TESTING AND ESTIMATION FOR A CIRCULAR STATIONARY MODEL

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I. OLKIN and S. J. PRESS

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1. Introduction

Tests of hypotheses for the means of a p-variate normal distribution given that the covariance matrix $\Sigma = (\sigma_{ij})$ has some special structure, or that the covariance matrix has a special structure, have been considered in the literature. Some of the covariance structures that have received attention are:

$H_1$: $\Sigma = \Sigma_0$, a known matrix,
$H_2$: $\Sigma = \sigma^2 \Sigma_0$,
$H_3$: $\sigma_{ii} = \sigma^2, \sigma_{ij} = \sigma^2 \rho, (i \neq j)$,
$H_4$: $\Sigma$ unrestricted.

The test of $H_2$ versus $H_4$, the sphericity test, was considered by Mauchly (1940), Girschick (1941), and more recently by Olkin (1966). The test of $H_3$ versus $H_4$, and the test for the equality of means when $H_3$ holds, form the basis of the hypotheses of Wilks (1946). An analog to Hotelling’s test that the means are zero when $H_3$ holds has been considered by Geisser (1963).

Alternative models which have been considered are those for which the covariance kernel is weakly stationary, i.e., $\sigma_{ij} = \sigma|_{i-j}$, or which arise from stochastic difference equations, e.g., $\sigma_{ij} = \sigma^2 p^{i-j}$. The model considered here combines a circular symmetry condition with weak stationarity.

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Although the genesis of the model stems from a physical situation (described below), there are other applications e.g., in time series, for which the model is suitable.

Consider a point source located at the geocenter of a regular polygon of \( p \) sides, from which a signal is transmitted. Identical signal receivers (with identical noise characteristics) are positioned at the \( p \) vertices, denoted sequentially by \( V_1, \ldots, V_p \). The signal received at vertex \( V_i \) is denoted by \( x_i \). The main assumption is that the signal strength is the same in all directions, and that covariances depend only on the number of vertices separating the two receivers, so that

\[
\text{Cov}(x_i, x_{i+k}) = \sigma_0^2 \rho_k, \quad k = 1, \ldots, p, \quad 1 \leq i + k \leq p
\]

\[
\rho_0 \equiv 1, \quad \rho_j = \rho_{p-j+1}, \quad j = 2, \ldots, p.
\]

If the points \( V_1, \ldots, V_p \) were on a line (1.1) would be equivalent to stationarity, but because \( V_p \) is adjacent to \( V_1 \), a circularity is introduced.

Specifically, the assumptions are that \( x = (x_1, \ldots, x_p) \) is a random vector having a \( p \)-variate normal distribution with mean vector \( \mu = (\mu_1, \ldots, \mu_p) \) and covariances given by (1.1). The covariance matrices differ in the middle terms for \( p \) even or odd. For example, for \( p = 4 \) and \( p = 5 \), we have the structure

\[
\sigma_0^2 \begin{bmatrix}
1 & \rho_1 & \rho_2 & \rho_1 \\
\rho_1 & 1 & \rho_1 & \rho_2 \\
\rho_2 & \rho_1 & 1 & \rho_1 \\
\rho_1 & \rho_2 & \rho_1 & 1 \\
\end{bmatrix}, \quad \sigma_0^2 \begin{bmatrix}
1 & \rho_1 & \rho_2 & \rho_2 & \rho_1 \\
\rho_1 & 1 & \rho_1 & \rho_2 & \rho_2 \\
\rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 \\
\rho_2 & \rho_2 & \rho_1 & 1 & \rho_1 \\
\rho_1 & \rho_2 & \rho_2 & \rho_1 & 1 \\
\end{bmatrix}.
\]
Such covariance matrices have the property of being symmetric and cyclic, and for brevity we refer to them as circular.

The circularity condition may be introduced from temporal rather than spatial considerations. This leads to a stochastic process in which the covariance kernel is $\text{Cov}(x_s', x_{s+L}) = \sigma^2 \rho_L$, where $\rho_{N+L} = \rho_{N-L} = \rho_L$. The stationary process which yields a Laurent covariance kernel $\text{Cov}(x_i', x_j') = \text{Cov}(x_{i+L}', x_{j+L}')$ has been studied in detail by Whittle (1951). Following his development Wise (1955) considered the modification to a circular process for which $\text{Cov}(x_i', x_{i+L}) = \sigma^2 \rho_L$, where $\rho_{N+L} = \rho_{N-L} = \rho_L$.

The concern in this paper is with: (i) tests for symmetries in the covariance matrix, and with (ii) tests of hypotheses for the means when the covariance matrix is circular. The particular symmetries of interest are $\rho_j = \rho_{p-j+1}'$, the circular symmetry model; $\rho_1 = \ldots = \rho_p = \rho$, which is the intraclass correlation model; and $\rho_1 = \ldots = \rho_p = 0$, which is the spherical model. For each hypothesis we obtain the likelihood ratio test and the asymptotic distribution of the likelihood ratio statistic (LRS) under the hypothesis and alternative.

2. Preliminaries and Reduction to Canonical Form

We first review the notation used. Row vectors are generally denoted by lower case letters, matrices by capital letters. The dimensionality of a matrix is indicated by the symbol $A: r \times s$. By $A > 0$ we mean that the (symmetric) matrix $A$ is positive definite. The special vector $(1, \ldots, 1)$ is denoted by $e$.

To emphasize that a matrix $A$ is circular and therefore has the form
\[
A = \begin{bmatrix}
a_1 & a_2 & a_3 & \cdots & a_p \\
a_p & a_1 & a_2 & \cdots & a_{p-1} \\
\vdots \\
a_2 & a_3 & a_4 & \cdots & a_1
\end{bmatrix}
\]

we sometimes write \(A_c(a_1', \ldots, a_p')\) or simply \(A_c\). Since circular matrices are basic to the ensuing analysis, several properties of such matrices are now reviewed. Whittle (1951) provides a general discussion of Laurent matrices, and specialization to the case when the circular matrix is positive definite (symmetric) is discussed by Wise (1955) and Press (1964).

Let \(r_k = \exp 2\pi i (k-1)/p\) for \(k = 1, \ldots, p\) denote the \(p\) roots of unity. Then

\[
\zeta_k = a_1 r_k^0 + \ldots + a_p r_k^{p-1}
\]

are the eigenvalues of \(A_c(a_1', \ldots, a_p')\). If \(A_c\) is symmetric, then \(a_{p-j+2} = a_j\) \((j = 2, \ldots, p)\). In this case the eigenvalues are real and are given by

\[
(2.1) \quad \zeta_k = \sum_{j=1}^{p} a_j \cos \frac{2\pi}{p} (k-1)(p-j+1), \quad k = 1, \ldots, p.
\]

If

\[
(2.2) \quad \gamma_{jk} = \frac{1}{\sqrt{p}} \left[ \cos \frac{2\pi}{p} (j-1)(k-1) + \sin \frac{2\pi}{p} (j-1)(k-1) \right],
\]

then \(\Gamma = (\gamma_{jk})\) is orthogonal and transforms \(A_c\) to diagonal form, namely,

\(A_c = \Gamma' D_c \Gamma\), where \(D_c = \text{diag}(\zeta_1', \ldots, \zeta_p')\). A critical point is the fact that this diagonalization is achieved by an orthogonal matrix whose elements are constants, independent of the elements of \(A_c\). (Note that the first row (and column) of \(\Gamma\) is \(p^{-1/2} e_1\).)
The condition of positive definiteness, namely, \( \gamma_j > 0 \), is equivalent to linear constraints in the elements \( a_j \) as given by (2.1). From (2.1), with \( a_j = \zeta_{p-j+2} \), \( j = 2, \ldots, p \), and from trigonometric identities, it follows that \( \gamma_j = \zeta_{p-j+2} \), \( j = 2, \ldots, p \).

