, y_T$. An additional measurement, $c$, on a concomitant variable is also made on each subject.
In many longitudinal studies, measurements are made on a set of mutually dependent variables. Section 6 deals with an extension of univariate to vector-variate repeated measurement time series models. Finally, two applications are given in Section 7.

2. Exact derivatives for the covariance structures of ARMA(n, m)-processes.

Consider the following parametric form for $\Sigma$ (cf. Part I (4.36)):

$$\Sigma = T_\alpha^{-1}[\Gamma \Pi' + T_\beta D_\psi T_\beta'](T_\alpha^{-1})',$$

(2.1)

where (cf. Part I, Section 4) $\Gamma$ is the first $s$ columns of $I_T$,

$$\Gamma = I_{T,s}$$

(2.2)

In order to simplify the expression $\frac{\partial \Sigma}{\partial \gamma_i}$, we shall introduce the $T \times T$ matrices $U$ and $V$, where

$$U = -T_\alpha^{-1} \left( \frac{\partial}{\partial \gamma_i} T_\alpha \right) T_\alpha^{-1} \left[ \Gamma \Pi' + T_\beta D_\psi T_\beta' \right] (T_\alpha^{-1})'$$

(2.3)

$$= T_\alpha^{-1} \left( \frac{\partial}{\partial \gamma_i} T_\alpha \right) \Sigma,$$

$$V = \frac{\partial T_\beta}{\partial \gamma_i} D_\psi T_\beta'.$$

(2.4)

Then

$$\frac{\partial \Sigma}{\partial \gamma_i} = U + U' + T_\alpha^{-1} \left\{ \Gamma \frac{\partial P}{\partial \gamma_i} \Gamma' + V + V' + T_\beta \left( \frac{\partial D_\psi}{\partial \gamma_i} \right) T_\beta' \right\} (T_\alpha^{-1})$$

(2.5)

Subsequently, further simplifications of (2.5) will be considered for each of the four ARMA models (cf. Part I, Section 5).

Let $\gamma_1 : (k_1 \times 1)$ be a vector of all the unknown parameters in Model 1 (cf. Part I, (4.36)) arranged in the following order:

$$\gamma_1' = (\psi^2, \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m),$$

(2.6)

where

$$k_1 = (1 + n + m)$$

(2.7)

Then

$$\frac{\partial \Sigma}{\partial \psi^2} = T_\alpha^{-1} \{ \Gamma P^* \Gamma' + T_\beta T_\beta' \} T_\alpha^{-1'},$$

(2.8)
where $P = \psi^2 P^*$ with $P$ defined in Part I, Section 4 (cf. (4.23)).

$$\frac{\partial \Sigma}{\partial \alpha_i} = U + U' + T^{-1}_\alpha \left\{ \Gamma \frac{\partial P}{\partial \alpha_i} \Gamma' \right\} (T^{-1}_\alpha)'$$, \hspace{1cm} (2.9)

where (cf. Part I, (4.14))

$$\frac{\partial T_\alpha}{\partial \alpha_i} = \sum_{j=1}^{T-i} J_{(i+j),j}, \hspace{1cm} i = 1, \ldots, n$$ \hspace{1cm} (2.10)

and $J_{ij}$ denotes a $T \times T$ matrix with all elements equal to zero except the $(i, j)$-th element which is equal to one.

The $(s \times s)$ matrix $\frac{\partial P}{\partial \alpha_i}$ is obtained as follows. Since (cf. Part I (4.23))

$$P = \Phi P \Phi' + \psi^2 bb'$$

we have

$$\frac{\partial P}{\partial \alpha_i} = \Phi \frac{\partial P}{\partial \alpha_i} \Phi' + \left[ \frac{\partial \Phi}{\partial \alpha_i} P \Phi' + \left( \frac{\partial \Phi}{\partial \alpha_i} P \Phi' \right)' + \psi^2 \frac{\partial}{\partial \alpha_i} (bb') \right]$$ \hspace{1cm} (2.11)

Define the $(s \times s)$ matrix $B^{**}$ as

$$B^{**} = \left[ \frac{\partial \Phi}{\partial \alpha_i} P \Phi' + \left( \frac{\partial \Phi}{\partial \alpha_i} P \Phi' \right)' + \psi^2 \frac{\partial}{\partial \alpha_i} (bb') \right]$$ \hspace{1cm} (2.12)

then (2.11) can be written as

$$\text{vec} \left( \frac{\partial P}{\partial \alpha_i} \right) - \text{vec} \left( \Phi \frac{\partial P}{\partial \alpha_i} \Phi' \right) = \text{vec} (B^{**})$$ \hspace{1cm} (2.13)

so that (cf. Section 4, Part I, (4.29))

$$\text{svec} \left( \frac{\partial P}{\partial \alpha_i} \right) = C^{-1} K_s^- \text{vec} (B^{**})$$ \hspace{1cm} (2.14)

where the $(\frac{1}{2}s(s+1) \times \frac{1}{2}s(s+1))$ matrix $C$ and $(\frac{1}{2}s(s+1) \times s^2)$ matrix $K_s^-$ are defined by (4.30) and (4.25), Part I, respectively.

To obtain (2.14) we still require expressions for $\frac{\partial \Phi}{\partial \alpha_i}$ and $\frac{\partial}{\partial \alpha_i} (bb')$ where the $(s \times s)$ state transition matrix $\Phi$ and $(s \times 1)$ vector $b$ are defined by (4.20) and (4.21), Part I, respectively. From Part I, (4.20) it follows that

$$\frac{\partial \Phi}{\partial \alpha_i} = -J_{ij}, \hspace{1cm} i = 1, \ldots, n, \hspace{0.2cm} j = 1, \hspace{1cm} (2.15)$$
and from Part I, (4.21) it follows that

\[
\frac{\partial}{\partial \alpha_i} (bb') = -\{J_{i1}b' + (J_{i1}b')'\},
\]

where \(J_{i1}\) denotes a \((s \times 1)\) column vector with all elements equal to zero except the \(i\)-th element which is equal to 1.

Further (cf. (2.1))

\[
\frac{\partial \Sigma}{\partial \beta_i} = T_{\alpha}^{-1} \left\{ \Gamma \left( \frac{\partial}{\partial \beta_i} P \right) \Gamma' + \psi^2 \left[ \frac{\partial T_{\beta}}{\partial \beta_i} T_{\beta}' + \left( \frac{\partial T_{\beta}}{\partial \beta_i} T_{\beta}' \right)' \right] \right\} (T_{\alpha}^{-1})',
\]

where (cf. Part I, (4.15))

\[
\frac{\partial T_{\beta}}{\partial \beta_i} = \sum_{j=1}^{T-i} J_{i+j,j}, \quad i = 1, \ldots, m,
\]

and

\[
\text{svec} \left( \frac{\partial P}{\partial \beta_i} \right) = C^{-1} K_\beta \text{ vec}(B^{**}),
\]

where (cf. (2.12))

\[
\frac{\partial B^{**}}{\partial \beta_i} = \psi^2 \{J_{i1}b' + (J_{i1}b')'\},
\]

since \(\Phi\) is not a function of the \(\beta_i\), \(i = 1, \ldots, m\).

The parametric structure for \(\Sigma\) appropriate to Model 2 (cf. Part 1 (5.2)) is given by

\[
\Sigma = T_{\alpha}^{-1} \left\{ \psi^2 T_{\beta} T_{\beta}' \right\} (T_{\alpha}^{-1})'
\]

If (cf. (2.1))

\[
P : (s \times s) = 0,
\]

then the covariance matrix given by (2.21) follows as a special case of (2.1). Let \(\gamma_2 : (k_2 \times 1)\) denote the vector of unknown parameters in Model 2, then

\[
\gamma_2 : (k_2 \times 1) = \gamma_1,
\]

with \(\gamma_1\) defined by (2.6) and \(k_2 = k_1\).

The derivatives \(\frac{\partial \Sigma}{\partial \gamma_i}, \ i = 1, \ldots, k_2\) are obtained in a similar manner to those of Model 1, with the omission of terms which contain the symmetric matrix \(P\).
For Model 3 (cf. Part I, Section 5, (5.6)) the appropriate parametric structure for $\Sigma$ is given by

$$
\Sigma = T_\alpha^{-1} \left\{ \Gamma P_1 \Gamma + T_\beta D_\psi T_\beta' \right\} (T_\alpha^{-1})',
$$

(2.24)

where the elements of the symmetric matrix $P_1$ are additional parameters to be estimated, i.e. $P_1: (s \times s)$ is not a function of the matrices $T_\alpha$, $T_\beta$ or $D_\psi$. Let $\gamma_3: (k_3 \times 1)$ be a vector of all the unknown parameters in Model 3, arranged on the following order:

$$
\gamma'(k_3 \times 1) = (\psi_{11}, \psi_{22}, \ldots, \psi_{TT}, \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m, \rho_{11}, \rho_{12}, \rho_{22}, \ldots, \rho_{ss}),
$$

(2.25)

where

$$
k_3 = T + n + m + \frac{1}{2} s(s + 1),
$$

(2.26)

and the $\rho_{ij}, \; i \leq j \leq s$ are the nonduplicated elements of $P_1$.

A general expression for $\frac{\partial \Sigma}{\partial \gamma_i}$ is given by (2.5). In order to evaluate (2.5), we require expressions for $\frac{\partial T_\alpha}{\partial \alpha_i}$ (given by (2.10)), $\frac{\partial T_\beta}{\partial \beta_i}$ (given by (2.18)), $\frac{\partial P_1}{\partial \rho_{ss}}$ and $\frac{\partial D_\psi}{\partial \psi_{ii}}$.

Since $P_1: (s \times s)$ is a symmetric matrix it follows that

$$
\frac{\partial P_1}{\partial \rho_{ij}} = J_{ij} + (1 - \delta_{ij})J_{ji},
$$

(2.27)

Further

$$
\frac{\partial D_\psi}{\partial \psi_{kk}} = J_{kk}; \; \; k = 1, \ldots, T.
$$

(2.28)

To conclude this section, we shall consider Model 4 (cf. Part I, Section 5, (5.13)). The parametric structure for $\Sigma$ appropriate to this model is given by

$$
\Sigma = T_\alpha^*^{-1} T_\beta^* D_\psi T_\beta^*' (T_\alpha^*^{-1})',
$$

(2.29)

where the $T \times T$ matrices $T_\alpha^*$ and $T_\beta^*$ are defined by (5.11) and (5.12) Section 5, Part I, respectively.

Let $\gamma_4: (k_4 \times 1)$ be a vector of all the unknown elements in Model 4, arranged in the following order:

$$
\gamma': (k_4 \times 1) = (\psi_{11}, \psi_{22}, \ldots, \psi_{TT}, \alpha_1, \alpha_3, \ldots, \alpha_T, \alpha_{23}, \alpha_{42}, \alpha_{43}, \ldots, \alpha_{23T}, \ldots, \alpha_{Tn}, \beta_{21}, \beta_{31}, \ldots, \beta_{T1}, \beta_{32}, \beta_{42}, \ldots, \beta_{T2}, \beta_{43}, \beta_{53}, \ldots, \beta_{T3}, \ldots, \beta_{Tm}),
$$

(2.30)

where

$$
k_4 = T + [(T - 1) + (T - 2) + \cdots + (T - n)]
$$

$$
+ [(T - 1) + (T - 2) + \cdots + (T - m)] + 4q
$$

$$
= T + (nT - \frac{1}{2} n(n + 1)) + (mT - \frac{1}{2} m(m + 1))
$$

(2.31)
The matrices $\frac{\partial \Sigma}{\partial \gamma_i}$ for $i = 1, \ldots, k_4$ can be determined within the general framework (2.5), provided that $T_\alpha$ and $T_\beta$ are replaced by $T_\alpha^*$ and $T_\beta^*$ respectively, where

$$\frac{\partial T_\alpha^*}{\partial \alpha_{i,(i-j)}} = J_{ij}, \quad 1 \leq i - j \leq n \quad (2.32)$$

and

$$\frac{\partial T_\beta^*}{\partial \beta_{i,(i-j)}} = J_{ij}, \quad 1 \leq i - j \leq m \quad (2.33)$$

3. Simplified expressions for the discrepancy function and its derivatives: Special cases.

In Part I, Section 2 it is shown that maximum normal likelihood estimates of the unknown parameter may be obtained by minimizing the discrepancy function

$$F(\gamma) = \ln |\Sigma| - \ln |S| + \text{tr} [S^* \Sigma^{-1}] - T, \quad (3.1)$$

where

$$S^* = S + (\overline{Y} - \Xi)C(\overline{Y} - \Xi)'$$

with $\overline{Y}$ and $S$ defined by (3.2) and (3.3) (Part I) respectively. From Part I, (3.7) it further follows that

$$\frac{\partial F}{\partial \gamma_i} = - \left\{ 2 \text{tr} \ C(\overline{Y} - \Xi)' \Sigma^{-1} \frac{\partial \Xi}{\partial \gamma_i} + \text{tr} \ \Sigma^{-1} (\Sigma - S^*) \Sigma^{-1} \frac{\partial \Sigma}{\partial \gamma_i} \right\} \quad (3.2)$$

