MINIMAL CONDITIONS FOR WEAK CONVERGENCE OF
THE SAMPLE SPECTRAL DISTRIBUTION FUNCTION

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In proving the weak convergence of the normalized empirical spectral distribution function of a stationary process to a Gaussian process, Rosenblatt and Grenander (1952), (1957) assumed that the eighth-order moment of the process is finite. This paper shows that existence of the fourth-order moment is sufficient when the process consists of independently identically distributed variables. For weak convergence of the normalized standardized empirical spectral distribution, only the second moment is needed.

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Let \( \{y_t\} \) be a stationary stochastic process with \( \mathbb{E}y_t = 0 \) and \( \mathbb{E}y_t y_{t+h} = \sigma(h) \). The spectral distribution function is

\[
H(\lambda) = \frac{1}{\pi} \left[ \sigma(0) \lambda + 2 \sum_{h=1}^{\infty} \frac{\sin \lambda h}{h} \sigma(h) \right], \quad 0 \leq \lambda \leq \pi.
\]

If \( y_1, \ldots, y_T \) is a sample from \( \{y_t\} \), define the sample covariance as \( c_h = \frac{1}{T-h} \sum_{t=1}^{T-h} y_t y_{t+h} / T, \quad h = 0, 1, \ldots, T - 1 \). The sample spectral distribution is

\[
H_T(\lambda) = \frac{1}{\pi} \left( c_0 \lambda + 2 \sum_{h=1}^{T-1} \frac{\sin \lambda h}{h} c_h \right), \quad 0 \leq \lambda \leq \pi.
\]

Under suitable conditions \( \sqrt{T} \left[ H_T(\lambda) - H(\lambda) \right] \) converges weakly to a Gaussian stochastic process. A frequently quoted condition is that of Grenander and Rosenblatt (1952), (1957), namely, \( \mathbb{E}y_t^8 < \infty \) when the \( y_t \)'s are independently identically distributed (iid). We shall show that when the \( y_t \)'s are iid, the condition \( \mathbb{E}y_t^4 < \infty \) suffices.

The standardized spectral distribution

\[
F(\lambda) = \frac{1}{\pi} \left[ \lambda + 2 \sum_{h=1}^{\infty} \frac{\sin \lambda h}{h} \frac{\sigma(h)}{\sigma(0)} \right]
\]

is useful for specifying patterns of dependence [Anderson (1993)]. The sample standardized spectral distribution is

\[
F_T(\lambda) = \frac{1}{\pi} \left[ \lambda + 2 \sum_{h=1}^{T-1} \frac{\sin \lambda h}{h} \frac{c_h}{c_0} \right].
\]
We shall show that $\sqrt{T} \left[ F_T(\lambda) - F(\lambda) \right]$ converges weakly if $\mathbb{E} y_t^2 < \infty$ when the $y_t$'s are iid.

**Theorem 1.** If $\mathbb{E} y_t^4 < \infty$ and the $y_t$'s are iid,

$$
(5) \quad \sqrt{T} [H_T(\lambda) - H(\lambda)] = \frac{\sqrt{T}}{\pi} \left\{ c_0 - \sigma(0) \right\} \lambda + 2 \sum_{h=1}^{T-1} \frac{\sin \lambda h}{h} c_h \overset{w}{\to} U(\lambda), \quad 0 \leq \lambda \leq \pi,
$$

where $U(\lambda)$ is a Gaussian process with $\mathbb{E} Z(\lambda) = 0$ and

$$
(6) \quad \mathbb{E} U(\lambda) U(\nu) = 2 \frac{\sigma^4}{\pi} \min(\lambda, \nu) + \frac{\lambda \nu}{\pi^2} \kappa_4,
$$

where $\sigma^2 = \sigma(0)$ and $\kappa_4 = \mathbb{E} y_t^2 - 3 \sigma^4$.

**Proof.** Convergence in distribution of $\sqrt{T} \left[ F_T(\lambda_1) - F(\lambda_1) \right], \cdots, \sqrt{T} \left[ F_T(\lambda_k) - F(\lambda_k) \right]$ for any $\lambda_1, \cdots, \lambda_k$ and $k$ was shown by Grenander and Rosenblatt. The moment condition is used for tightness. We want to show that (5) converges uniformly in $\lambda$. Our proof is patterned after the method used by Harold Cramér in his lectures on Stochastic Processes in Stockholm, 1947-48, to develop Brownian motion and is similar to that used by Grenander and Rosenblatt (1952), (1957).

**Lemma 1.** If the $y_t$ are iid and $\mathbb{E} y_t^2 < \infty$, then for any $\varepsilon > 0$ and $\eta > 0$ there exists a $k_0$ such that for $k > k_0$

$$
(7) \quad \Pr \left\{ \sup_{0 \leq \lambda \leq \pi} \left| \frac{\sqrt{T}}{\pi} \sum_{h=2^k}^{2^{k+1}} \frac{\sin \lambda h}{h} c_h \right| > \varepsilon \right\} < \eta
$$

for all $T > 2^k$.