2.1 Reduction to Canonical Form

Given a sample \((x_{1\alpha}, \ldots, x_{p\alpha})\), \( \alpha = 1, \ldots, N \), of size \( N \) from a \( p \)-variate \( N(\mu, \Sigma) \) distribution, we may (by sufficiency) consider the mean vector \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_p) \) and covariance matrix \( S/n, S = (s_{ij}) \),

\[
s_{ij} = \sum_{\alpha=1}^{N} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j), \quad n = N - 1,
\]
as our starting point. Denoting by \( X(Z) \) the law of the random matrix (or vector) \( Z \), we note that \( \bar{x} \) and \( S \) are independently distributed, with

\[
X(\sqrt{n} \bar{x}) = N(\sqrt{n} \mu, \Sigma), \quad X(S) = W(\Sigma, p; n),
\]
i.e., \( S \) has density function

\[
p(S) = c(p, n) \left| \Sigma \right|^{-\frac{n}{2}} \left| S \right|^{-\frac{n-p-1}{2}} \exp\left\{-\frac{1}{2} \text{tr} \Sigma^{-1} S\right\}, \quad S > 0, \Sigma > 0,
\]

\[
c(p, n) = 2^{-\frac{np}{2}} \pi^{-\frac{p(p-1)}{2}} \prod_{1 \leq i < j \leq p} \Gamma^2 \left( \frac{n-i+1}{2} \right)^{-1}.
\]

Making the transformations

\[
y = \sqrt{n} \bar{x}, \quad V = \Gamma' S \Gamma,
\]

where \( \Gamma \) is defined by (2.2), we have that \( y \) and \( V \) are independently distributed with
(2.3) \[ \mathcal{L}(y) = N(\eta, \Sigma), \quad \mathcal{X}(v) = W(\Sigma, \nu; n), \]
\[ \eta = \sqrt{n} \mu \Gamma, \quad \Sigma = \Gamma \Sigma \Gamma. \]

When \( \Sigma \) is circular, \( \Sigma = D_\Sigma = \text{diag}(\gamma_1, \ldots, \gamma_p) \), \( \gamma_j = \gamma_{p-j+2}, \) \( j = 2, \ldots, p \), and

(2.4) \[ \mathcal{L}(y) = N(\eta, D_\gamma), \quad \mathcal{X}(v) = W(D_\gamma, \nu; n). \]

Because \( \gamma_j = \gamma_{p-j+2}, \) \( j = 2, \ldots, p \), we can achieve a further reduction by noting that the minimal sufficient statistic for \( (\eta, \gamma_1, \ldots, \gamma_p) \) is given by

(2.5a) \[ (y, v_1, v_2, \ldots, v_m, v_{m+1}) = \]
(2.5b) \[ (y, v_1, v_2, \ldots, v_m) = (y, v_{11}, v_{22} + v_{pp}, \ldots, v_{mm} + v_{m+2,m+2}, v_{m+1,m+1}). \]

when \( p = 2m \) and \( p = 2m + 1 \), respectively. For later use define
\[ v_j = \gamma_{p-j+2}, \] \( j = 2, \ldots, p. \)

Since the \( y \)'s and \( v \)'s are independently distributed when \( \Sigma \) is circular, we have as a canonical model:

(2.6) \[ \mathcal{L}(y_1) = N(\eta_1, \gamma_1), \quad \mathcal{L}(y_j) = N(\eta_j, \gamma_j), \quad \gamma_j = \gamma_{p-j+2}, \] \( j = 2, \ldots, p; \)

(2.7a) \[ \chi^2 \left( \frac{v^2}{\gamma_k} \right) = \chi^2 \left( \frac{v_k}{\gamma_k} \right) = \chi^2 \left( \frac{v_{m+1}}{\gamma_{m+1}} \right) = \chi^2_n, \]

when \( p = 2m, \)

(2.7b) \[ \chi^2 \left( \frac{v^2}{\gamma_k} \right) = \chi^2_n, \quad \chi^2 \left( \frac{v_k}{\gamma_k} \right) = \chi^2_{2n}, \quad k = 2, \ldots, m + 1, \]

when \( p = 2m + 1. \)
For each of the hypotheses considered the starting point is either (2.3) or (2.6), and (2.7), from which the LRT is obtained. In all the problems considered, the LRS is distributed as a product of independent beta variates when the hypothesis is true. Consequently, except for some special cases, it is not feasible to obtain exact distributions in closed form, and we obtain an asymptotic approximation accurate to \( O(N^{-3}) \). Under the alternative hypothesis, we use the delta method to obtain the asymptotic distribution of a suitably normalized function of the LRS.

2.2 Distributional Preliminaries

In this section we present some distributional results which permit a representation of the LRS as a product of independent beta variables. In order to obtain asymptotic approximations, we modify the procedure of Box (1949) to suit the needs of the statistics considered.

If \( z_1, \ldots, z_m \) are independently distributed with \( \mathcal{N}(z_i) = N(\xi_i, 1) \), then \( \mathcal{X}(\Sigma z_i^2) = \chi_m^2(\delta) \) denotes the noncentral chi-square distribution with \( m \) degrees of freedom and noncentrality parameter \( \delta = \Sigma \xi_i^2 \). (We write \( \chi_m^2 = \chi_m^2(0) \).)