Using the fact that $\frac{\partial \Sigma}{\partial \gamma_i} = - \Sigma \frac{\partial \Sigma^{-1}}{\partial \gamma_i} \Sigma$, and alternative expression for (3.2) is given by

$$\frac{\partial F}{\partial \gamma_i} = - \left\{ 2 \text{tr} \ C(\overline{Y} - \Xi)' \Sigma^{-1} \frac{\partial \Xi}{\partial \gamma_i} - \text{tr} (\Sigma - S^*) \frac{\partial \Sigma^{-1}}{\partial \gamma_i} \right\} \quad (3.3)$$

Expressions for $\frac{\partial \Sigma}{\partial \gamma_i}$ will depend on the type of response function chosen. Suppose, for example, (cf. Part I, Section 3) that $\Xi$ is the $T \times 1$ column vector $B\theta$ where $B$ is a $T \times m$ known design matrix of rank $m$ and $\theta$ an $m \times 1$ vector of unknown parameters, then (cf. Part I, (3.4) and (3.3) above)

$$\frac{\partial F}{\partial \theta_i} = -2 \text{tr} (\overline{Y} - B\theta)' \Sigma^{-1} BJ_{i1}, \quad (3.4)$$

where $J_{i1}$ is a column vector with all elements zero except the $i$-th element which is equal to 1. Using the result

$$\text{tr} \ AJ_{rs} = [A]_{s,r}, \quad (3.5)$$

6
it follows that
\[ \frac{\partial F}{\partial \theta_i} = -2[(\vec{Y} - B\theta)'\Sigma^{-1} B]_{1,i} \]  
(3.6)

Setting these partial derivatives equal to zero, \( i = 1, 2, \ldots, m \), it follows that
\[ (\vec{Y} - B\hat{\theta})'\hat{\Sigma}^{-1} B = 0' \]  
(3.7)

and therefore
\[ \hat{\theta} = (B'\hat{\Sigma}^{-1} B)B'\hat{\Sigma}^{-1} \vec{Y} \]  
(3.8)

If the response function is nonlinear in the parameters, it will not be possible to obtain a closed form solution such as (3.8). Note, also, that (3.8) is a function of \( \hat{\Sigma} \). In the remaining part of this section results will be given that may be used to simplify (3.1) and \( \frac{\partial F}{\partial \gamma_i} \), where \( \gamma_i, i = 1, 2, \ldots, k \) will denote the unknown parameters for the covariance structure only.

Suppose the covariance structure \( \Sigma(\gamma) \) is generated by a stationary AR(p)-process then (cf. Part I, (4.36) and (4.37))
\[ \Sigma = \psi^2 T_\alpha^{-1} \left[ \begin{array}{cc} P^* + I_p & 0 \\ 0 & I_{T-p} \end{array} \right] T_\alpha'^{-1}, \]  
(3.9)

where \( \psi^2 P^* = P \), the initial state covariance matrix. Let
\[ L'L = (P^* + I_p)^{-1}, \]  
(3.10)

then
\[ \Sigma^{-1} = \psi^{-2}T_\alpha' \left[ \begin{array}{cc} L'L & 0 \\ 0 & I_{T-p} \end{array} \right] T_\alpha \]  
(3.11)

From (3.9),
\[ |\Sigma| = \psi^{2T}|P^* + I_p|, \]  
(3.12)

where \( P^* \) may be determined using the results of Part I, Section 4 (cf. (4.23) to (4.31)). In particular, for an AR(2)-process
\[ P^* + I_2 = \frac{1}{|C|} \left( \begin{array}{cc} 1 + \alpha_2 & \alpha_1 \alpha_2 \\ \alpha_1 \alpha_2 & (1 + \alpha_2) - \alpha_1^2(1 - \alpha_2) \end{array} \right), \]  
(3.13)

where
\[ |C| = (1 + \alpha_2)(1 - \alpha_1^2 - \alpha_2^2) + 2\alpha_1 \alpha_2 \]  
(3.14)

Using (3.13),
\[ |P^* + I_2| = \frac{1}{(1 - \alpha_2)|C|} \]  
(3.15)
and
\[(P^* + I_2)^{-1} = (1 - \alpha_2) \begin{pmatrix} 1 + \alpha_2 - (1 - \alpha_2)\alpha_1^2 & -\alpha_1\alpha_2 \\ -\alpha_1\alpha_2 & 1 + \alpha_2 \end{pmatrix} \)  \quad (3.16)

Substitution of (3.15) in (3.12) gives
\[|\Sigma| = \frac{\psi^2 T}{(1 - \alpha_2)|C|} \]  \quad (3.17)

Furthermore use of (3.16) and (3.11) shows that
\[\Sigma^{-1} = \psi^{-2}\tilde{T}_\alpha'\tilde{T}_\alpha, \]  \quad (3.18)

where
\[\tilde{T}_\alpha = \begin{pmatrix} LT_{11} & 0 \\ T_{12} & T_{22} \end{pmatrix}, \]  \quad (3.19)

with $T_{11}$, $T_{12}$, and $T_{22}$ submatrices of order $2 \times 2$, $T \times 2$ and $(T - 2) \times (T - 2)$ formed from $T_\alpha$ and where
\[LT_{11} = \left(\frac{1 - \alpha_2}{1 + \alpha_2}\right)^{\frac{1}{2}} \begin{pmatrix} (1 + \alpha_2)^2 - \alpha_1^2 \alpha_2 \alpha_1 & -\alpha_1^2 \alpha_2 \\ \alpha_1(1 + \alpha_2) & 1 + \alpha_2 \end{pmatrix} \]  \quad (3.20)

The results for an AR(1) process follows immediately by setting $\alpha_2 = 0$ in (3.13) to (3.20).

Use of (3.17) and (3.18) shows (cf. (3.1)), that the discrepancy function of an AR(2)-process may be written as
\[F(\gamma) = T \ln \psi^2 - \ln(1 - \alpha_2)[(1 + \alpha_2)(1 - \alpha_1^2 - \alpha_2^2) + 2\alpha_1\alpha_2] - \ln |S| + \text{tr} [S^*\psi^{-2}\tilde{T}_\alpha'\tilde{T}_\alpha] - T. \]  \quad (3.21)

Partial derivatives for the AR(2) process are obtained as follows.
\[\frac{\partial F}{\partial \psi^2} = -\text{tr} (\Sigma(\gamma) - S^*) \frac{\partial}{\partial \psi^2} \left\{\psi^{-2}\tilde{T}_\alpha'\tilde{T}_\alpha\right\} = \text{tr} (\Sigma(\gamma) - S^*)\psi^{-2}\Sigma \]  \quad (3.22)

By setting $\frac{\partial F}{\partial \psi^2} = 0$ we obtain
\[\text{tr} \Sigma(\tilde{\gamma}) = \text{tr}(S^*). \]  \quad (3.25)

Let $\gamma_0 = [\Sigma]_{i,i}, \ i = 1, 2, \ldots, T$, then from (3.25)
\[\tilde{\gamma}_0 = \frac{1}{T} \sum_{i=1}^{T} \delta_{ii}^* \]  \quad (3.26)
It also follows that
\[
\frac{\partial F}{\partial \alpha_k} = -\text{tr} (\Sigma - S^*) \psi^{-2} \frac{\partial}{\partial \alpha_k} \left\{ \tilde{T}_\alpha \tilde{T}_\alpha' \right\},
\]
and using the result that \( \text{tr} AB = \text{tr} A'B' \) for any two square matrices of the same order,
\[
\frac{\partial F}{\partial \alpha_k} = -2 \text{tr} (\Sigma - S^*) \psi^{-2} \left\{ \frac{\partial}{\partial \alpha_k} \tilde{T}_\alpha' \right\} T_\alpha, \quad k = 1, 2.
\]

In particular, for a stationary AR(1) process it can easily be shown by setting \( \alpha_2 = 0 \) in (3.20) that
\[
\left( \frac{\partial}{\partial \alpha_1} \tilde{T}_\alpha' \right) T_\alpha = \sum_{j=2}^{T} J_{j,j-1} + \alpha_1 \sum_{j=2}^{T-1} J_{j,j}
\]
Therefore
\[
\frac{\partial F}{\partial \alpha_1} = -2 \left\{ \sum_{j=2}^{T} (\Sigma - S^*)_{j,j-1} + \alpha_1 \sum_{j=2}^{T-1} (\Sigma - S^*)_{j,j} \right\}
\]
From (3.25) with \( \frac{\partial F}{\partial \alpha_1} = 0 \), we have
\[
\sum_{j=2}^{T} \tilde{T}_{j,j-1} + \tilde{\alpha}_1 \sum_{j=2}^{T-1} \tilde{\gamma}_0 = \sum_{j=2}^{T} s_{j,j-1} + \tilde{\alpha}_1 \sum_{j=2}^{T-1} s_{jj}
\]
Results for Model II (ARMA-process with zero initial state covariance matrix) may be obtained in a similar manner. Suppose that \( P = 0 \) and that the process is AR(2), then
\[
\Sigma^{-1} = \psi^{-2} T_\alpha' T_\alpha
\]
and
\[
\frac{\partial}{\partial \alpha_k} T_\alpha = \sum_{j=k+1}^{T} J_{j,j-k}.
\]
Next, a number of results for stationary ARMA(1,1) processes will be given. From Part I, Section 3 we have
\[
\Sigma = \psi^2 T_\alpha'^{-1} (\Gamma p^* \Gamma' + T_\beta T_\beta'^{-1}) T_\alpha'^{-1},
\]
where
\[
p^* = \frac{(\beta - \alpha)^2}{1 - \alpha^2},
\]
\[
|\Sigma| = \psi^2 |\{ \Gamma p^* \Gamma' (T_\beta T_\beta')^{-1} + I \} (T_\beta T_\beta')|,
\]
Since \( |T_\alpha| = |T_\beta| = 1 \),
\[
|\Sigma| = \psi^2 |\beta^* p^* \beta^* + 1|,
\]
where

\[ \beta^* = T_\beta^{-1} \Gamma = T_\beta^{-1} I_{T,1} \]  

(3.37)

Du Toit (1979) gave the following result for stationary ARMA processes. If \( \Phi \) is the \( s \times s \) state transition matrix defined in Part I, Section 4 (4.20), then

\[ [T_\alpha^{-1}]_{k,i} = [\Phi^{k-i}]_{1,1} \]  

(3.38)

In particular for an ARMA\((1, 1)\) process, \( \Phi = -\alpha_1 \), and therefore the elements of the first column of \( T_\alpha^{-1} \) are given by

\[ [T_\alpha^{-1}]_{k,1} = (-\alpha)^{k-1}, \quad k = 1, 2, \ldots, T \]  

(3.39)

From (3.37) and by substituting \( \beta \) for \( \alpha \) in (3.39) it follows that

\[ \beta^{*,1} = (1 - \beta \beta^2 \ldots (-1)^{T-1} \beta^{T-1}) \]  

(3.40)

so that, for an ARMA\((1, 1)\) process

\[ |\Sigma| = \psi^2 T \left\{ \frac{(\beta - \alpha)^2}{1 - \alpha^2} \left( \frac{1 - \beta^{2T}}{1 - \beta^2} \right) + 1 \right\} \]  

(3.41)

Using a result of Mardia et al (1982), namely

\[ [A + BCB']^{-1} = A^{-1} - A^{-1} B [C^{-1} + B'A^{-1}B]^{-1} B' A^{-1} \]  

(3.42)

with \( A, B \) and \( C \) of appropriate dimensions it further follows for an ARMA\((1, 1)\) process that

\[ \Sigma^{-1} = \psi^{-2} T_\alpha [\Gamma p^* \Gamma' + T_\beta^* T_\beta]^{-1} \]  

(3.43)

where

\[ [\Gamma p^* \Gamma' + T_\beta^* T_\beta]^{-1} = (T_\beta^* T_\beta)^{-1} - \beta^* \left[ \frac{1}{p^*} + \beta^{*,1} \beta^* \right]^{-1} \beta^{*,1} \]  

(3.44)

To conclude this section, some results will be given for non-stationary ARMA\((n,n)\) processes. Since (cf. Part I, Section 5, (5.11) to (5.13))

\[ \Sigma = T_\alpha^{*-1} T_\beta^{*} D_\psi T_\beta^{*} (T_\alpha^{*})^{-1}, \]  

(3.45)

we have

\[ |\Sigma| = \prod_{i=1}^{T} [D_\psi]_{i,i}. \]  

(3.46)
Suppose the process is non-stationary AR(1), then
\[ \Sigma^{-1} = T^*_\alpha D_\psi T^*_\alpha \] (3.47)

and
\[ \frac{\partial T^*_\alpha}{\partial \alpha_{j_1}} = J_{j_1,j-1} \quad j = 2, 3, \ldots, T. \] (3.48)

Consequently (cf. (3.2))
\[
\frac{\partial F}{\partial \alpha_{j_1}} = \text{tr} (\Sigma - S^*) \left\{ J_{j_1,j} - T^*_{\alpha} + T^*_{\alpha} D_\psi J_{j_1,j-1} \right\} \\
= 2 \text{tr} (\Sigma - S^*) T^*_{\alpha} D_\psi J_{j_1,j-1}
\]

so that
\[
\frac{\partial F}{\partial \alpha_{j_1}} = \left[ (\Sigma - S^*) T^*_{\alpha} D_\psi \right]_{j_1,j} \quad j = 2, \ldots, T
\] (3.49)

In summary, a number of results and ideas were given in this section which may be used as tools for the simplification of the discrepancy function and its partial order derivatives. The actual simplification depends on the choice of response function and type of ARMA model.