**Proof of Lemma 1.** Consider

$$
(8) \quad \psi_m(\lambda) = \frac{\sqrt{T}}{\sum_{h=m+1}^{2m} \frac{\sin \lambda h}{h} c_h}
$$
for arbitrary $m$. (We define $c_h = 0$ for $h \geq T$.) Then

\[
|\psi_m(\lambda)|^2 \leq \sqrt{T} \sum_{h=m+1}^{2m} \frac{e^{i\lambda h}}{h} \frac{c_h}{c_g} \]

\[
= T \sum_{g, h=m+1}^{2m} \frac{e^{i\lambda(g-h)}}{gh} c_g c_h
\]

\[
= T \sum_{h=m+1}^{2m} \frac{c_h^2}{h^2} + 2TR \left[ \sum_{g > h=m+1}^{2m} \frac{e^{i\lambda(g-h)}}{gh} c_g c_h \right]
\]

\[
= T \sum_{h=m+1}^{2m} \frac{c_h^2}{h^2} + 2TR \left[ \sum_{j=1}^{m-1} e^{i\lambda j} \sum_{h=m+1}^{2m-j} \frac{c_h c_h+j}{h(h+j)} \right]
\]

\[
\leq \frac{T}{m^2} \sum_{h=m+1}^{2m} c_h^2 + 2T \sum_{j=1}^{m-1} \left| \sum_{h=m+1}^{2m-j} \frac{c_h c_h+j}{h(h+j)} \right|
\]

\[
= \Psi_m^2,
\]

say. Note that $\Psi_m^2$ is independent of $\lambda$. Define

\[
U_j = T \sum_{h=m+1}^{2m-j} \frac{c_h c_h+j}{h(h+j)}
\]

\[
= \frac{1}{T} \sum_{h=m+1}^{2m-j} \sum_{t=1}^{T-h} \sum_{s=1}^{T-(h+j)} \frac{y_t y_{t+h} y_{s+h+j}}{h(h+j)}
\]

\[
= X_j + Y_j,
\]

where $X_j = X_{1j} + X_{2j} + X_{3j} + X_{4j},$

\[
X_{1j} = \frac{1}{T} \sum_{h=m+1}^{2m-j} \frac{1}{h(h+j)} \sum_{t=1}^{T-(h+j)} y_t^2 y_{t+h} y_{t+h+j},
\]

\[
X_{2j} = \frac{1}{T} \sum_{h=m+1}^{2m-j} \frac{1}{h(h+j)} \sum_{t=h+j+1}^{T-h} y_t^2 y_{t+h} y_{t-(h+j)},
\]

\[
X_{3j} = \frac{1}{T} \sum_{h=m+1}^{2m-j} \frac{1}{h(h+j)} \sum_{t=1}^{T-(2h+j)} y_t y_{t+h} y_{t+2h+j},
\]

\[
X_{4j} = \frac{1}{T} \sum_{h=m+1}^{2m-j} \frac{1}{h(h+j)} \sum_{t=1}^{T-(h+j)} y_t y_{t+h} y_{t+h+j}.
\]
\begin{align}
(14) \quad X_{4j} &= \frac{1}{T} \sum_{h=m+1}^{2m-j} \frac{1}{h(h+j)} \sum_{t=j+1}^{T-h} y_t y_{t+h} y_{t-j}, \\
(15) \quad Y_j &= \frac{1}{T} \sum_{h=m+1}^{2m-j} \sum_{t=1}^{T-h} \sum_{s=1}^{T-(h+j)} \frac{y_t y_{t+h} y_{s} y_{s+h+j}}{h(h+j)}.
\end{align}

Since each term in $Y_j$ has mean zero and is uncorrelated with the others, the variance of $Y_j$ is the sum of the variances of the individual terms. We have

\begin{align}
(16) \quad EY_j^2 &= \frac{1}{T^2} \sum_{h=m+1}^{2m-j} \sum_{t=1}^{T-h} \sum_{s=1}^{T-(h+j)} \frac{1}{h^2(h+j)^2} \sigma^8 \\
&< \frac{1}{T^2} \sum_{h=m+1}^{2m-j} \sum_{t=1}^{T-h} \sum_{s=1}^{T-(h+j)} \frac{1}{h^2(h+j)^2} \sigma^8 \\
&= \frac{(T-h)(T-h-j)}{T^2} \sum_{h=m+1}^{2m-j} \frac{1}{h^2(h+j)^2} \sigma^8.
\end{align}

By using Schwarz inequality, we have

\begin{equation}
(17) \quad E|Y_j| \leq \sqrt{\sum_{h=m+1}^{2m-j} \frac{1}{h^2(h+j)^2} \sigma^4}.
\end{equation}

Note that the highest moment of the $y_t$'s that is involved is the second since the four indices of the $y_t$'s in each summand of $Y_j$ are different.