If \( u \) and \( v \) are independently distributed, \( \mathcal{X}(u) = \chi_{2a}^2, \mathcal{X}(v) = \chi_{2b}^2(\delta), \) then the random variable \( z = u/(u + v) \) has the noncentral beta distribution with \( a \) and \( b \) degrees of freedom and noncentrality parameter \( \delta \), denoted by \( \beta(a, b; \delta) \), i.e., with density

\[
p(z) = \sum_{j=0}^{\infty} \frac{e^{-\frac{\delta}{2}} \left(\frac{\delta}{2}\right)^j}{j!} \frac{z^{a-1} (1 - z)^{b+j-1}}{B(a, b + j)}, \quad 0 < z < 1.
\]

(The central beta distribution has \( \delta = 0 \) and is denoted by \( \beta(a, b) \).)
The following lemmas permit us to represent certain statistics as products of independent beta variates and are used repeatedly in the analysis. The proofs of the first two lemmas are based on equating moments. In so doing, it is helpful to use the Dirichlet integral

\[
\int_{0}^{r} \frac{r}{\prod_{i=1}^{m} x_i^{c_i-1}} f(\sum_{i=1}^{m} x_i) dx = \frac{1}{\Gamma(c_1) \cdots \Gamma(c_m)} \int_{0}^{\infty} z^{c_1-1} f(z) dz,
\]

where \( c = \sum_{i=1}^{m} c_i \), and the duplication formula

\[
\Gamma'(rk + a) = \frac{1}{r} \Gamma'(rk) \left( \frac{a}{2} \right) \prod_{k=1}^{r-1} \Gamma \left( k + \frac{a - 1}{r} \right),
\]

where \( r \) is an integer.

**Lemma 1**

Let \( U_1, \ldots, U_{m_1}, V_1, \ldots, V_{m_2} \) be independently distributed, \( \chi^2_n, \chi^2_{2n} \). If

\[
L = \frac{M}{2^{m_2} \left( U_1 + V_1 \right)^M}, \quad M = m_1 + 2m_2,
\]

then \( \chi^2(L) = \chi^2_{\left( \prod_{j=1}^{M-1} x_j \right)}, \) where \( x_1, \ldots, x_{M-1} \) are independently distributed,

\[
\chi^2(x_j) = \beta \left( \frac{n+1}{2}, \frac{j}{M} \right), \quad j = 1, \ldots, M - m_2 - 1, \quad \chi^2(x_j) = \beta \left( \frac{n-1}{2}, \frac{j}{M} - \frac{1}{2} \right), \quad j = M - m_2, \ldots, M - 1.
\]

**Lemma 2**

Let \( W_0, W_1, \ldots, W_r \) be independently distributed, \( \chi^2_{2a_j} \), \( j = 0, 1, \ldots, r \). If

\[
L = r^r \left( \prod_{j=1}^{r} W_j \right) \left( W_0 + \sum_{j=1}^{r} W_j \right)^r,
\]
then $\mathcal{P}(L) = \mathcal{P}\left(\prod_{1}^{x} X_j\right)$, where $X_1, \ldots, X_r$ are independently distributed,

$$\mathcal{P}(X_j) = \beta\left(\frac{a_j}{a}, \frac{a + 1 - 1}{a} - \frac{a_j}{r}, a = \sum_{0}^{r} a_j\right).$$

**Lemma 3**

Let $Z_1, \ldots, Z_r$ be independently and identically distributed, each as $\beta(a, 1)$. If $L = \prod_{1}^{r} Z_i^M$, then $P\left\{-2 \log L \leq t\right\} = P\left\{X_{2r}^2 \leq \frac{a}{M} t\right\}$. 

**Proof**

$$EL^{-t} = (EZ^{-t}M)^r = (1 - \frac{th}{q})^{-r},$$ so that $\mathcal{P}\left\{-2 \frac{a}{M} \log L \right\} = X_{2r}^2$. 

If $\chi(Z) = \beta(a, b)$, then $E\chi' = B(a + 1, b)/B(a, b)$. Suppose $W, 0 \leq W \leq 1$, is a random variable whose distribution is that of a product of independent beta variables, so that

$$(2.8) \quad EW^h = k \prod_{1}^{a} \left(1 + \frac{h x(1 + h) + \xi_j}{x(1 + h) + \eta_j}\right), \quad h = 0, 1, \ldots,$$

where $k$ is determined from $EW^0 = 1$, and $(x, \xi_j, \eta_j)$ are known quantities. The procedure of Box (1949) for finding the asymptotic distribution of $W$ is based on expanding the characteristic function to yield a linear combination of chi-square variables

$$(2.9) \quad P\left\{-2 \rho \log W \leq t\right\} = (1 + \omega) P\left\{X_{2r}^2 \leq t\right\} - \omega P\left\{X_{r+4}^2 \leq t\right\} + O(N^{-3}),$$

where $f = 2\Sigma(\eta_j - \xi_j)$, $\rho = (x - b)/x$, and $b = \frac{1}{x} \Sigma \frac{1}{2} = \frac{1}{2} x (x - 2) + \frac{1}{2}$. The factor $\omega$ is obtained from

$$\omega = \frac{1}{6(x - b)^2} \prod_{1}^{a} \left[B_3(b + \xi_j) - B_3(b + \eta_j)\right].$$
where $B_3(z) = (2z^3 - 3z^2 + z)/2$ is the Bernoulli polynomial of degree 3.

Because the computations become tedious, the following representation may prove useful:

\[
(2.10) \quad B_3(z + 1) - B_3(z) = 3z^2, \quad B_3\left(z + \frac{1}{2}\right) - B_3(z) = \frac{3z(2z - 1)}{4},
\]

\[
\sum_{j=1}^{r} \left[ B_3\left(z + \frac{j}{c}\right) - B_3(z) \right] = \frac{r(r + 1)}{hc} \left[ \frac{r(r + 1)}{c^2} + \frac{(2r + 1)(2z - 1)}{c} + 6z^2 - 6z + 1 \right].
\]

To obtain the noncentral asymptotic distribution of $c_n g(V)$, where $\chi(n) = W(\xi, \eta; n)$ and $c_n$ is a normalizing constant, we use the standard delta method (Cramér (1946), pp. 354, 366). First let

\[
Z = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{2}} n, \quad \text{so that} \quad \chi(n) = W(nI, \eta; n) \quad \text{and} \quad g(V) = g\left(n \sum_{i=1}^{n} z_{i} \right) = h(nZ).
\]

If $h$ is scale invariant, i.e., $h(cZ) = h(Z)$, and satisfies mild regularity conditions (which is the case for the problems considered) then

\[
(2.11) \quad \chi\left\{\sqrt{n} [h(Z) - h(I)]\right\} \rightarrow N(0, 2\text{tr}h^2),
\]

where $H = (h_{ij}), \quad h_{ij} = \frac{1}{2} \frac{\partial h}{\partial z_{ij}} \bigg|_{Z=I}$ for $i \neq j$, and $h_{ii} = \frac{\partial h}{\partial z_{ii}} \bigg|_{Z=I}$. The asymptotic variance is given by

\[
\nu_\infty = \Sigma \Sigma h_{ij} h_{kj} n \text{Cov}(z_{ij}, z_{kj}).
\]

Since $\text{Cov}(z_{ij}, z_{kj}) = \frac{1}{n} [\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}]$, where $\delta_{ij}$ is the Kronecker delta, we obtain

\[
(2.12) \quad \nu_\infty = 2 \Sigma h_{ij}^2 = 2\text{tr}h^2.
\]

3. Tests for Symmetries in the Covariance Structure

The hypotheses

\[
H_1 : \Sigma = \sigma^2 I, \quad H_2 : \sigma_{ii} = \sigma^2, \quad \sigma_{ij} = \sigma^2 \rho(i \neq j),
\]

\[
H_3 : \Sigma = \Sigma_c, \quad H_4 : \Sigma > 0,
\]
represent various degrees of specialization in the covariance matrix. The
tests for $H_1$ versus $H_4$ (sphericity), $H_2$ versus $H_4$ (Wilks' hypotheses)
are known. The test for sphericity given homogeneity of variances and
covariances ($H_1$ versus $H_2$) is the test that the intraclass correlation
is zero and is a standard $F$-test. Our concern is mainly with tests
involving circularity: sphericity given circularity ($H_1$ versus $H_3$),
homogeneity of covariances given circularity ($H_2$ versus $H_3$), and test
for circularity ($H_3$ versus $H_4$).