Since the choice of an appropriate ARMA model is not always a simple decision, the next section is devoted to the identification of a number of ARMA models via the graphical display of the correlation matrix.

4. Graphical display of ARMA correlation structures.

4.1. Introduction.

Suppose that the \( T \times 1 \) vector \( e \) is generated by an ARMA process and let
\[ \text{cov}(e) = \Sigma \] (4.1)

Since our aim is to make of graphical representation of the correlation structure \( R \), it will be convenient to express the covariance matrix of the variables \( e_1, e_2, \ldots, e_T \) by
\[ \Sigma = D_\sigma RD_\sigma, \] (4.2)

where the diagonal elements of the diagonal matrix \( D_\sigma \) denote the standard deviations of the variables.

The graphical display of variables is based on an inverse relationship between correlations and distances: Each variable will be treated as a point in a Euclidean space in such
a way that, the higher the correlation between two variables, the closer they are in this Euclidean space. In Section 4.2 a transformation function from correlations to distances (and vice versa) which makes this inverse relationship possible, will be given. Section 4.3 deals with the properties and graphical display of repeated measurement autoregressive processes of order 1 and 2 (AR(1) and AR(2) processes), Section 4.4 with moving average process of order 1 and 2 (MA(1) and MA(2) processes). In Section 4.5 the properties and graphical display of repeated measurement ARMA(1, 1)-processes are considered. Results of a simulation study is given in Section 4.6 and finally, some general remarks and conclusions are made in Section 4.7.

It should be stressed that this report deals specifically with situations where the number of time points \( T \) are relatively small whereas the number of repetitions \( N \) are large.

Procedures for the estimation of the unknown co-ordinates of the points that represent the variables in space, for setting up of confidence intervals and for performing statistical inference, are based on the method of maximum likelihood for normal distributions. Use is made of the results given in Part I Section 3, for this purpose.

4.2 Transformations from correlations to distances.

It will be shown in subsequent sections that time series processes up to order two may be adequately represented in one or two dimensional space. Let us assume therefore that the variables \( e_1, e_2, \ldots, e_T \) may be represented by the points \( (x_{11}, x_{12}), (x_{21}, x_{22}), \ldots, (x_{T1}, x_{T2}) \) in geometric space.

The Euclidean distance between the points \( i \) and \( j \) which represents the variables \( e_i \) and \( e_j \) in two-dimensional space is therefore given by

\[
d_{ij} = \left[ (x_{i1} - x_{j1})^2 + (x_{i2} - x_{j2})^2 \right]^{1/2}
\]

Du Toit and Boraine (1987) introduced the following functional relationship between positive correlations and distances:

\[
\rho_{ij} = \exp(-\tau_1 d_{ij} \tau_2); \quad \tau_1, \tau_2 > 0
\] (4.3)

From the definition (4.3) it follows that \( 0 \leq \rho_{ij} \leq 1 \) and that

\[
d_{ij} = [-\tau_1 \ln \rho_{ij}]^{1/\tau_2},
\]

where \( \rho_{ij} = [R]_{i,j} \). It also follows from (4.3) that there exists an inverse relationship between correlations and distances in the sense that small distances will yield larger correlation values and vice versa for large distances.
At this point it should also be noted that ARMA-covariance structures often contain negative as well as positive correlations. We therefore modify (4.3) as follows

$$\rho_{ij} = \text{sign}(r_{ij}) \exp(-\tau_1 d_{ij} \tau_2),$$  \hfill (4.4)

where \(\text{sign}(r_{ij}) = -1\) if the \((i, j)\)-th element of the sample correlation matrix is negative and \(\text{sign}(r_{ij}) = 1\) otherwise. Using (4.4), variables that are highly correlated (in absolute value) will still be represented as points that are closely located in space. From (4.4) it follows that

$$d_{ij} = [-\tau_1 \ln |\rho_{ij}|]^{1/\tau_2}$$  \hfill (4.5)

Now consider the null hypothesis,

$$H_0 : \Sigma = D_{\sigma} R(X^*) D_{\sigma},$$  \hfill (4.6)

where \(X^*_{T \times m}\) is the configuration of points in \(m\)-dimensional space. Then (4.6) is equivalent to testing the null hypothesis

$$H'_0 : \Sigma = \Sigma(\gamma),$$  \hfill (4.7)

where (cf. (4.2) and (4.6))

$$\gamma = (\sigma_1, \sigma_2, \ldots, \sigma_T, x_{11}, x_{12}, \ldots, x_{Tm}, \tau_1, \tau_2)'$$  \hfill (4.8)

The results reported in Part I, Section 2 may therefore be employed to test the hypothesis (3.6). If the null hypothesis is accepted, it may be assumed that the correlation structure is adequately represented graphically by points in an \(m\)-dimensional space. We will make use of this fact in considering the graphical display of repeated measurement time series structures in the sections to follow.

4.3 Graphical display of AR(1) and AR(2) processes.

An ARMA\((m, n)\) process is defined by (cf. Part I, Section 3)

$$\epsilon_t + \alpha_1 \epsilon_{t-1} + \cdots + \alpha_m \epsilon_{t-m} = u_t + \beta_1 u_{t-1} + \cdots + \beta_n u_{t-n}$$  \hfill (4.9)

where \(\alpha_1, \alpha_2, \ldots, \alpha_m\) denote the AR parameters, \(\beta_1, \beta_2, \ldots, \beta_n\) the moving average parameters and \(\ldots, u_{-2}, u_{-1}, u_0, u_1, \ldots\) form a series of uncorrelated identically normally distributed random variables with zero means and homogeneous variances \(\psi^2\).

A characteristic of all stationary ARMA-processes is that the covariance matrix for the process may be written as

$$\Sigma = \gamma_0 R = D \sqrt{\tau_0} R D \sqrt{\tau_0}$$  \hfill (4.10)
with $\gamma_0 = \text{var}(e_t)$ and

$$[R]_{i,j} = \rho_k, \quad k = |i - j|, \quad (4.11)$$

where $\rho_k$ denotes the correlation between $e_{t+k}$ and $e_t$, $t = 1, 2, \ldots, T$, $k = 0, 1, 2, \ldots, T-1$.

Since (cf. (4.11)) the correlations of all variables at lag $k$ are equal, it implies that the distances between the points that represent these variables in Euclidean space, will also be equal. This implies that

$$d_{ij} = d_k, \quad k = |i - j| \quad (4.12)$$

First, consider the AR(1) process. Since (Box and Jenkins, 1976)

$$\rho_k = (-\alpha_1)^k, \quad k \geq 1, \quad (4.13)$$

it follows that (with $\tau_1 = \tau_2 = 1$ in (4.5))

$$d_k = -\ln |\alpha_1|^k = kc, \quad (4.14)$$

with $c = -\ln |\rho_1|$ and $k = |i - j|$.

From (4.14) it therefore follows that the graphical representation of the correlation matrix of an AR(1)-process is one-dimensional and that the points representing the variables fall on a straight line with equally spaced intervals. Figure 4.1 is an illustration of this situation for $T = 5$.

```
0   c   2c   3c   4c
```

**Figure 4.1.** Graphical display of a $5 \times 5$ correlation matrix of an AR(1)-process. $|\alpha_1| = \exp(-c)$

The co-ordinates $x_{11}, x_2, \ldots, x_{T1}$ representing the points $e_1, e_2, \ldots, e_T$ are given by $x_{i1} = (i - 1)c$, $i = 1, 2, \ldots, T$. The unknown parameters in this correlation structure is

$$\gamma = (\sigma, c)^T \quad (4.15)$$

Initial values for $\sigma$ and $s$ may be obtained by setting $\vartheta = 2$ and $c = -\ln r_{21}$, where $s$ denotes the sample standard deviation of $e_1$ and $r_{21}$ denotes the sample correlation between variables $e_2$ and $e_1$.

Next consider the AR(2)-process. The relationship between correlations and distances is not as evident as in the case of the AR(1)-process. Figure 4.2 gives a graphical representation of the allowable $\phi_1$ and $\phi_2$ values for the AR(2)-process.
Figure 4.2. Set of allowable $\alpha_1, \alpha_2$-values for an AR(2)-process.

Note that the absolute value of correlations generated by any $(\alpha_1, \alpha_2)$-value left of the dotted line $A$ is equal to the correlations generated by the corresponding $(-\alpha_1, \alpha_2)$-value right of the dotted line. With the aid of Figure 4.2 we group the AR(2)-processes into 4 main types:

1. $0 \leq |\alpha_1| \leq 0.5$ and $-1 \leq \alpha_2 \leq -0.5$
2. $0 \leq |\alpha_1| \leq 0.5$ and $-0.5 \leq \alpha_2 \leq 0$
3. $0.5 \leq |\alpha_1| \leq 1.0$ and $-0.5 \leq \alpha_2 \leq 0$
4. $0.5 \leq |\alpha_1| \leq 2$ and $0 \leq \alpha_2 \leq 1$

The autocorrelations of a stationary AR(2) process may be calculated from the following set of recursive equations:

$$\rho_k = -\alpha_1 \rho_{k-1} - \alpha_2 \rho_{k-2} \quad k \geq 2$$  \hspace{1cm} (4.16)
with \( \rho_0 = 1 \) and

\[
\rho_1 = \frac{-\alpha_1}{1 + \alpha_2}
\]  \hspace{1cm} (4.17)

(a) \((\alpha_1 = -0.1, \alpha_2 = -0.75)\)

(b) \((\alpha_1 = -0.48, \alpha_2 = -0.45)\)

(c) \((\alpha_1 = -0.6, \alpha_2 = -0.25)\)

(d) \((\alpha_1 = -1.20, \alpha_2 = 0.23)\)

**Figure 4.3.** (a)–(d) Graphical representation of 4 main types of AR(2)-processes.

Associated with each of these main types is a typical graphical representation as depicted in Figure 4.2.

**Table 4.2** contains the results for a number of different AR(2)-processes: Columns 1 and 2 are the \( \alpha_1 \) and \( \alpha_2 \) parameters used, columns 2 to 7 contain the first 5 autocorrelations, column 8 is the estimated \( \alpha_2 \) value used in the transformation function (4.4) and the last column is the values of the discrepancy function \( F \). The computer program AUFIT was used to obtain the estimated coordinates. All input correlation matrices were taken as 10 \( \times \) 10 matrices.
Table 4.2:
Parameter values ($\gamma$) for different AR(2)-processes

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\rho_3$</th>
<th>$\rho_4$</th>
<th>$\rho_5$</th>
<th>$\alpha_2$</th>
<th>$F(\gamma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.20</td>
<td>-0.60</td>
<td>0.498</td>
<td>0.700</td>
<td>0.436</td>
<td>0.510</td>
<td>0.354</td>
<td>0.913</td>
<td>0.0024</td>
</tr>
<tr>
<td>-0.10</td>
<td>-0.75</td>
<td>0.399</td>
<td>0.790</td>
<td>0.379</td>
<td>0.631</td>
<td>0.344</td>
<td>0.965</td>
<td>0.0009</td>
</tr>
<tr>
<td>-0.40</td>
<td>-0.55</td>
<td>0.888</td>
<td>0.905</td>
<td>0.848</td>
<td>0.841</td>
<td>0.800</td>
<td>0.794</td>
<td>0.0144</td>
</tr>
<tr>
<td>-0.10</td>
<td>-0.23</td>
<td>0.139</td>
<td>0.294</td>
<td>0.072</td>
<td>0.086</td>
<td>0.028</td>
<td>0.998</td>
<td>0.0003</td>
</tr>
<tr>
<td>-0.10</td>
<td>-0.24</td>
<td>0.132</td>
<td>0.253</td>
<td>0.060</td>
<td>0.064</td>
<td>0.020</td>
<td>1.00</td>
<td>0.0002</td>
</tr>
<tr>
<td>-0.48</td>
<td>-0.45</td>
<td>0.871</td>
<td>0.868</td>
<td>0.806</td>
<td>0.783</td>
<td>0.738</td>
<td>0.7902</td>
<td>0.0147</td>
</tr>
<tr>
<td>-0.60</td>
<td>-0.25</td>
<td>0.799</td>
<td>0.729</td>
<td>0.633</td>
<td>0.570</td>
<td>0.504</td>
<td>0.827</td>
<td>0.0083</td>
</tr>
<tr>
<td>-0.80</td>
<td>-0.12</td>
<td>0.909</td>
<td>0.847</td>
<td>0.785</td>
<td>0.733</td>
<td>0.683</td>
<td>0.899</td>
<td>0.0043</td>
</tr>
<tr>
<td>-0.51</td>
<td>-0.10</td>
<td>0.567</td>
<td>0.388</td>
<td>0.253</td>
<td>0.174</td>
<td>0.117</td>
<td>0.888</td>
<td>0.0006</td>
</tr>
<tr>
<td>-0.80</td>
<td>0.15</td>
<td>0.695</td>
<td>0.405</td>
<td>0.215</td>
<td>0.106</td>
<td>0.049</td>
<td>1.314</td>
<td>0.0006</td>
</tr>
<tr>
<td>-0.90</td>
<td>0.18</td>
<td>0.762</td>
<td>0.504</td>
<td>0.309</td>
<td>0.178</td>
<td>0.098</td>
<td>1.333</td>
<td>0.0021</td>
</tr>
<tr>
<td>-1.20</td>
<td>0.23</td>
<td>0.976</td>
<td>0.941</td>
<td>0.903</td>
<td>0.862</td>
<td>0.820</td>
<td>1.288</td>
<td>0.0483</td>
</tr>
</tbody>
</table>