In this section we assume the means are unknown. A slight modification
in the development yields analogous results when the means are known. From
the canonical model (2.6), (2.7), the hypotheses on $\Sigma$ are translated to
hypotheses on $\mathcal{Y} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_p)$ as represented by the regions

$$
\omega_1 = \{ \mathcal{Y}: \mathcal{Y}_1 = \cdots = \mathcal{Y}_{m+1}, 0 < \mathcal{Y}_1 < \infty \},
\omega_2 = \{ \mathcal{Y}: 0 < \mathcal{Y}_1 < \infty, \mathcal{Y}_2 = \cdots = \mathcal{Y}_{m+1}, 0 < \mathcal{Y}_2 < \infty \},
\omega_3 = \{ \mathcal{Y}: 0 < \mathcal{Y}_j < \infty, j = 1, 2, \ldots, m+1 \},
\omega_4 = \{ \mathcal{Y}: \Sigma > 0 \}.
$$

It is straightforward to obtain the maxima of the likelihood functions $L(v, y)$.

**Spherical Model**

From (2.6) and (2.7),

$$
(3.1) \quad \sup_{\omega_1} L(v, y) = ce^{-\frac{1}{2} \frac{pN}{2}} \left( \frac{pN}{2} (m+1) \right)^{-\frac{pN}{2}} \left( \sum_{i=1}^{m+1} v_i \right)^{-\frac{pN}{2}}
$$

**Intraclass Correlation Model**

From (2.6) and (2.7),

$$
(3.2) \quad \sup_{\omega_2} L(v, y) = ce^{-\frac{1}{2} \frac{pN}{2}} \left[ \frac{(p-1)^{p-1}}{N \left( \sum_{i=1}^{m+1} v_i \right)^{p-1}} \right]^{-\frac{N}{2}}
$$
Circular Model

From (2.6) and (2.7a) or (2.7b)

\[ \text{sup}_{\omega_3} L(v, y) = c e^{-\frac{1}{2} \frac{dN}{N} \frac{pN}{2} 2^{N(p-1)} \left( \prod_{1}^{p} v_j \right)^{-\frac{N}{2}}} \]

General \( \Sigma \)

In this case we start with (2.3),

\[ \text{sup}_{\omega_4} L(V, Y) = c |V|^{-\frac{N}{2}} \frac{1}{2} dN \frac{pN}{N^2} \]

The constant \( c \) in (3.1)-(3.4) is the same, and is a function of \( V \).

The various LRS in this section are denoted by

\[ \lambda_{ij} = \text{sup}_{\omega_i} L / \text{sup}_{\omega_j} L. \]

3.1 Test for Sphericity Versus Circularity

From (3.1) and (3.3), the LRS is given by

\[ \lambda_{13} = \frac{p^p}{N} 2^{-2(p-m-1)} \prod_{1}^{p} v_j \left( \sum_{1}^{m+1} v_j \right)^{-p} \]

When the hypothesis is true, from Lemma 1, we have the representation

\[ \mathcal{L}(\lambda_{13}) = \chi_{p-1} \left( \prod_{1}^{p} T_j \right), \]

where the \( T_j \) are independent,

\[ \mathcal{L}(T_j) = \beta \left( \frac{n}{2}, \frac{j}{p} \right), \quad j = 1, \ldots, m, \]

\[ \mathcal{L}(T_j) = \beta \left( \frac{n+1}{2}, \frac{j}{p} - \frac{1}{2} \right), \quad j = m+1, \ldots, p-1. \]

In its canonical form, this problem is that of testing for homogeneity of variances, for which Bartlett (1937) suggested replacing \( N \) by \( n \). This problem has been considered by many authors (see e.g., Box (1949), Rao (1952)). For completeness, we merely note the result, obtained as a
specialization of the multivariate problem, as given by Anderson (1958, p. 255).

If $V_{13} = \lambda_{13}$, then

$$P\left\{-2p \log V_{13} \leq z\right\} = (1 + o) P\left\{\chi^2_f \leq z\right\} - o P\left\{\chi^2_{f+4} \leq z\right\} + o(n^{-3}),$$

where

$$p = 2m: \quad f = \frac{p}{2}, \quad \rho = 1 - \frac{p^2 + 6p - 4}{6p^2 n}, \quad \omega = \frac{p(1 - \rho)^2}{8p^2 n},$$

$$p = 2m + 1: \quad f = \frac{p - 1}{2}, \quad \rho = 1 - \frac{p + 4}{6p n}, \quad \omega = \frac{(p - 1)(1 - \rho)^2}{8p^2 n}.$$

To obtain the noncentral asymptotic distribution let (when $p = 2m + 1$)

$$z_{jj} = \frac{v_{jj}}{\mathcal{C}_j}. \quad \text{Define}$$

$$h(z) = \frac{2}{N} \log \lambda_{13} = \log \left[\frac{z^2}{\frac{p}{2}} \left(\frac{z_{j} + \frac{z_{j+m+2}}{2}}{\frac{p}{2}}\right) \frac{p}{\sqrt{\frac{p}{2} \mathcal{C}_j}} \right],$$

where $z_j = z_{jj}$. Then from (2.14), $h(I) = \mathcal{Z} \log \mathcal{C}_i - p \log \mathcal{Z}$, where

$$\mathcal{Z} = \sum \frac{\mathcal{C}_i}{p}, \quad \text{so that} \quad h_{ii} = 1 - \frac{\mathcal{C}_i}{\mathcal{Z}}, \quad h_{ij} = 0 (i \neq j).$$

Consequently,

$$\mathcal{Z}\left\{\mathcal{Z}\left[\mathcal{Z}(z) - h(I)\right]\right\} \rightarrow \mathcal{N}\left(0, \frac{p}{\mathcal{Z}} \Sigma \frac{\mathcal{C}_i - \mathcal{Z}}{\mathcal{Z}}^2 \right).$$

The case $p = 2m$ yields the same result.

3.2 Test for Intraclass Structure Versus Circularity

From (3.2) and (3.3), the LRS is given by

$$\lambda^{\frac{2}{N}} = (p - 1)^p z^{-2(p-2)} y_{j} \left(\frac{m+1}{2} \right)^{p-1} \left(\frac{\Sigma y_{j}}{2} \right)^{-(p-1)}.$$

From Lemma 1, we have the representation $\chi\left(\lambda_{13}^{\frac{2}{N}}\right) = \chi\left(\frac{p-2}{1} \frac{T_j}{\mathcal{C}_j}\right)$, where the $T_j$ are independent and
\[ \mathcal{L}(T_j) = \beta \left( \frac{n}{2}, \frac{j}{p - 1} \right), \quad j = 1, \ldots, p - m - 1, \]
\[ \mathcal{L}(T_j) = \beta \left( \frac{n + 1}{2}, \frac{j - 1}{p - 1} - \frac{1}{2} \right), \quad j = p - m, \ldots, p - 2. \]

We note that this problem also reduces to a test for homogeneity of variances. This is because \( \chi_1 \) is unrestricted in both \( \omega_2 \) and \( \omega_3 \), so that the hypothesis is, in effect, that \( \chi_2 = \ldots = \chi_{m+1} \). As in §3.1 (Anderson (1958), p. 255) we obtain the asymptotic distribution when the hypothesis is true to be
\[
P \left( -2p \log V_{23} \leq z \right) = (1 + \omega) P \left( \chi_2^2 \leq z \right) - \omega P \left( \chi_{p+4}^2 \leq z \right) + \mathcal{O}(n^{-3}), \quad \text{where} \quad V_{23} = \frac{\chi_2^2}{n},
\]
\[
p = 2m: \quad f = \frac{p - 2}{2}, \quad \rho = 1 - \frac{p + 3}{6(p - 1)n}, \quad \omega = \frac{(p - 2)(1 - \rho)^2}{8p^2},
\]
\[
p = 2m + 1: \quad f = \frac{p - 3}{2}, \quad \rho = 1 - \frac{p + 1}{6n(p - 1)}, \quad \omega = \frac{(p - 3)(1 - \rho)^2}{8p^2}.
\]

The noncentral asymptotic distribution obtained as in §3.1, namely,
\[
\chi \left\{ \sqrt{n} \left[ h(2) - h(1) \right] \right\} \rightarrow N \left( 0, \frac{p}{2} \left( \chi_1^2 - \bar{\chi}^2 \right) \right),
\]
where \( h(2) = \left( \frac{2}{N} \right) \log \chi_{23}, \quad h(1) = \frac{p}{2} \log \chi_1 - (p - 1) \log \bar{\xi}, \quad \text{where} \quad \bar{\xi} = \frac{\sum \chi_i^2}{p - 1}, \)

3.3 Circular Versus General Structure

From (3.3) and (3.4), the IRS is given by
\[
(3.5) \quad \frac{2}{N} \lambda_{3k} = \frac{2^{2(p-m-1)} |R|}{\prod_{j=1}^p \sqrt{v_j}} = \frac{2^{2(p-m-1)} |R|}{\prod_{j=1}^p \sqrt{v_j}},
\]
where \( R \) is the correlation matrix. In order to obtain a representation as a product of independent beta random variables, let \( V = TT' \), where \( T \) is lower triangular. Then \( |V| = \prod_{j=1}^p t_{ii}^2, \quad \sqrt{v_j} = \sum_{\alpha=1}^j t_{ij}, \quad j = 2, \ldots, p \). If we let
\[ \bar{G}_{m+1} = \frac{t_{m+1}}{t_{m+1} + q_{m+1}}, \quad G_j = \frac{2^t t_j t_{p-j+2}}{t_j + t_{p-j+2} + q_j + q_{p-j+2}}, \quad j = 2, \ldots, m+1, \]

where \( t_j = t_{j2} \), \( q_j = \sum_{\alpha=1}^{j-1} t_{j\alpha} \), \( j = 2, \ldots, p \), then

\[
\frac{2}{N} \lambda_{34} = \begin{cases} \left( \frac{m}{2} \right) G_j \overline{G}_{m+1} & p = 2m, \\
\left( \frac{m}{2} \right) G_j G_{m+1} & p = 2m + 1. \end{cases}
\]

When \( H_3 \) is true (the covariance matrix is circular), it may be directly verified that all the \( t_{ij} \) are independently distributed, so that the \( G_j \) are independent. Also, because \( \gamma_j = \gamma_{p-j+2} \), \( j = 2, \ldots, p \), the elements of the \( j \)th and \((p - j + 2)\)nd rows of \( T \) have the same scale, namely,

\[
\zeta(t_{j\alpha}^2 / \gamma_j) = \zeta(x_1^2), \quad \alpha = 1, \ldots, j - 1,
\]

\[
\zeta(t_{j\beta}^2 / \gamma_j) = \zeta(x_{n-j+1}^2), \quad j = 2, \ldots, p.
\]

Because \( G_j \) is invariant under a scale change \( t_j \rightarrow ct_j \), we may (under \( H_3 \)) assume \( \gamma_j = 1 \).

From Lemma 2, we then obtain (for both even and odd \( p \)), that

\[
\zeta(\lambda_{34}^2) = \zeta\left( \frac{p-1}{2} T_4 \right), \quad \text{where the } T_j \text{ are independent,}
\]

\[ (3.6) \]

\[
\zeta(T_j) = \beta \left( \frac{n - j}{2}, \frac{j}{2} \right), \quad j = 1, \ldots, m,
\]

\[
\zeta(T_j) = \beta \left( \frac{n - j}{2}, \frac{j + 1}{2} \right), \quad j = m + 1, \ldots, p - 1.
\]

When \( H_3 \) is true, we obtain from the representation (3.5),
where \( c \) is a normalizing constant so that \( E_{34}^{0} = 1 \). From (2.8) we make the association

\[
\begin{align*}
\lambda_{34} & = \frac{1}{2} \quad \gamma_{j} = \frac{j + 1}{2}, \quad j = 1, \ldots, p - 1, \\
\eta_{1} = \ldots = \eta_{m} = -\frac{1}{2}, \quad \eta_{m+1} = \ldots = \eta_{p-1} = 0.
\end{align*}
\]

Hence (when \( H_{3} \) is true)

\[
P \left( -2p \log \lambda_{34} \leq z \right) = (1 + \omega)P \left( \chi_{p}^{2} \leq z \right) - \omega P \left( \chi_{p+4}^{2} \leq z \right) + O(N^{-3}),
\]

where \( f \) and \( b \) are determined from (2.10), and (2.11):

\[
f = \frac{p(p + 1) - 2(m + 1)}{2}, \quad b = \frac{[p(p + 1)(2p + 7) - 18(m + 1)]}{24f}.
\]

From (2.9) and (2.10),

\[
6(x - b)^{2} = \sum_{1}^{p} \left[ B_{3}(b - \frac{j}{2}) - B_{3}(b) \right] + (m + 1) \left[ B_{3}(b) - B_{3}(b - \frac{1}{2}) \right]
\]

\[
= a_{0} + a_{1}b + a_{2}b^{2},
\]

where \( a_{2} = -\frac{3f}{2}, \quad a_{1} = 3bf, \quad a_{0} = \frac{24(m + 1) - p(p + 1)^{2}(p + 8)}{32}, \) so that

\[
6(x - b)^{2} = \frac{3}{2} b^{2}f + \frac{24(m + 1) - p(p + 1)^{2}(p + 8)}{32}.
\]

Therefore, when \( p = 2m, \quad f = (p^{2} - 2)/2, \quad b = [2p^{3} + 2p^{2} - 2p - 18]/[12(p^{2} - 2)]; \)

when \( p = 2m + 1, \quad f = (p^{2} - 1)/2, \quad b = (2p + 9)/12. \) In either case, \( \omega \) simplifies
To obtain the noncentral asymptotic distribution, let \( Z = \Sigma \frac{1}{2} Z' \Sigma^{-1} \).