Note that $\ln(x_2 + 1)$ was used in Figures 4(a)-(d) for the vertical axis. This was done so that the same axial system could be used for the graphical displays. From (4.4) it follows that $d_k \to \infty$ as $\rho_k \to 0$. This is the case when $\alpha_1 = 0$ and $\alpha_2 \neq 0$, since it follows from Table 4.1 that $\rho_k = 0$ for $k$ uneven. In practice this problem may be overcome by replacing $r_{ij}$ with a small number $\varepsilon$, say $\varepsilon = 10^{-6}$ for all $r_{ij}$ for which $|r_{ij}| < 10^{-6}$.

Initial values for the coordinates may be obtained with the aid of the following figure:

![Diagram](image)

From Table 4.2 with $\alpha_1 = -0.1$, $\alpha_2 = -0.75$ it follows that $\rho_1 = 0.399$ and $\rho_2 = 0.790$. Therefore $d_1 = -\ln \rho_1 = 0.90$ and $d_2 = -\ln \rho_2 = 0.24$. The height $h$, which is the value
of the coordinates $x_{i2}, \ i = 1, 2, \ldots, T$ is $0.90 \sin \theta = 0.367$, with \( \theta = \arccos(0.24/0.90) = 74.53^\circ \).

The coordinates $x_{11}, x_{31}, x_{51} \ldots$ for uneven numbered variables are therefore 0, 0.24, 0.48, 0.72, \ldots and the coordinates for even numbered variables $x_{21}, x_{41}, x_{61}, \ldots$ 0.12, 0.36, 0.60, \ldots. As initial values for $\tau_1$ and $\tau_2$ (cf. 4.4) we use $\tau_1 = \tau_2 = 1$. Note that the values of the coordinates of the vertical axis depend on the values of $\alpha_1$ and $\alpha_2$, e.g. as the value of $|\alpha_1|$ becomes smaller the distance between the line with even numbers and the line with uneven numbers becomes larger. For the main type 4 AR(2)-processes the graphical display tends to be one-dimensional as is the case for AR(1)-processes. This type of process may however be distinguished from an AR(1)-process in that the distances between points are not equally spaced and the $\tau_2$-parameter is usually larger than one. An alternative method for the graphical display of AR(1) and AR(2)-processes is given in Figures 4.4 and 4.5 respectively.

![Diagram](image)

**Figure 4.4.** Graphical display of an AR(1)-process.

4.4 Graphical displays of MA(1) and MA(2) processes.

From (4.9) with $m = 0$ we obtain ($n = 1$) the MA(1) process $e_t = u_t + \theta_1 u_{t-1}$ and ($n = 2$) the MA(2) process $e_t = u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2}$. It can be shown (Box and Jenkins,
that the autocorrelations at lag $k$ for these processes are as follows:

$$
\begin{align*}
\rho_1 &= \frac{\alpha_1}{1 + \alpha_2}, \\
\rho_k &= 0, \quad k \geq 2 \\
&\quad -0.5 \leq \rho_1 \leq 0.5 \\
\rho_1 &= \alpha_1(1 - \alpha_1^2)/(1 + \alpha_1^2 + \alpha_2^2) \\
\rho_2 &= \alpha_2/(1 + \alpha_1^2 - \alpha_2) \\
\rho_k &= 0, \quad k \geq 3
\end{align*}
$$

It should be noted from Table 5.1 that for MA(1)-processes $\rho_k = 0$, $k > 1$ and for MA(2)-processes $\rho_k = 0$, $k > 2$. This poses a problem if we want to display these processes graphically by using the inverse relationship that exists between distances and correlations. Consider for example the graphical display of $e_1$, $e_2$, and $e_3$ for a MA(1)-process:
Since \( \rho_2 = 0 \), it is required that the distance between \( e_1 \) and \( e_3 \) should be infinitely large, but since \( \rho_1 \neq 0 \) it is simultaneously required that the distances between \( e_1 \) and \( e_2 \) as well as between \( e_2 \) and \( e_3 \) should be relatively small. It is obvious that it would be difficult to make a graphical representation of MA-processes in one or two dimensional space.

It was, however, shown by du Preez (1989) that the inverse of the correlation matrix of a MA-process may be approximated by

\[
P^{-1} \approx D_\phi \Omega D_\phi,
\]

where \( D_\phi \) is a diagonal matrix whose diagonal elements \( \phi_{ii} \) have the property

\[
\phi_{ii} = \phi_{T+i-1,T+i-1}; \quad i = 1, 2, \ldots, T/2
\]

The \( T \times T \) matrix \( \Omega \) has the same properties than the correlation matrix of the AR-process given that \( \alpha_i = \beta_i; \ i = 1, 2. \)

For MA-processes it is necessary to modify (4.5) to

\[
\omega_{ij} = \text{sign}(r^{ij}) \exp(-\tau_1 d_{ij} \tau_2),
\]

where \( r^{ij} \) denote the \((i,j)\)-th element of the inverse of the sample correlation matrix and \( \omega_{ij} \) the \((i,j)\)-th element of the matrix \( \Omega \) defined in (4.20).

The vector of unknown parameters \( \gamma \) for the fit of an MA-process is

\[
\gamma = (\sigma, \phi_{11}, \phi_{22}, \ldots, \phi_{T/2,T/2}, x_{11}, x_{12}, \ldots, x_{T2})'
\]

We then test the nullhypothesis

\[
\Sigma = \gamma_0 [P^{-1}(\gamma)]^{-1}
\]

For the MA(1)-process, the graphical representation of the inverse correlation matrix is one dimensional being equally spaced points on a straight line.

For the MA(2)-process to be invertible, the set of allowable \( \beta_1 \) and \( \beta_2 \)-values correspond with the set of \( \alpha_1 \) and \( \alpha_2 \) values shown in Figure 4.2 and hence we also distinguish between 4 main types of MA-processes.

**Example.** (Main type 1)

Let \( \theta_1 = -0.1, \theta_2 = -0.75 \) and \( T = 10. \) The fitted coordinates, \( \psi_{1t} \)-values and a graphical display of the data is given below.
$F(\hat{\gamma} = 0.0048)$

<table>
<thead>
<tr>
<th>Variable</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>coordinates</td>
<td>0.0</td>
<td>0.36</td>
<td>0.53</td>
<td>0.90</td>
<td>0.96</td>
<td>1.33</td>
<td>1.39</td>
<td>1.75</td>
<td>1.92</td>
<td>2.2</td>
</tr>
<tr>
<td>ln($x_2 + 1$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>coordinates</td>
<td>2.2</td>
<td>0.0</td>
<td>2.2</td>
<td>0.0</td>
<td>2.2</td>
<td>0.0</td>
<td>2.2</td>
<td>0.0</td>
<td>2.2</td>
<td>0.0</td>
</tr>
<tr>
<td>$\hat{\psi}$-values</td>
<td>0.81</td>
<td>0.81</td>
<td>0.66</td>
<td>0.66</td>
<td>0.63</td>
<td>0.63</td>
<td>0.66</td>
<td>0.66</td>
<td>0.810</td>
<td>0.81</td>
</tr>
</tbody>
</table>

Graphical display of MA(2)-process.

4.5. Graphical display of ARMA(1,1) processes.

From (4.9) with $m = n = 1$ we obtain the ARMA(1,1)-process

$$e_t + \alpha_1 e_{t-1} = u_t + \theta_1 u_{t-1}.$$  \hspace{1cm} (4.25)

The autocorrelation at lag $k$ for process (4.25) is given by

$$\rho_k = (1 - \alpha_1 \beta_1)(\beta_1 - \alpha_1)/(1 + \beta_1^2 - 2\alpha_1 \beta_1)$$

$$\rho_k = (-\alpha_1)^{k-1} \rho_1, \quad k \geq 2$$  \hspace{1cm} (4.26)

Note that higher order autocorrelations may be expressed in terms of $\alpha_1$ and $\rho_1$. Applying the transformation function (4.5) with $\tau_1 = \tau_2 = 1$ we obtain

$$d_k = -(k - 1) \ln |\alpha_1| - \ln |\rho_1|, \quad k \geq 2$$  \hspace{1cm} (4.27)
Hence, points representing the variables $e_2, e_3, e_4, \ldots$ are equally spaced, the distance between the points $e_{k+1}$ and $e_k$, $k \geq 2$ being $-\ln |\alpha_1|$. However, the distance between any two points $e_{k+1}$ and $e_k$ is a measure of the correlation $\rho_1$. But from (4.5) for $\tau_1 = \tau_2 = 1$ and (4.26) it follows that $d_1 = -\ln |\rho_1|$ which is not equal to $-\ln |\alpha_1|$ unless $\beta_1 = 0$ in which case the process is AR(1). Suppose that we set $\tau_2 = 1$ and regard $\tau_1$ as a free parameter to be estimated, then

$$d_1 = \tau_1^* (-\ln |\rho_1|),$$

with $\tau_1^* = 1/\tau_1$. It will always be possible to find a value $\tau_1$ such that

$$d_1 = \tau_1^* (-\ln |\alpha_1|)$$

For the graphical display of the points representing the variables at lags 2, 3, \ldots, $k$ we therefore use the Euclidean distance measure, whereas variables at lag 1 are represented by an adjusted Euclidean distance $d_1^* = \tau_1 d_1$.

The correlation structure of the ARMA(1, 1)-process is therefore approximated by

$$\rho_k = \exp(-d_k), \quad k \geq 2$$

$$= \exp(\tau_1 d_k), \quad k = 1$$

Initial values for the parameters are obtained by setting the coordinates $x_{11}, x_{21}, x_{31}, \ldots, x_{T1}$ equal to 0, $-\ln \alpha, -2\ln \alpha, \ldots, -(T-1)\ln \alpha$ respectively, with $\alpha = |r_{31}/r_{21}|$ and by taking $\tau_1 = 1$. A graphical display of the correlation structure of an ARMA(1, 1)-process with $\alpha_1 = 0.3$ and $\beta_1 = -0.18$ is given in Figure 4.6.

It is evident from Figure 4.6 that the points representing lags 2, 3, 4, \ldots are equally spaced and fall on a straight line. The location of the point $A$ representing a lag of 1 deviates from the general pattern. The size of this deviation depends upon the influence of the MA parameter $\beta_1$. If this parameter is small, point $A$ will tend to fall on the same line than the points $B, C, D, \ldots$ and hence the process will be identified as an AR(1)-process.


The results in Sections 4.3 to 4.5 were obtained by computing the population correlation matrices according to expressions (4.13), (4.16) and (4.26) for autocorrelations. In all cases it was found that the correlation structures could be exactly (or very closely) represented by some configuration of points in one or two dimensional space.

ARMA$(n, m)$ processes for $n \leq 1$ and $m \leq 1$ were simulated using different $\alpha$ and $\beta$ values. Each “sample” correlation matrix was based on a data matrix of $N = 200$
rows and $T = 10$ columns. Each row being a set of simulated time series deviations $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{10}$. For the AR, MA and ARMA(1,1)-processes discussed in the previous 3 sections it was found that acceptable fits were obtained throughout and it was possible to identify the various processes from their respective graphical representations. As an example, Table 4.2 contains results for 10 simulated correlation matrices for an AR(2)-process with $\alpha_1 = -0.1$ and $\alpha_2 = -0.75$. As an illustration the coordinates obtained by fitting a model to the data of correlation matrix number 7, is given in Figure 4.7.

**Table 4.2**

Results obtained from 10 simulated AR(2) correlation matrices.