Denote the \( j^{th} \) row of \( \Sigma' \) by \( s_j \), so that \( v_{ij} = s_j A_{ij} s_j' = \text{tr}(ZA_j) \), where \( A_j = s_j s_j' \). When \( p = 2m + 1 \), we have

\[
h(z) = \frac{2}{N} \log \lambda_{34} = \log \frac{2^{p-1} |\Sigma| |Z|}{\left( \text{tr} ZA_1 \right) \frac{p}{2} \left[ \text{tr} Z(A_j + A_p^{-j+2}) \right]} ,
\]

and

\[
(3.7) \quad \mathcal{N} \left( \sqrt{N} \left[ \frac{2}{N} \log \lambda_{34} - h(I) \right] \right) \rightarrow N(0, \nu_\infty),
\]

where

\[
h(I) = \log \left\{ \left| |P| \frac{p}{2} \left[ \frac{\sigma_i \sigma_{i-j+2}}{\sigma_i^2 + \sigma_{i+j+2}^2} \right] \right\} ,
\]

\( \sigma_i^2 = \sigma_{jj} \) and \( P = (\rho_{ij}) \) is the correlation matrix. To obtain \( \nu_\infty = 2 \text{tr} H^2 \), note that

\[
H = I - \frac{p}{2} \sum_{j=1}^{p} \frac{A_j + A_{p-j+2}}{\text{tr} (A_j + A_{p-j+2})} \equiv I - B,
\]

where \( A_{p+1} = 0 \). Since \( \text{tr} B = p \), \( \text{tr} H^2 = \text{tr} B^2 - p \). From

\[
\text{tr} A_i A_j = \text{tr}(s_i s_i')(s_j s_j') = \sigma_{ij}^2, \quad \text{tr} A_i = \sigma_i^2,
\]

we immediately obtain

\[
\text{tr} B^2 = \sum_{i,j} \frac{\text{tr} A_i A_j}{\text{tr} (A_i + A_{p-i+2}) \text{tr} (A_j + A_{p-j+2})}
\]

\[
= \sum_{i,j} \frac{\sigma_{ij}^2}{\left( \sigma_i^2 + \sigma_{p-i+2}^2 \right) \left( \sigma_j^2 + \sigma_{p-j+2}^2 \right)}.
\]
where \( \sigma_{p+1}^2 = 0 \). Consequently, 
\( \nu_\infty = 2(\text{tr} B^2 - p) \).

The result (3.7) is valid for \( p = 2m \), if we replace \( 2\sigma_{m+1}^2 \) by \( \sigma_{m+1}^2 \).

4. Tests for Means Given that the Covariance Matrix is Circular

In this section we are concerned with testing that the means are zero and that the means are equal, given that the covariance matrix is circular.

From (2.3), the test that \( \mu_1 = \ldots = \mu_p = \mu^* \) is equivalent to
\( \eta = \sqrt{N} \mu \Gamma = \sqrt{N} \mu^* e_{\Gamma^t} \), where \( \Gamma^t \) is given by (2.2). But \( \Gamma^t \) is orthogonal with first row \( \frac{1}{p^{0.5}}e_1 \), so that \( e_{\Gamma^t} = \sqrt{p} \begin{pmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \). Thus we have as our starting point (2.6) and (2.7), and regions

\[
\omega_1^* = \left\{ \eta, \xi : \eta = 0, 0 < \xi_j \right\}, \]

\[
\omega_2^* = \left\{ \eta, \xi : \eta = (\eta_1, 0, \ldots, 0), 0 < \xi_j \right\}, \]

\[
\omega_3^* = \left\{ \eta, \xi : -\infty < \eta_j < \infty, 0 < \xi_j \right\}.
\]

The LRS for testing \( (\eta, \xi) \in \omega_1^* \) versus \( \omega_3^* \) is denoted by \( f_{ij} \).

4.1 Test that Means are Zero

From (2.6) and (2.7) the LRS is given by
\[
\frac{2}{\sqrt{N}} \sum_{i=1}^{p} \left( \frac{v_j}{v_j + w_j} \right),
\]

where for \( p = 2m + 1 \), \( v_1 = y_1^2, w_j = y_j^2 + y_{p-j+2}^2, j = 2, \ldots, m + 1 \); and for \( p = 2m \), \( v_1 = y_1^2, w_j = y_j^2 + y_{p-j+2}^2, j = 2, \ldots, m \), \( w_{m+1} = y_{m+1}^2 \). From §2.1 and the fact that if \( \chi(X) = \beta(a, 1) \), then \( \chi(X^2) = \beta(a^2, 1) \), we obtain the representation
\[ \mathcal{L} \left( \begin{pmatrix} \mathcal{L} \left( \sum_{j} T_j \right) \bar{T}_{m+1} \right) \right), \quad p = 2m, \]

\[ \mathcal{L} \left( \bar{T}_{m+1} \right), \quad p = 2m + 1, \]

where the \( T_j \) are independently distributed as:

(4.1) \( \mathcal{L}(T_1) = \beta \left( \frac{n}{2}, \frac{1}{2} \right), \quad \mathcal{L}(T_j) = \beta \left( \frac{n}{2}, 1 \right), \quad j = 2, \ldots, m, \quad \mathcal{L}(T_{m+1}) = \beta \left( \frac{n}{2}, \frac{1}{2} \right). \)

When the hypothesis of zero means is true, we obtain from the representation (4.2),

(4.2) \[ E_{13}^{h} = \frac{\left[ \Gamma \left( A - \frac{1}{2} \right) \right]^{m+1}}{\left[ \Gamma \left( A + \frac{1}{2} \right) \right]^{2m-p+2}} \left[ \Gamma \left( A + \frac{1}{2} \right) \right]^{p-m-1}, \quad A = \frac{N}{2} (1 + h). \]

From (2.8) make the association: \( x = \frac{N}{2}, \quad p = 1 - \frac{2b}{N}, \quad \xi_1 = \cdots = \xi_{m+1} = -\frac{1}{2}, \)