$N = 200, T = 10$

<table>
<thead>
<tr>
<th>Correlation matrix number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{r}_2$</td>
<td>1.05</td>
<td>1.03</td>
<td>1.01</td>
<td>0.91</td>
<td>1</td>
<td>0.91</td>
<td>0.97</td>
<td>0.90</td>
<td>1</td>
<td>1.04</td>
</tr>
<tr>
<td>$\chi^2$-statistic</td>
<td>37.4</td>
<td>46.5</td>
<td>32.4</td>
<td>33.2</td>
<td>26.9</td>
<td>40.3</td>
<td>33.0</td>
<td>30.8</td>
<td>16.0</td>
<td>26.9</td>
</tr>
<tr>
<td>$P$-value</td>
<td>0.71</td>
<td>0.33</td>
<td>0.88</td>
<td>0.85</td>
<td>0.97</td>
<td>0.58</td>
<td>0.86</td>
<td>0.91</td>
<td>1.00</td>
<td>0.97</td>
</tr>
</tbody>
</table>
4.7. Concluding remarks.

In this section repeated time series correlation structures were displayed graphically by using normal maximum likelihood. As in non-metric scaling we assume an inverse relationship between correlations and distances. The methods discussed were first tried out on the theoretical values of the correlation matrices of ARMA-processes up to order 2. The results obtained were then substantiated by means of simulation studies.

A great deal of work can still be done in this field, for example the graphical display of non-stationary processes such as the process

$$\rho_{ij} = \rho^{\mid\phi_i - \phi_j\mid}$$

for which $\rho_{ij} = \exp(-\tau_1 d_{ij}^2), \tau_1 = \tau_2 = 1$

$$d_{ij} = \mid\phi_i - \phi_j\mid \ln \mid\rho\mid$$

(4.29) (4.30)

In situations where time intervals between successive observations may vary or where data is missing, the time series process may be specified in terms of (4.29) and (4.30). For an AR(1) model the covariance between residuals which are $k$ time units apart is given by $\sigma_u^2 \exp(-\tau_1 k \tau_2)$. The case $\tau_2 = 1$ corresponds to a continuous time AR(1) process, where $k \geq 0$. Higher order processes may be fitted by assuming that

$$\rho_k = \sigma^2 \exp(-\tau_1 (k_1 + \cdots + k_p) \tau_2).$$

Unknown parameters are $\sigma^2, \tau_1, k_1, \cdots, k_p, \tau_2$. It is hoped that this contribution will stimulate further research in this field and will bring forward more efficient and elucidating graphical displays of repeated measurement time series correlation structures.
5. ARMA(1,1) model with concomitant variable.

Suppose that each of a sample of $N$ subjects is measured $T$ times at regular intervals on, for example a learning task yielding measurements $y_1, y_2, \ldots, y_T$ and that an additional measurement $c$ on, for example, an ability measure is also made on each subject. In this section it will be shown how to relate the repeated measurement characteristics $y_1, y_2, \ldots, y_T$ to the concomitant variable $c$.

The model to be considered is the ARMA(1,1) model with arbitrary initial state variance (Part I, Section 5) extended to cope with a concomitant variable and imposing a trend on white noise variances. The methodology developed in this section may readily be extended to higher order ARMA models and to the case of several concomitant variables.

Let

$$y = f(\theta) + e,$$

where (cf. Part I, Section 2), $f(\theta)$ is a vector-valued response function and where it is assumed that $e$ is generated by and ARMA(1,1) covariance structure with arbitrary initial state variance, that is (cf. Part I, Section 4, (4.5)),

$$e = T_\alpha^{-1}[T_\beta u + I_{T,1}x_1],$$

with

$$x_1 = \alpha^*_1 e_0 + \beta^* u_0.$$  \hspace{1cm} (5.3)

We do not necessarily require that $\alpha^*_1 = \alpha_1$, or $\beta^*_1 = \beta_1$ and therefore it is assumed that

$$\text{var}(x_1) = \sigma_{xx}.$$ \hspace{1cm} (5.4)

Suppose $u \sim N(0, D_\psi)$, then (cf. Part I, (5.6))

$$\Sigma = \text{cov}(e) = T_\alpha^{-1}[I_{T,1} \sigma_{xx} I_{T,1}^\prime + T_\beta D_\psi T_\beta^\prime] T_\alpha^{-1}. $$ \hspace{1cm} (5.5)

It is further assumed that

$$\psi_{tt} = \gamma_1 \exp\{-\gamma_2 t\}, \quad \gamma_1, \gamma_2 \geq 0,$$ \hspace{1cm} (5.6)

where $\psi_{tt} = [D_\psi]_{t,t}, \quad t = 1, 2, \ldots, T.$

Model (5.1) may be extended to include a concomitant variable as follows. Let

$$E \left( \begin{array}{c} y \\ c \end{array} \right) = \left( \begin{array}{c} f(\theta) \\ \mu_c \end{array} \right),$$ \hspace{1cm} (5.7)
where it is assumed that \( c \) is independent of \( u \) and
\[
\begin{bmatrix}
  x_1 \\
  c
\end{bmatrix}
\sim N
\begin{bmatrix}
  \left( \begin{array}{cc}
  0 & \sigma_{xc} \\
  \mu_c & \sigma_{cc}
  \end{array} \right)
\end{bmatrix}.
\] (5.8)

From (5.2) and (5.8) it follows that
\[
\text{cov}(y, c) = \text{cov}(e, c) = T^{-1}_{\alpha} I_{T, 1} \sigma_{xc}.
\] (5.9)

Let \( \sigma_{yc} = \text{cov}(y, c) \) and \( t^* = T^{-1}_{\alpha} I_{T, 1} \) then the covariance matrix \( \Sigma^* \) of the \((T+1) \times 1\) vector \( \begin{bmatrix} y \\ c \end{bmatrix} \) is given by
\[
\Sigma^* = \begin{bmatrix}
  \Sigma & \sigma_{yc} \\
  \sigma'_{yc} & \sigma_{cc}
\end{bmatrix}.
\]

Note that \( \sigma_{xc} = \text{cov}(x_1, c) \) may alternatively be expressed as
\[
\sigma_{xc} = \sigma_x \sigma_c \rho_{xc}.
\] (5.10)

\( \Sigma^* \) has eight unknown parameters, these being
\[
\gamma' = (\mu_c, \alpha_1, \beta_1, \sigma_x, \sigma_c, \rho_{xc}, \gamma_1, \gamma_2).
\] (5.11)

In summary, results in this section were obtained by assuming that there exists a relationship between the concomitant variable and the initial state variable, which in the case of a stationary ARMA model, contains information about the past behavior of the process. A significant correlation between the state variable and concomitant variable could indicate that repeated measurements on a characteristic may be influenced by values of the concomitant variable.

6. Vector-variate repeated measurement models.


Suppose that \( N \) repeated measurements are made on each of the \( p \times 1 \) vectors \( y_1, y_2, \ldots, y_T \). Let
\[
y^* = \xi^* + e^*,
\] (6.1)

where
\[
y^* : (pT \times 1) = \begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_T
\end{bmatrix},
\] (6.2)
\[ E(y^*) = \xi^* \]  
(6.3)

and

\[ e^*: (pT \times 1) = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_T \end{pmatrix}. \]  
(6.4)

It is further assumed that the error variates \( e_i, \ i = 1, 2, \ldots, T \) are generated by a vector ARMA\((n,m)\) process defined by

\[
\sum_{i=0}^{n} A_i e_{k-i} = \sum_{j=0}^{m} B_j u_{k-j}, \quad k = \ldots, -1, 0, 1, \ldots,
\]  
(6.5)

where

\[ A_0 = B_0 = I_p. \]  
(6.6)

If \( \ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots \) are independent \( N(0, \Psi) \) variates, then

\[ \text{cov}(u^*) = I_T \otimes \Psi, \]  
(6.7)

where \( \Psi \) is a \( p \times p \) covariance matrix.

**Example.** Suppose the vector process is ARMA\((1,1)\), \( p = 2 \) and that \( T = 4 \). Then

\[
A_1 = \begin{pmatrix} \alpha_{11,1} & \alpha_{12,1} \\ \alpha_{21,1} & \alpha_{22,1} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \beta_{11,1} & \beta_{12,1} \\ \beta_{21,1} & \beta_{22,1} \end{pmatrix}.
\]

Commencing at time \( t = 1 \) it follows from (6.5) that (cf. Part I, Section 4)

\[
e_1 + A_1 e_1 + e_2 & = u_1 + (B_1 u_0 - A_1 e_0) \\
A_1 e_2 + e_3 & = B_1 u_1 + u_2 \\
A_1 e_3 + e_4 & = B_1 u_2 + u_3
\]

Let

\[ x_{11} = B_1 u_0 = A_1 e_0 \]  
(6.8)

then (cf. Part I, (4.6) to (4.8))

\[ e_1 = u_1 + x_{11} \]

and similarly

\[ e_0 = u_0 + x_{10}, \]  
(6.9)
with $x_{10} = B_1 u_{-1} - A_1 e_{-1}$.

From (6.8) and (6.9) it follows that

$$x_{11} = -A_1 x_{10} + (B_1 - A_1) u_0$$  \hspace{1cm} (6.10)

where $u_0$ and $x_{10}$ are independent.

Let $\text{cov}(x_{11}) = P_{11} : (2 \times 2)$, then for a vector stationary ARMA(1,1) process and using (6.7) and (6.10) it follows that

$$P_{11} = -A_1 P_{11} A_1 + (B_1 - A_1) \Psi (B_1 - A_1)'$$  \hspace{1cm} (6.11)

Suppose further that

$$T_A = \begin{pmatrix} I & 0 & 0 & 0 \\ A_1 & I & 0 & 0 \\ 0 & A_1 & I & 0 \\ 0 & 0 & 0 & A_1 \end{pmatrix}$$  \hspace{1cm} (6.12)

and

$$T_B = \begin{pmatrix} I & 0 & 0 & 0 \\ B_1 & I & 0 & 0 \\ 0 & B_1 & I & 0 \\ 0 & 0 & 0 & B_1 \end{pmatrix}$$  \hspace{1cm} (6.13)

then

$$T_A e^* = T_B u^* + I_{6,2} x_{11}.$$  \hspace{1cm} (6.14)

From (6.11) and (6.7) with $\text{cov}(e^*) = \Sigma^*$,

$$\Sigma^* = T_A^{-1} [I_{8,2} P_{11} I_{8,2} + T_B (I \otimes \Psi) T_B'] T_A^{-1}$$  \hspace{1cm} (6.15)

The results for univariate ARMA($n, m$) models can be extended directly to accommodate vector ARMA($n, m$) models. For example, let $s = \max(n, m)$ where

$$A_{n+1} = A_{n+2} = \cdots = A_m = 0, \hspace{1cm} n < m$$

and

$$B_{m+1} = B_{m+2} = \cdots = B_n = 0, \hspace{1cm} n > m$$

Define the $pT \times pT$ matrix $T_A$ as follows,

$$T_A = \begin{pmatrix} I & 0 & 0 & 0 & \ldots \\ A_1 & I & 0 & 0 & \ldots \\ A_2 & A_1 & I & 0 & \ldots \\ A_3 & A_2 & A_1 & I & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$  \hspace{1cm} (6.16)
and let $T_B$ be similarly defined as (6.16) with the $A_i$ replaced by $B_i, \ i = 1, 2, \ldots, s$. Let

$$
I : pT \times ps \text{ denote the first } ps \text{ columns of the identity matrix of order } pT \text{ and let}
$$

$$
x_k^* = \begin{pmatrix}
x_{1k}^*
\vdots
x_{sk}^*
\end{pmatrix}
$$

$k = \ldots, -1, 0, 1, 2, \ldots$

(6.17)

Then, following the procedure outlined in Part I, Section 4

$$
e^* = T_A^{-1} [I x_1^* + T_B u^*]
$$

(6.18)

The state-space equations are given by (cf. Part I, 4.22)

$$
x_{k+1}^* = \Phi^* x_k^* + B^* u_0,
$$

(6.19)

where

$$
B^* = \begin{pmatrix}
B_1 - A_1 \\
B_2 - A_2 \\
\vdots \\
B_s - A_s
\end{pmatrix}
$$

(6.20)

and (cf. Part I, (4.20))

$$
\Phi^* : (ps \times ps) = \begin{pmatrix}
-A_1 & I & 0 & \ldots & 0 \\
-A_2 & 0 & I & \ldots & 0 \\
\vdots \\
-A_s & 0 & 0 & \ldots & 0
\end{pmatrix}
$$

(6.21)

Let

$$
cov(x_k^*) = P^* = \begin{pmatrix}
P_{11} & P_{12} & \ldots & P_{1s} \\
P_{21} & P_{22} & \ldots & P_{2s} \\
\vdots \\
P_{s1} & P_{s2} & \ldots & P_{ss}
\end{pmatrix}
$$

(6.22)

then if $\Sigma^* = cov(e^*)$,

$$
\Sigma^* = T_A^{-1} [I P^* \Gamma' + T_B (I \otimes \Psi) T_B'] T_A^{-1}
$$

(6.23)

where the elements of $P^*$ may be calculated according to the method described in Section 4 of Part I.

For a stationary bivariate ARMA(1,1) process the unknown parameters in (6.23) are

$$
\gamma' = (\alpha_{111}, \alpha_{121}, \alpha_{211}, \alpha_{221}, \beta_{111}, \beta_{121}, \beta_{211}, \beta_{221}, \psi_{11}, \psi_{12}, \psi_{22})
$$
where $\alpha_{i,j}, \beta_{i,j}, i \neq j$ denotes crosscorrelations at lag $1$. The number of parameters may be decreased if it can be assumed that $\alpha_{121} = \alpha_{211}$ and similarly $\beta_{121} = \beta_{211}$.

Although it is not difficult to program the covariance structure (6.20), it may in general be difficult to find initial estimates for the unknown parameters. In applications where $p > T$, the task of parameter identification may pose a serious problem. Furthermore, it may not be realistic to assume that the correlation structure between the $p$ variables is generated by an ARMA-type stochastic process. In many practical applications it may be more realistic to assume that $\text{cov}(\mathbf{y}_i) = \Sigma_M, \ i = 1, 2, \ldots, T$, the $p \times p$ covariance matrix of the observed scores and that $\text{cov}(\mathbf{\tilde{y}}_j) = \Sigma_T$, where $\Sigma_T$ is the $T \times T$ covariance structure of an ARMA process with

$$
\mathbf{\tilde{y}}'_j = ([y_{1j}], [y_{2j}], \ldots, [y_{Tj}]), \quad j = 1, 2, \ldots, p.
$$

This assumption gives rise to the multimethod-multitrait ARMA model which will be discussed in the next section.

Different vector ARMA models follow from the definition of the stationary model according to the methods described in Part I, Section 5. For example $P^* = 0$ implies that the stochastic process started at the commencement of measurements.


Browne (1984) proposed a covariance structure which is convenient for the analysis of multi-method multi-trait experiments. This structure is the direct product decomposition

$$
\Sigma = \Sigma_T \otimes \Sigma_M + D_\eta,
$$

(6.24)

where $\Sigma$ is the $T_p \times T_p$ covariance matrix of observed scores, $\Sigma_T$ is a $T \times T$ time covariance matrix (cf. Part I, Sections 4 and 5), $\Sigma_M$ is a $p \times p$ method covariance matrix and $D_\eta$ is a diagonal error covariance matrix. Relationships between times and relationships between variables are then separated into two covariance or correlation matrices which may be inspected separately.

Let

$$
\mathbf{y}^* : T_p \times 1 = (y'_1, y'_2, \ldots, y'_p),
$$

(6.25)

where $y'_j : T \times 1 = (y_{1j}, y_{2j}, \ldots, y_{Tj})$, are $T$ repeated measurements on variable $j, \ j = 1, 2, \ldots, p$.

Suppose that the response pattern over time for variable $j$ may be described by the following regression model

$$
\mathbf{y}_j = f(\theta_j) + \mathbf{e}_j,
$$

(6.26)

30
where \( \text{cov}(e_j) = \Sigma_T \) and \( \Sigma_T : T \times T \) has one of the four time series covariance structures given in Part I, Section 5.

If the response function is linear in the parameters, then

\[
f(\theta_j) = X\theta_j.
\]

Suppose that each of the \( p \) variables have the same type of linear response function over time, for example, \( E(y_{ij}) = \theta_{1j} + \theta_{2j}t_i, \quad j = 1, 2, \ldots, p \) and \( i = 1, 2, \ldots, T \). In this case

\[
E(y^*) = X^*\theta^*,
\]

where

\[
X^* = I_p \otimes X,
\]

\[
X = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_p \end{pmatrix}
\]

and

\[
\theta^* = (\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22}, \ldots, \theta_{1p}, \theta_{2p})'.
\]

Under the assumption that \( y^* \) has a \( N(E(y^*), \Sigma) \) distribution, with the \( T_p \times T_p \) covariance matrix \( \Sigma \) defined by (6.24), maximum likelihood estimates of the unknown parameters, standard errors and goodness-of-fit statistic may be obtained as outlined in Part I, Sections 2 and 3. Furthermore, random samples may be drawn from \( g \) subpopulations each having a population mean of, for example, \( X^* \theta^*_\ell, \quad \ell = 1, 2, \ldots, g \). In general the elements of the \( pT \times pT \) matrix \( \Sigma \) can be computed as follows. Let \( [\Sigma]_{ik,j\ell} = \text{cov}(y_{ik}, y_{j\ell}) \), where \( y_{ik} \) denotes variable \( i, \ i = 1, 2, \ldots, p \) with respect to time \( t_k, \ k = 1, 2, \ldots, T \).

Then

\[
[\Sigma]_{ik,j\ell} = [\Sigma_M]_{i,j}[\Sigma_T]_{k,\ell} + \delta_{ij}\delta_{k\ell}[D_\eta]_{i,k}
\]

\[(6.29)\]

**Example.** Suppose \( p = 4 \) and \( T = 5 \). Suppose further that \( \Sigma_T \) has the following ARMA(1,1) structure (cf. Part I, Model 2, (5.4) and (5.5)).

\[
\Sigma_T = T^{-1}_\alpha(T_\beta D_\psi T_\beta')T^{-1}_\alpha,
\]

where

\[
[D_\psi]_{i,i} = \lambda_1 + \lambda_2 \lambda_3^{t_i}, \quad i = 1, 2, \ldots, T.
\]
In this case \( \Sigma \) (cf. (6.24)) is a 20 \( \times \) 20 matrix with unknown parameters

\[
\gamma' : 40 \times 1 = ([\Sigma_M]_{1,1}, [\Sigma_M]_{1,2}, \ldots, [\Sigma_M]_{4,4}), \\
\alpha_1, \beta_1, \lambda_1, \lambda_2, \lambda_3, \eta_{11}, \eta_{22}, \eta_{33}, \ldots, \eta_{20,20}.
\]

For \( p \) and \( T \) general the number of unknown parameters for the example above is \( \frac{1}{2}p(p + 1) + 5 + T \).

7. Applications.

Two examples will be considered to illustrate some of the features of the methods described in the preceding sections. All results to be presented were computed using AUFIT (du Toit and Browne, 1982 and 1989) without providing subroutines to evaluate derivatives. In all cases, normal theory maximum likelihood results are reported.


The model considered is an ARMA(1,1) model with one concomitant variable (cf. (5.7) to (5.11)) with a trend imposed on the white noise variances. A data set of Kanfer and Ackerman (1989, Experiment 1) is used in the analysis. Each of \( N = 137 \) subjects carried out a computerised Air Traffic Controller task. A continuous queue of planes of various sizes had to be brought in to land using three landing strips of different lengths. The task was to bring them in safely. Subjects carried out the task continuously for 60 minutes and the measurement taken was the number of correct landings per 10 minutes, thereby yielding \( T = 6 \) scores. The ASVAB test (Wilfgong, 1980) was also administered to each subject yielding a concomitant ability measure (c).

The sample means, standard deviations and correlation matrix for the Kanfer-Ackerman learning data are shown in Table 7.1.
Both the trial means and standard deviations show clear monotonic increasing trends while the trial correlation matrix has a pattern of decreasing elements usually associated with a Guttman Simplex. As pointed out by Rogosa and Willett (1985), however, the usual Simplex model is not the only model that results in a correlation pattern of this type. It is noticeable, also, that the ASVAB ability measure correlates most highly with the first trial and the correlations show a tendency to decrease as trial number increases.

We shall assume that the learning trial means $\xi_t, \ t = 1, \ldots, T$, lie on a smooth monotonic increasing curve tending to an asymptote. The deviations of a person's trial scores from the mean curve are assumed to follow a first order autoregressive time series with first order moving average residuals, that is an ARMA(1,1) process. The deviation from the mean on trial $t$ then is related to the deviations from the mean on the previous trials and the change in performance on trial $t$ is related to the change in performance on the previous trial. Different persons' deviations from the mean are regarded as different realisations of the same time series. We then have a repeated time series model instead of the usual single time series model.

The stationarity assumptions usually made to estimate the process covariance matrix from a single realisation are therefore not essential here, although some restrictions on parameters will be imposed for model parsimony.
We shall employ the Gompertz curve for the learning trial means:

\[ \xi_t = \theta_1 \exp \left[ - \theta_2 \exp\{-(t-1)\theta_3\} \right], \quad t = 1, \ldots, T. \tag{7.1} \]

Other members of the Richards (1959) family such as the exponential and logistic curves would also be suitable. The parameters are interpretable as follows: \( \theta_1 \) \((\theta_1 \geq 0)\) is the asymptote, \( \theta_2 \) \((\theta_2 \geq 0)\) governs the distance from the asymptote on the first trial and \( \theta_3 \) \((\theta_3 \geq 0)\) governs the rate at which the asymptote is approached.

Autoregressive weights will be assumed to have the same value, \( \alpha \) from one trial to another and moving average weights the same value, \( \beta \). The data model for observations after the first is

\[ y_t - \xi_t = \alpha(y_{t-1} - \mu_{t-1}) + u_t + \beta u_{t-1}, \quad t = 2, \ldots, T, \]

where the \( u_t, \ t = 1, \ldots, T, \) are disturbances, distributed independently with \( E(u_t) = 0 \) and \( \text{var}(u_t) = \psi_{tt}, \ t = 1, \ldots, T, \) and the means \( \xi_t \) follow the trend in (7.1). Because no observations have been made prior to \( y_1 \), the initial state variable \( x_1 \) (cf. (5.3)) is introduced to replace observations and disturbances before the first trial.

To allow for relationships between the concomitant ability variable, \( c \), and performance on the learning task it is assumed that \( x_1 \) is correlated with \( c \). Let (cf. (5.10)) \( E(c) = \mu_c \), and \( \text{corr}(x_1, c) = \rho_{x_1,c} \) with

\[ -1 \leq \rho_{x_1,c} \leq 1. \tag{7.2} \]

Since few changes in performance occur after the learning task has been mastered we assume that \( \psi_{tt} \) follow a decreasing exponential trend

\[ \psi_{tt} = \gamma_1 \exp \{ -(t-1)\gamma_2 \}, \quad t = 1, 2, \ldots, T; \gamma_1, \gamma_2 \geq 0. \tag{7.3} \]

**Table 7.2:**

Time series model: Parameter estimates.

<table>
<thead>
<tr>
<th></th>
<th>( \theta_1 )</th>
<th>( \theta_2 )</th>
<th>( \theta_3 )</th>
<th>( \mu_c )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
<td>34.83</td>
<td>1.07</td>
<td>0.77</td>
<td>0.76</td>
<td>0.96</td>
<td>-0.12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( \sigma_{x_1} )</th>
<th>( \sigma_c )</th>
<th>( \rho_{x_1,c} )</th>
<th>( \gamma_1 )</th>
<th>( \gamma_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
<td>3.76</td>
<td>5.62</td>
<td>1.00</td>
<td>40.42</td>
<td>0.23</td>
</tr>
</tbody>
</table>

The estimate of the autoregressive weight \( \alpha \) is close to one indicating a substantial influence of one trial on the next while the estimate of the moving average weight is negative but near
zero. This indicates a small negative effect of the disturbance from trial \( t - 1 \) on trial \( t \). The estimate of the correlation \( \rho_{t_1,c} \) between initial state and the ASVAB concomitant variable lies on the upper bound of one in (7.3). This is in keeping with the higher correlations of ASVAB with initial trials than later trials evident in Table 7.1.

The number of parameters for the fitted model is 11 and the \( \chi^2 \) goodness-of-fit statistic yielded a value of 9.76 with degrees of freedom equal to 24.

**Example 2. The relation between television viewing and personality needs.**

In South Africa, the last Western industrialized country to introduce television, the popularity of this mass medium is very high. We were very fortunate in South Africa because the research possibilities were realized and exploited even before the introduction of television in 1976. A series of research projects on television were launched by the Institute for Communication Research of the Human Science Research Council. The largest of these projects, on the influence of television on school-going young people, constituted an investigation on a wide range of relevant matters.

The results which are discussed in this section are concerned with an important aspect of the popularity of television, namely the relation between certain personality needs in adolescents and their viewing behaviour (Van Vuuren et al, 1982).

Data used in this investigation were collected from 1974 to 1978. Pupils in provincial schools were involved on a country-wide basis. Questionnaires and psychometric tests were completed under the supervision of teachers trained by school psychologists. It was possible to use the data of approximately 2,000 adolescents.

A Biographical Questionnaire, a Television Questionnaire and the Group Personality Projection Test (GPPT) developed by Cassell and Kahn (1961), were used to collect the data. The Group Personality Projection Test gives an indication of the degree of anxiety-producing tension present in the individual, as well as the degree of activeness of certain psychological need states. It consists of 90 multiple-choice items, requiring responses to stick-figure drawings portraying a widely diversified range of usual life activities, each with a minimal amount of situational structuring. In the subsequent analyses the following fields were considered:

1. **Anxiety-producing tension.** This field is concerned with obtaining an index of the anxiety-producing tension present in the individual.
2. **Nurturance.** This score measures the individual's inclination to give aid to others and to provide initiative in leadership or guidance.
3. **Affiliation and psychosexual needs.** The score is indicative of the need for group membership and for psychosexual contacts.
4. Succorance. The results indicate the subjects’ need to seek aid and, in general to play an infant role.