\( \eta_1 = \cdots = \eta_{p-m-1} = \frac{1}{2}, \quad \eta_{p-m} = \cdots = \eta_{m+1} = 0. \) Hence

\[ P \left\{ -2p \log f_{13} \leq z \right\} = (1 + \omega) P \left\{ X_f^2 \leq z \right\} - \omega P \left\{ X_{f+1}^2 \leq z \right\} + o(N^{-3}), \] where \( f \) and \( b \) are determined from (2.9), (2.10) : \( f = p, \quad b = \frac{2m + 2 + p}{4p}. \)

\[ 6(x - b)^2 = (m + 1)B_3 \left( b - \frac{1}{2} \right) - (2m + 2 - p)B_3(b) - (p - m - 1)B_3 \left( b + \frac{1}{2} \right) \]

\[ = -(p - m - 1) \left[ B_3 \left( b + \frac{1}{2} \right) - B_3(b) \right] - (m + 1) \left[ B_3(b) - B_3 \left( b - \frac{1}{2} \right) \right] \]

\[ = a_0 + a_1 b + a_2 b^2, \]
where \( a_2 = -\frac{3p}{2}, \ a_1 = +3pb, \ a_0 = -\frac{3(m+1)}{4}. \) Hence
\[
\omega = \frac{p}{4} \left( \frac{2m + 2 - p}{2Np - (2m + 2 + p)} \right)^2.
\]

To obtain the noncentral asymptotic distribution, note that under the alternative \( \beta_{13}^N \) is the product of independent random variables. Using the delta method on each term, we obtain
\[
\mathcal{L} \left\{ \sqrt{N} \left[ \log \frac{v_j}{v_j + y_j^2 + y_{p-j+2}^2} - \log \frac{1}{1 + \delta_j} \right] \right\} \rightarrow N \left( 0, \ 1 - \frac{1}{(1 + \delta_j)^2} \right),
\]
where \( \delta_1 = \eta_{1/2}^2, \ \delta_j = (\eta_j^2 + \eta_{p-j+2}^2)/\nu_j, \) for \( j = 2, \ldots, m + 1, \) and \( \delta_{m+1} = \eta_{m+1}^2/\nu_{m+1}. \) A convolution then yields
\[
(4.3) \quad \mathcal{L} \left\{ \sqrt{\frac{N}{2}} \left[ \log \left( \frac{\beta_{13}^N}{\mathcal{L} \left( \frac{2}{1 + \delta_j} \right)} \right) \right] \right\} \rightarrow N \left( 0, \ p - \frac{p}{1 (1 + \delta_j)^2} \right).
\]

4.2 Tests that the Means are Equal

From (2.6) and (2.7) the ILS is given by
\[
(4.4) \quad \frac{2}{\mathcal{L} \left( T_{23} \right)} = \frac{p}{2} \left( \frac{v_j}{v_j + w_j} \right).
\]

By Lemma 2 and \( \delta_{4.1}, \)
\[
(4.5) \quad \mathcal{L} \left( \frac{2}{\mathcal{L} \left( T_{23} \right)} \right) = \begin{cases} \mathcal{L} \left( \left( \prod_{j=1}^{m} T_j \right)^{m+1} \right), & p = 2m, \\ \mathcal{L} \left( \left( \prod_{j=1}^{m+1} T_j \right)^{m+1} \right), & p = 2m + 1, \\ \end{cases}
\]

where the \( T_j \) are independently distributed, and are defined by (4.1).
When \( p = 2m + 1 \), the exact result
\[
P\left\{ -2 \log \frac{r_{23}}{z} \right\} = P\left\{ \chi^2_{p-1} \leq \frac{N-1}{N} z \right\}
\]
is obtained from Lemma 3 with the association \((r, q, M) = (m, \frac{n}{2}, N)\).

When \( p = 2m \), we have from (4.2) that
\[
B_{23}^h = \left[ \frac{\Gamma \left( \frac{A - \frac{1}{2}}{2} \right)}{\Gamma \left( \frac{A + \frac{1}{2}}{2} \right)} \right]^m, \quad A = \frac{N}{2} (1 + h).
\]

From (2.8) make the association
\[
- x = \frac{N}{2}, \quad \rho = 1 - \frac{2b}{N}, \quad \xi_1 = \ldots = \xi_m = -\frac{1}{2}, \quad \eta_1 = \ldots = \eta_m = \frac{1}{2}.
\]

Hence \( f = p - 1, \ b = (2p - 1)/4(p - 1) \).

\[
f = 2 \left[ \frac{m - 1}{2} + \frac{m}{2} \right] = p - 1
\]

\[
b = \frac{1}{f} \left[ \frac{m}{4} - \frac{m - 1}{4} + \frac{f}{2} \right] = \frac{2p - 1}{4(p - 1)}.
\]

From (2.9) and (2.10),
\[
6(x - b)^2 \omega = -m \left[ B_3 \left( b + \frac{1}{2} \right) - B_3 \left( b - \frac{1}{2} \right) \right] + \left[ B_3 \left( b + \frac{1}{2} \right) - B_3 (b) \right]
\]

\[= a_0 + a_1 b + a_2 b^2,
\]

where \( a_2 = \frac{3f}{2}, \ a_1 = 3fb, \ a_0 = -3p/8 \); hence

\[
\omega = \frac{p - 1}{4 \left[ 2n(p - 1) - 1 \right]^2}.
\]
The noncentral asymptotic distribution is obtained as in §4.1 to yield

\[(4.6) \quad \mathcal{L} \left( \sqrt{N} \left[ \log \frac{2}{\bar{f}_{23}} - \log \frac{p}{2} \frac{1}{1 + \delta_j} \right] \right) \rightarrow N \left( 0, p - 1 - \frac{p-1}{1 \left(1 + \delta_j\right)^2} \right). \]

5. Tests for Means and Covariances

There are various combinations of tests which can be performed. We only consider two tests, (i) that the means are zero and the covariance matrix is circular, (ii) that the means are equal and the covariance matrix is circular, both against general alternatives. If we let

\[\Omega_1 = \left\{ \mu, \Sigma: \mu = 0, \Sigma = \Sigma_c \right\},\]

\[\Omega_2 = \left\{ \mu, \Sigma: \mu = \mu^*, \Sigma = \Sigma_c \right\},\]

\[\Omega = \left\{ \mu, \Sigma: -\infty < \mu_j < \infty, \Sigma > 0 \right\},\]

then we wish to test that \((\mu, \Sigma) \in \Omega_1 \) or \(\Omega_2\) versus \(\Omega\). Denote the IRS by \(I_1\) and \(I_2\), respectively. Because of a nesting of the various regions in §3 and §4, we obtain (Anderson (1958), Lemma 10.3.1) that

\[(5.1) \quad I_1 = f_{13, 34}^{\lambda_j}, \quad I_2 = f_{23, 34}^{\lambda_j}.\]