These fields are all indicators of development. A plot of the means over time of each variable (field) exhibits a more or less linear trend. This fact was incorporated in the linear model (cf. Section 6.2, (6.25 to 6.29))

\[
y^* = X^* \theta^* + e^*,
\]

where

\[
X^* = I_4 \otimes X,
\]

\[
X = \begin{pmatrix} 1 & t_1 \\
1 & t_2 \\
1 & t_3 \\
1 & t_4 \\
1 & t_5 
\end{pmatrix}
\]

and

\[
\theta^* = (\theta_{T0}, \theta_{T1}, \theta_{N0}, \theta_{N1}, \theta_{A0}, \theta_{A1}, \theta_{S0}, \theta_{S1})'
\]

\(\theta_{T1}\) denotes, for example, the slope with regard to the Tension variable. The sample means and sample covariance matrix are shown as part of the computer output given in Section 6.3.

Further, let \(\xi_L\) and \(\xi_H\) respectively denote the mean vectors for the light and heavy viewers, where

\[
\xi_L = X^* \theta^*_L
\]

and

\[
\xi_H = X^* \theta^*_H.
\]

We would like to test the following null hypothesis (cf. (6.24)):

\[H_0 : \Sigma = \Sigma_M \otimes \Sigma_T + D_\eta, \quad \xi_L = \xi_H\]

against

\[H_a : \Sigma = \Sigma_M \otimes \Sigma_T + D_\eta, \quad \xi_L \neq \xi_H.
\]
Using the computer program AUFIT, the following set of hypotheses may be tested.

\[ H_{01} : \Sigma = \Sigma_M \otimes \Sigma_T + D_\eta \]
\[ \xi_L = X^* \theta_L^* \]
\[ \xi_H = X^* \theta_H^* \]

against \( H_a \) : the elements of \( \Sigma \) and the elements of \( \xi_L \) and \( \xi_H \) are unrestricted.

\[ H_{02} : \Sigma = \Sigma_M \otimes \Sigma_T + D_\eta \]
\[ X^* \theta_L^* = X^* \theta_H^* \quad (\xi_L \neq \xi_H) \]

against \( H_a \) : the elements of \( \Sigma \) and the elements of \( \xi_L \) and \( \xi_H \) are unrestricted.

In AUFIT the test statistic for testing whether \( H_{01} \) is true is \( \chi_1^2 \) with \( \{20(21)/2 + 2(20)\} - \{4(5)/2 + 7 + 20 + 2(8)\} = 197 \) degrees of freedom. These degrees of freedom are obtained by noting that \( \Sigma \) is a 20 \( \times \) 20 matrix and \( \xi_L \) and \( \xi_H \) each have 20 elements.

The total number of elements in the unrestricted model are therefore 20(21)/2 + 2(20). When the model is restricted, \( \Sigma_M \) has 4(5)/2 unknown parameters, \( \Sigma_T \) has 7 (the AR(1), MA(1) and five white noise variances), \( D_\eta \) has 20 parameters and \( \xi_L \) and \( \xi_H \) both have 8 (the \( \theta_0 \) and \( \theta_1 \) parameters for each variable).

To test \( H_{02} \), the test statistic is \( \chi_2^2 \) with 205 degrees of freedom, since \( \xi_L = \xi_H = X^* \theta^* \) has 8 unknown parameters.

The test statistic for testing \( H_0 \) is therefore \( \chi^2 = \chi_2^2 - \chi_1^2 \) with 205 - 197 = 8 degrees of freedom.

In order to avoid indeterminancy, \( [D_\psi]_{1,1} \) was fixed at a value of one. Furthermore the elements of \( D_\eta \) were constrained to be larger or equal to zero. At the optimal solution, parameter 51 was on the boundary. The total number of active parameters were thus reduced by 2 and therefore the degrees of freedom for \( \chi_1^2 \) and \( \chi_2^2 \) increased by 2.

From the computer output given below it follows that the \( \chi^2 \)-value for testing \( H_0 \) is 218.4 - 208.6 = 9.8. The probability of exceedance (df = 8) is 0.20.

From the above analyses is can be concluded that the general trend over time with regard to the personality variables is similar for both light and heavy viewers. It would therefore appear that the number of hours that television is watched per week does not affect the personality traits in question.

To summarise, the television data was used to show that in repeated measurement designs, use can be made of the structure of both the means and covariance matrices.
Typical graphical display of group means.

Computer Output.
### Sample Covariance Matrix (Within Groups)

<table>
<thead>
<tr>
<th></th>
<th>TENS_74</th>
<th>TENS_75</th>
<th>TENS_76</th>
<th>TENS_77</th>
<th>TENS_78</th>
<th>NURT_74</th>
<th>NURT_75</th>
<th>NURT_76</th>
<th>NURT_77</th>
<th>NURT_78</th>
</tr>
</thead>
<tbody>
<tr>
<td>TENS_74</td>
<td>126.538</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TENS_75</td>
<td>66.994</td>
<td>152.313</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TENS_76</td>
<td>48.282</td>
<td>88.699</td>
<td>150.934</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TENS_77</td>
<td>48.442</td>
<td>82.120</td>
<td>92.471</td>
<td>169.079</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TENS_78</td>
<td>45.715</td>
<td>75.868</td>
<td>87.522</td>
<td>112.162</td>
<td>181.918</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NURT_74</td>
<td>0.345</td>
<td>1.061</td>
<td>-4.400</td>
<td>-2.845</td>
<td>-3.402</td>
<td>78.702</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NURT_75</td>
<td>-1.709</td>
<td>4.652</td>
<td>-0.925</td>
<td>4.727</td>
<td>2.633</td>
<td>22.756</td>
<td>81.546</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NURT_77</td>
<td>0.538</td>
<td>1.727</td>
<td>1.407</td>
<td>8.187</td>
<td>2.283</td>
<td>15.732</td>
<td>23.253</td>
<td>28.721</td>
<td>76.724</td>
<td></td>
</tr>
<tr>
<td>SUCC_78</td>
<td>5.060</td>
<td>10.842</td>
<td>8.785</td>
<td>12.275</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AFFI_74</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AFFI_75</td>
<td>19.304</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AFFI_76</td>
<td>8.204</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AFFI_77</td>
<td>6.379</td>
<td>10.687</td>
<td>25.836</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AFFI_78</td>
<td>6.245</td>
<td>8.946</td>
<td>12.875</td>
<td>24.974</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SUCC_75</td>
<td>-0.113</td>
<td>0.635</td>
<td>1.928</td>
<td>0.204</td>
<td>-0.124</td>
<td>21.387</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SUCC_76</td>
<td>-0.224</td>
<td>-0.998</td>
<td>0.528</td>
<td>-0.005</td>
<td>0.298</td>
<td>10.466</td>
<td>19.893</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SUCC_77</td>
<td>-0.853</td>
<td>0.098</td>
<td>0.630</td>
<td>0.121</td>
<td>1.160</td>
<td>8.191</td>
<td>10.473</td>
<td>18.076</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SUCC_78</td>
<td>0.455</td>
<td>0.653</td>
<td>0.807</td>
<td>0.028</td>
<td>1.035</td>
<td>6.488</td>
<td>7.833</td>
<td>8.661</td>
<td>15.572</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.064</td>
<td>1.247</td>
<td>0.817</td>
<td>0.729</td>
<td>1.103</td>
<td>5.156</td>
<td>6.334</td>
<td>6.603</td>
<td>8.132</td>
<td>16.446</td>
</tr>
</tbody>
</table>
### Estimates of Unknown Parameters (H_0)

<table>
<thead>
<tr>
<th>Parameters for means. T:B1_LI = slope, light viewers, Tension</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A:B0_LI</strong></td>
</tr>
<tr>
<td>ESTIM :</td>
</tr>
<tr>
<td>S.E. :</td>
</tr>
</tbody>
</table>

| **S:B0_LI** | **S:B1_LI** | **T:B0_HE** | **T:B1_HE** | **N:B0_HE** | **N:B1_HE** |
| ESTIM : | 13.81753 | -0.66160 | 37.47592 | -1.03756 | 29.30376 | 0.04774 |
| S.E. : | 0.33328 | 0.08755 | 0.81289 | 0.21853 | 0.60841 | 0.17063 |

A: BO_HE = Intercept, heavy viewers, Affiliation.

#### Correlations (of variables)

| **DGAM4.4** | **RHO1.1** | **RHO2.1** | **RHO2.2** | **RHO3.1** | **RHO3.2** | **RHO3.3** | **RHO4.1** | **RHO4.2** |
| ESTIM : | 0.33557 | 1.00000 | 0.03339 | 1.00000 | -0.15438 | -0.44949 | 1.00000 | 0.31027 |
| S.E. : | 0.01477 | 0.0 | 0.03037 | 0.0 | 0.02725 | 0.03446 | 0.0 | 0.02700 |

#### ARMA parameters

| **RHO4.1** | **RHO4.4** | **WHNO11** | **WHNO12** | **WHNO13** | **WHNO14** | **WHNO15** | **AR(1)** | **MA(1)** |
| ESTIM : | -0.00684 | 1.00000 | 122.43897 | 101.17718 | 81.15805 | 74.07666 | 95.84882 | -0.83851 |
| S.E. : | 0.02775 | 0.0 | 10.37338 | 10.15533 | 8.90018 | 8.26456 | 6.55251 | 0.02038 |

0.03855
Error variance

\[ n \]

\[
\begin{array}{cccccccc}
\text{ETA1} & \text{ETA2} & \text{ETA3} & \text{ETA4} & \text{ETA5} & \text{ETA6} & \text{ETA7} & \text{ETA8} \\
0.0 & 11.00611 & 10.29383 & 17.10372 & 23.23341 & 0.00000 & 31.13078 & 31.13078 \\
0.0 & 9.61325 & 8.39476 & 7.61859 & 7.66995 & 0.0 & 3.65966 & 3.65966 \\
0.0 & 29.52382 & 30.29109 & 2.87067 & 5.12465 & 6.18013 & 5.50932 & 4.31152 \\
0.0 & 4.02890 & 4.95567 & 1.53413 & 1.46068 & 1.34799 & 1.29714 & 1.70705 \\
0.0 & 3.36115 & 2.14277 & 0.14418 & 0.89757 & 0.80879 & 1.09219 & \\
\end{array}
\]

\text{NOTE: PARAMETER NO. 51 ON BOUNDARY}

\[ \text{SUMMARY OF ANALYSIS } (H_0) \]

\[ \text{DISCREPANCY FUNCTION (M.L.)} : 0.4584 \]
\[ \text{DISCREPANCY FUNCTION (QUADRATIC FORM)} : 0.4435 \]
\[ \text{NUMBER OF ACTIVE PARAMETERS} : 51 \]
\[ \text{DISCREPANCY FUNCTION WITH AKAIKE PENALTY} : 0.683 \]
\[ \text{DISCREPANCY FUNCTION WITH SCHWARZ PENALTY} : 1.144 \]
\[ \text{CHISQ STATISTIC (LIKELIHOOD RATIO)} : 208.556 \]
\[ \text{CHISQ STATISTIC (QUADRATIC FORM)} : 201.796 \]

\[ \text{DEGREES OF FREEDOM} : 199 \]
\[ \text{SATURATED MODEL} : 1.099 \]
\[ \text{SATURATED MODEL} : 3.363 \]
\[ \text{PROBABILITY OF EXCEEDENCE} : 0.307 \]
\[ \text{PROBABILITY OF EXCEEDENCE} : 0.431 \]

\[ \text{SAMPLE GROUP MEANS} \]

\[ \text{LIGHT} \]
\[ \text{HEAVY} \]
\[ \text{TENS}_74 \quad 37.002 \quad 36.320 \]
\[ \text{TENS}_75 \quad 35.332 \quad 35.869 \]
\[ \text{TENS}_76 \quad 34.114 \quad 33.741 \]
\[ \text{TENS}_77 \quad 32.402 \quad 33.709 \]
\[ \text{TENS}_78 \quad 33.214 \quad 32.218 \]
\[ \text{BUH}_74 \quad 27.375 \quad 29.624 \]
\[ \text{BUH}_75 \quad 27.720 \quad 29.176 \]
NURT_76  28.200  28.847  
NURT_77  29.475  29.929  
NURT_78  29.385  29.471  
AFFI_74  16.335  16.565  
AFFI_75  17.115  16.976  
AFFI_76  17.940  17.749  
AFFI_77  17.965  17.773  
AFFI_78  18.135  18.263  
SUCCE_74  13.332  12.998  
SUCCE_75  12.484  12.329  
SUCCE_76  11.392  11.523  
SUCCE_77  10.988  11.181  
SUCCE_78  10.892  10.491  

(Sample means - continued)

REPRODUCED MEAN MATRIX

<table>
<thead>
<tr>
<th></th>
<th>LIGHT</th>
<th>HEAVY</th>
</tr>
</thead>
<tbody>
<tr>
<td>TENS_74</td>
<td>36.64</td>
<td>36.64</td>
</tr>
<tr>
<td>TENS_75</td>
<td>35.62</td>
<td>35.40</td>
</tr>
<tr>
<td>TENS_76</td>
<td>34.60</td>
<td>34.36</td>
</tr>
<tr>
<td>TENS_77</td>
<td>33.58</td>
<td>33.33</td>
</tr>
<tr>
<td>TENS_78</td>
<td>32.56</td>
<td>32.29</td>
</tr>
<tr>
<td>NURT_74</td>
<td>27.29</td>
<td>29.35</td>
</tr>
<tr>
<td>NURT_75</td>
<td>27.87</td>
<td>29.45</td>
</tr>
<tr>
<td>NURT_76</td>
<td>28.45</td>
<td>29.45</td>
</tr>
<tr>
<td>NURT_77</td>
<td>29.03</td>
<td>29.49</td>
</tr>
<tr>
<td>NURT_78</td>
<td>29.61</td>
<td>29.54</td>
</tr>
<tr>
<td>AFFI_74</td>
<td>16.97</td>
<td>17.02</td>
</tr>
<tr>
<td>AFFI_75</td>
<td>17.42</td>
<td>17.44</td>
</tr>
<tr>
<td>AFFI_76</td>
<td>17.87</td>
<td>17.86</td>
</tr>
<tr>
<td>AFFI_77</td>
<td>18.32</td>
<td>18.28</td>
</tr>
<tr>
<td>AFFI_78</td>
<td>13.16</td>
<td>12.93</td>
</tr>
<tr>
<td>SUCCE_74</td>
<td>12.49</td>
<td>12.32</td>
</tr>
<tr>
<td>SUCCE_75</td>
<td>11.83</td>
<td>11.71</td>
</tr>
<tr>
<td>SUCCE_76</td>
<td>11.17</td>
<td>11.10</td>
</tr>
<tr>
<td>SUCCE_77</td>
<td>10.51</td>
<td>10.49</td>
</tr>
<tr>
<td>SUCCE_78</td>
<td>10.51</td>
<td>10.49</td>
</tr>
</tbody>
</table>

STANDARD DEVIATIONS (PERSONALITY VARIABLES)

<table>
<thead>
<tr>
<th></th>
<th>LIGHT</th>
<th>HEAVY</th>
</tr>
</thead>
<tbody>
<tr>
<td>TENS_74</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>TENS_75</td>
<td>0.622</td>
<td>0.622</td>
</tr>
<tr>
<td>TENS_76</td>
<td>0.607</td>
<td>0.607</td>
</tr>
<tr>
<td>TENS_77</td>
<td>0.630</td>
<td>0.630</td>
</tr>
<tr>
<td>TENS_78</td>
<td>0.651</td>
<td>0.651</td>
</tr>
<tr>
<td>NURT_74</td>
<td>0.081</td>
<td>0.081</td>
</tr>
<tr>
<td>NURT_75</td>
<td>0.153</td>
<td>0.153</td>
</tr>
<tr>
<td>NURT_76</td>
<td>0.254</td>
<td>0.254</td>
</tr>
<tr>
<td>NURT_77</td>
<td>0.455</td>
<td>0.455</td>
</tr>
<tr>
<td>NURT_78</td>
<td>0.222</td>
<td>0.222</td>
</tr>
<tr>
<td>AFFI_74</td>
<td>0.191</td>
<td>0.191</td>
</tr>
<tr>
<td>AFFI_75</td>
<td>0.144</td>
<td>0.144</td>
</tr>
<tr>
<td>AFFI_76</td>
<td>0.521</td>
<td>0.521</td>
</tr>
<tr>
<td>AFFI_77</td>
<td>0.098</td>
<td>0.098</td>
</tr>
<tr>
<td>AFFI_78</td>
<td>0.191</td>
<td>0.191</td>
</tr>
<tr>
<td>SUCCE_74</td>
<td>0.182</td>
<td>0.182</td>
</tr>
<tr>
<td>SUCCE_75</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>SUCCE_76</td>
<td>0.440</td>
<td>0.190</td>
</tr>
<tr>
<td>SUCCE_77</td>
<td>0.182</td>
<td>0.083</td>
</tr>
<tr>
<td>SUCCE_78</td>
<td>0.182</td>
<td>0.100</td>
</tr>
</tbody>
</table>

H_0

STANDARD DEVIATIONS (TIME VARIABLES)

<table>
<thead>
<tr>
<th></th>
<th>LIGHT</th>
<th>HEAVY</th>
</tr>
</thead>
<tbody>
<tr>
<td>TENS_74</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>TENS_75</td>
<td>0.039</td>
<td>0.039</td>
</tr>
<tr>
<td>TENS_76</td>
<td>0.154</td>
<td>0.154</td>
</tr>
<tr>
<td>TENS_77</td>
<td>0.310</td>
<td>0.310</td>
</tr>
</tbody>
</table>

CORRELATIONS (PERSONALITY VARIABLES)

<table>
<thead>
<tr>
<th></th>
<th>TENSION</th>
<th>NURTUR</th>
<th>AFFILI</th>
<th>SUCCOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>TENSION</td>
<td>1.000</td>
<td>-0.000</td>
<td>-0.000</td>
<td>-0.000</td>
</tr>
<tr>
<td>NURTUR</td>
<td>-0.039</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>AFFILI</td>
<td>-0.154</td>
<td>1.000</td>
<td>0.000</td>
<td>-0.000</td>
</tr>
<tr>
<td>SUCCOR</td>
<td>-0.310</td>
<td>-0.305</td>
<td>1.000</td>
<td>-0.000</td>
</tr>
</tbody>
</table>

COVARIANCE MATRIX (TIME VARIABLES)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1974</td>
<td>122.439</td>
<td>134.789</td>
<td>132.566</td>
<td>132.501</td>
<td>157.263</td>
</tr>
<tr>
<td>1975</td>
<td>64.151</td>
<td>81.195</td>
<td>85.629</td>
<td>87.802</td>
<td>87.802</td>
</tr>
<tr>
<td>1976</td>
<td>45.105</td>
<td>68.083</td>
<td>85.629</td>
<td>87.802</td>
<td>157.263</td>
</tr>
<tr>
<td>1977</td>
<td>37.821</td>
<td>57.089</td>
<td>71.801</td>
<td>97.802</td>
<td>157.263</td>
</tr>
</tbody>
</table>

STANDARD DEVIATIONS (TIME VARIABLES)

<table>
<thead>
<tr>
<th></th>
<th>LIGHT</th>
<th>HEAVY</th>
</tr>
</thead>
<tbody>
<tr>
<td>TENS_74</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>TENS_75</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>TENS_76</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>TENS_77</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>TENS_78</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>NURT_74</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>NURT_75</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>NURT_76</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>NURT_77</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>NURT_78</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>AFFI_74</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>AFFI_75</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>AFFI_76</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>AFFI_77</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>AFFI_78</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>SUCCE_74</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>SUCCE_75</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>SUCCE_76</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>SUCCE_77</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>SUCCE_78</td>
<td>0.014</td>
<td>0.014</td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td><strong>STD. DEV.</strong></td>
<td>11.065</td>
<td>11.610</td>
</tr>
</tbody>
</table>

**CORRELATION: (TIME VARIABLES)**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1974</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1975</td>
<td>0.499</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1976</td>
<td>0.422</td>
<td>0.607</td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1977</td>
<td>0.354</td>
<td>0.509</td>
<td>0.646</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>1978</td>
<td>0.273</td>
<td>0.392</td>
<td>0.497</td>
<td>0.608</td>
<td>1.000</td>
</tr>
</tbody>
</table>

**ERROR VARIANCES**

<table>
<thead>
<tr>
<th>ETA1</th>
<th>ETA2</th>
<th>ETA3</th>
<th>ETA4</th>
<th>ETA5</th>
<th>ETA6</th>
<th>ETA7</th>
<th>ETA8</th>
<th>ETA9</th>
<th>ETA10</th>
<th>ETA11</th>
<th>ETA12</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>VARIANCE</strong></td>
<td>11.006</td>
<td>10.294</td>
<td>17.104</td>
<td>23.233</td>
<td>0.000</td>
<td>31.131</td>
<td>31.131</td>
<td>30.917</td>
<td>29.524</td>
<td>30.291</td>
<td>2.871</td>
</tr>
<tr>
<td>ETA13</td>
<td>ETA14</td>
<td>ETA15</td>
<td>ETA16</td>
<td>ETA17</td>
<td>ETA18</td>
<td>ETA19</td>
<td>ETA20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>VARIANCE</strong></td>
<td>6.180</td>
<td>5.509</td>
<td>4.312</td>
<td>5.303</td>
<td>2.925</td>
<td>3.361</td>
<td>2.143</td>
<td>0.144</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Estimates of Unknown Parameters (H₂O₂)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate (Estim)</th>
<th>Standard Error (S.E.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>V:O₂ LI</td>
<td>0.60933</td>
<td>0.055631</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>0.28147</td>
<td>0.12808</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>16.13714</td>
<td>0.24278</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>0.43204</td>
<td>0.06721</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>37.55631</td>
<td>-0.63241</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>0.60933</td>
<td>0.16366</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>28.16402</td>
<td>0.45750</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>0.28147</td>
<td>0.12808</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>16.13714</td>
<td>0.24278</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>0.43204</td>
<td>0.06721</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>37.55631</td>
<td>-0.63241</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>0.60933</td>
<td>0.16366</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>28.16402</td>
<td>0.45750</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>0.28147</td>
<td>0.12808</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>16.13714</td>
<td>0.24278</td>
</tr>
<tr>
<td>V:O₂ LI</td>
<td>0.43204</td>
<td>0.06721</td>
</tr>
<tr>
<td>ETA1</td>
<td>ETA2</td>
<td>ETA3</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>10.89372</td>
<td>10.42793</td>
<td>17.18864</td>
</tr>
<tr>
<td>9.62525</td>
<td>8.38468</td>
<td>7.61319</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ETA9</th>
<th>ETA10</th>
<th>ETA11</th>
<th>ETA12</th>
<th>ETA13</th>
<th>ETA14</th>
<th>ETA15</th>
<th>ETA16</th>
<th>ETA17</th>
</tr>
</thead>
<tbody>
<tr>
<td>29.35064</td>
<td>30.00294</td>
<td>2.90971</td>
<td>5.14553</td>
<td>6.19252</td>
<td>5.52394</td>
<td>4.31132</td>
<td>5.26054</td>
<td>2.93361</td>
</tr>
<tr>
<td>4.03566</td>
<td>4.97002</td>
<td>1.53896</td>
<td>1.45944</td>
<td>1.34676</td>
<td>1.29606</td>
<td>1.70644</td>
<td>1.24661</td>
<td>1.01138</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ETA18</th>
<th>ETA19</th>
<th>ETA20</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.37113</td>
<td>2.14944</td>
<td>0.14520</td>
</tr>
<tr>
<td>0.89695</td>
<td>0.80804</td>
<td>1.09196</td>
</tr>
</tbody>
</table>

**NOTE:** PARAMETER NO. 51 ON BOUNDARY

---

**SUMMARY OF ANALYSIS (H2O)**

<table>
<thead>
<tr>
<th>DISCREPANCY FUNCTION (M.L.)</th>
<th>DISCREPANCY FUNCTION (QUADRATIC FORM)</th>
<th>NUMBER OF ACTIVE PARAMETERS</th>
<th>DISCREPANCY FUNCTION WITH AKAIKE PENALTY</th>
<th>DISCREPANCY FUNCTION WITH SCHWARZ PENALTY</th>
<th>CHISQ STATISTIC (LIKELIHOOD RATIO)</th>
<th>CHISQ STATISTIC (QUADRATIC FORM)</th>
<th>DEGREES OF FREEDOM</th>
<th>SATURATED MODEL (M.L.)</th>
<th>SATURATED MODEL (QUADRATIC FORM)</th>
<th>PROBABILITY OF EXCEEDENCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>:</td>
<td>0.4799</td>
<td>43</td>
<td>0.669</td>
<td>1.058</td>
<td>218.369</td>
<td>212.181</td>
<td>207</td>
<td>1.099</td>
<td>3.363</td>
<td>0.280</td>
</tr>
</tbody>
</table>


References


Du Preez, P.H. (1980). 'n Onderzoek na die televisiekykpatrone en programvoorkeure van 'n groep standerd agt-leerlinge en die invloed wat televisie op hul studiegewoontes en houdings mag hê. (An investigation into the television viewing patterns and programme preferences of a group of Standard eight-pupils and the influence television might have on their study habits and attitudes.) Pretoria: Human Sciences Research Council.