5.1 Simultaneous Test that Means are Zero and Covariance Matrix is Circular

Relating (5.1) to (3.9) and (4.1), we obtain

\[\frac{2}{\sqrt{N}} |R| = 2^{2(p-m-1)} \prod_{j=1}^{p} \left( \frac{v_j}{v_j + w_j} \right),\]
where $R$ is the sample correlation matrix corresponding to $V$. Using the development of §3.3, we obtain the representation
\[
\mathcal{L}(\mathbf{L}_1^N) = \mathcal{L}(\mathbf{U}_j^1),
\]
where the $U_j$'s are independently distributed,
\[
(5.2) \quad \mathcal{L}(U_j) = \begin{cases} 
\beta \left( \frac{n - j + 1}{2}, \frac{j}{2} \right), & j = 1, \ldots, m + 1, \\
\beta \left( \frac{n - j + 1}{2}, \frac{j + 1}{2} \right), & j = m + 2, \ldots, p.
\end{cases}
\]

When the hypothesis is true, we obtain from (5.2),
\[
\text{EL}_1^h = c \frac{\prod_{j=1}^{p} \Gamma \left( A - \frac{j}{2} \right)}{\left[ \Gamma(A) \right]^{m+1} \left[ \Gamma \left( A + \frac{1}{2} \right) \right]^{p-m-1}}, \quad A = \frac{N}{2}(1 + h),
\]
where $c$ is a normalizing constant so that $E L_1^0 = 1$.

From (2.8) we make the association
\[
x = \frac{N}{2}, \quad e = 1 - \frac{2b}{N}, \quad \xi_j = - \frac{j}{2}, \quad j = 1, \ldots, p,
\]
\[
\eta_1 = \ldots = \eta_{m+1} = 0, \quad \eta_{m+2} = \ldots = \eta_{p-m-1} = \frac{1}{2}.
\]
Hence
\[
P \left\{ - 2 \rho \log L_1 \leq z \right\} = (1 + a) P \left\{ \chi_f^2 \leq z \right\} - a P \left\{ \chi_{f+b}^2 \leq z \right\} + o(N^{-3}),
\]
where $f$ and $b$ are determined from (2.10), and (2.11):
\[ f = p - m - 1 + \frac{1}{2} p (p + 1), \quad b = \frac{1}{24p} [p(p + 1)(2p + 7) + 6(p - m - 1)]. \]

From (2.9) and (2.10):

\[
6(x - b)^2 \omega = \sum_{1}^{p} \left[ B_3 \left( b - \frac{j}{2} \right) - B_3(b) \right] - (p - m - 1) \left[ B_3 \left( b + \frac{1}{2} \right) - B_3(b) \right]
\]

\[= a_0 + a_1 b + a_2 b^2, \]

where \[a_2 = -\frac{3p}{2}, \quad a_1 = 3bf, \quad a_0 = -p(p + 1)(p^2 + p + 4)/32.\]

Hence

\[
6(x - b)^2 \omega = -\frac{p(p + 1)(p^2 + p + 4)}{32} + \frac{3}{2} bf.
\]

**Remark**

The statistics \( \chi_{13} \) and \( \chi_{34} \) are not independent (which is often the case with nested hypotheses) so that the subhypotheses should not be tested individually.

Because \( \text{plim} \log |R| = 0 \) under the hypothesis that \( \Sigma \) is circular, the asymptotic distribution of \( \frac{2}{\text{\(L_2\)}} \text{\(2^{-2(p-m-1)}\)} \) is the same as the asymptotic distribution of \( \frac{2}{\text{\(L_3\)}} \) given by (4.3).

### 5.2 Simultaneous Test that Means are Equal and Covariance Matrix is Circular

Relating (5.1) to Sections (3.3) and (4.2), we obtain

\[
\frac{2}{\text{\(L_2\)}} = 2^{2(p-m-1)} |R| \frac{p}{2} \left( \frac{v_j}{v_j + w_j} \right),
\]

where \( R \) is the correlation matrix. Following the development of \( \chi_{33} \), we obtain the representation \( \chi \left( \frac{2}{\text{\(L_2\)}} \right) = \chi \left( \frac{p}{2} U_j \right) \), where the \( U_j \)'s are defined by (5.2).

When the hypothesis is true, we obtain from (5.2)
\[ EL_2^h = c \frac{\prod_{i=1}^{p} \prod_{j=1}^{m} \left( A - \frac{j}{2} \right)}{2^{p-m+1}} A = \frac{N}{2}(1 + h), \]

and \( c \) is a normalizing constant so that \( EL_2^0 = 1 \).

From (2.8) make the association

\[ x = \frac{N}{2}, \quad p = 1 - \frac{2b}{N}, \quad \xi_j = -\frac{(j + 1)}{2}, \quad j = 2, \ldots, p, \]

\[ \eta_1 = \ldots = \eta_m = 0, \quad \eta_{m+1} = \ldots = \eta_{p-m-1} = \frac{1}{2}. \]

Hence

\[ P\left\{ -2p \log L_2 \leq z \right\} = (1 + \omega)P\left\{ \chi_{p}^2 \leq z \right\} - \omega P\left\{ \chi_{p+m}^2 \leq z \right\} + O(N^{-3}), \]

where \( f \) and \( b \) are determined from (2.10), and (2.11):

\[ f = \frac{1}{2} \left( p(p + 1) + 2(p - m - 2) \right), \quad b = \frac{1}{24f} \left[ p(p + 1)(2p + 7) + 6(p - m) - 24 \right]. \]

From (2.12) and (2.13):

\[ 6(x - b)^2 = \sum_{i=1}^{p} \left[ B_3\left( b + \frac{j}{2} \right) - B_3(b) \right] - (p - m) \left[ B_3\left( b + \frac{1}{2} \right) - B_3(b) \right] \]

\[ = a_0 + a_1 b + a_2 b^2, \]

where \( a_2 = -\frac{3}{2} f, \quad a_1 = 3bf, \quad a_0 = \frac{-p(p + 1)^2(p + 3)}{32} + \frac{3}{4} \). Hence

\[ 6(x - b)^2 = \frac{48fb^2 + 24 - p(p + 1)^2(p + 8)}{32}. \]

The asymptotic distribution of \( \frac{L_2^2}{N} - \frac{2(p-m-1)}{2} \) is the same as the asymptotic distribution of \( B_{23}^m \) given by (4.6).

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REFERENCES


Bartlett, M. S. "Properties of Sufficiency and Statistical Tests."

Box, G. E. P. "A General Distribution Theory for a Class of Likelihood

University Press, 1946.

Geisser, S. "Multivariate Analysis of Variance for a Special Covariance

Girshick, M. A. "The Distribution of the Ellipticity Statistic $L_e$ when
the Hypothesis is False." *Terr. Magn. Atmos. Elect.*, XLVI (1941),
pp. 455-457.


Mauchly, J. W. "Significance Test for Sphericity of a Normal $n$-Variate

Press, S. J. *Some Hypotheses Testing Problems Involving Multivariate
Normal Distributions with Unequal and Intraclass Structured Matrices.*
Technical Report No. 12, Stanford University, Stanford, California,
1964.

Rao, C. R. *Advanced Statistical Methods in Biometric Research.* New York:
John Wiley and Sons, Inc., 1952.