To treat $X_j$ we use the following lemma:

**Lemma 2.** If the random variables $X_1$ and $X_2$ are uncorrelated with mean zero, then

\begin{equation}
(18) \quad E|X_1| \leq \left[ E(X_1 + X_2)^2 \right]^{\frac{1}{2}} = \left[ EX_1^2 + EX_2^2 \right]^{\frac{1}{2}}.
\end{equation}

Exchanging the summations in (11) allows us to write

\begin{equation}
(19) \quad X_{1j} = \frac{1}{T} \sum_{t=1}^{T-j-(m+1)} y_t^2 Z_{1t},
\end{equation}
where

\begin{equation}
Z_{1t} = \sum_{h \in S_{1t}} \frac{y_{t+h}y_{t+h+j}}{h(h+j)}
\end{equation}

and \( S_{1t} \) is an integer set contained in \( \{m+1, m+2, \ldots, 2m - j\} \). Consider \( Z_{1t} \) and \( \sum_{h=m+1}^{2m-j} y_{t+h}y_{t+h+j}/[h(h+j)] \) as \( X_1 \) and \( X_1 + X_2 \), respectively, in Lemma 1; then by using Lemma 1, we obtain

\begin{equation}
E|Z_{1t}| \leq \left[ E\left( \sum_{h=m+1}^{2m-j} \frac{y_{t+h}y_{t+h+j}}{h(h+j)} \right)^2 \right]^{\frac{1}{2}}
\end{equation}

\begin{equation}
= \left[ \sum_{h=m+1}^{2m-j} \frac{1}{h^2(h+j)^2} \right]^{\frac{1}{2}} \sigma^2.
\end{equation}

Since \( y_t \) and \( Z_{1t} \) in (19) are independent and \( T - j - (m + 1) < T \),

\begin{equation}
E|X_{1j}| \leq \left[ \sum_{h=m+1}^{2m-j} \frac{1}{h^2(h+j)^2} \right]^{\frac{1}{2}} \sigma^2.
\end{equation}

To prove the same property for \( X_{2j}, X_{3j}, \) and \( X_{4j} \), we follow the same procedure as that used for \( X_{1j} \). We can define \( Z_{kt} \) of \( X_{kj}, k = 2, 3, \) and 4, similar to that for \( k = 1 \). Since for any fixed \( t \), \( Z_{kt} \) consists of at most \( 2m - j - (m + 1) + 1 \) uncorrelated random variables with coefficients \( 1/[h^2(h+j)^2]\), \( m + 1 \leq h \leq 2m - j \), we obtain

\begin{equation}
E|Z_{kj}| \leq \left[ \sum_{h=m+1}^{2m-j} \frac{1}{h^2(h+j)^2} \right]^{\frac{1}{2}} \sigma^2, \quad k = 1, 2, 3, 4.
\end{equation}

Furthermore, the number of possible \( Z_{kt} \) terms in each \( X_{kj} \), \( k = 1, 2, 3, 4 \), is less than \( T \); so

\begin{equation}
E|X_{kj}| \leq \left[ \sum_{h=m+1}^{2m-j} \frac{1}{h^2(h+j)^2} \right]^{\frac{1}{2}} \sigma^2, \quad k = 1, 2, 3, 4.
\end{equation}

Therefore, for \( m \leq T \)

\begin{equation}
E|\sum_{j=1}^{m-1} X_j| \leq \sum_{j=1}^{m-1} E|X_j|
\end{equation}

\begin{equation}
\leq \sum_{j=1}^{m-1} \sum_{k=1}^{4} E|X_{kj}|
\end{equation}

\begin{equation}
\leq 4 \sum_{j=1}^{m-1} \left[ \sum_{h=m+1}^{2m-j} \frac{1}{h^2(h+j)^2} \right]^{\frac{1}{2}} \sigma^2.
\end{equation}
Then
\[
\mathcal{E}\Psi_m^2 \leq \frac{1}{m^2} T \sum_{h=m+1}^{2m} \mathcal{E}c_h^2 + 2 \sum_{j=1}^{m-1} \left[ \mathcal{E}|Y_j| + \sum_{i=1}^{4} \mathcal{E}|X_{ji}| \right]
\]
\[
\leq \frac{\sigma^4}{m} + 2\text{const} \sum_{j=1}^{m-1} \left[ \sum_{h=m+1}^{2m-j} \frac{1}{h^2(h+j)^2} \right]^{\frac{1}{2}} \sigma^4
\]
\[
\leq \frac{\sigma^4}{m} + \text{const} \sum_{j=1}^{m-1} \frac{1}{m^{3/2}}
\]
\[
< \frac{\text{const}}{m^{1/2}}.
\]

Note that the last inequality does not depend on \(T\).

From (26) Rosenblatt and Grenander (1957) show that with arbitrarily high probability \(\sum_{n=k}^{\log T+1} \Psi_{2n}^2\) is arbitrarily small for \(k\) sufficiently large. Lemma 1 follows from this fact.

The rest of the proof of Theorem 1 follows the proof of Theorem 2.2 of Durlauf (1991).

Tightness follows from Lemma 1. \(\blacksquare\)

Note that the condition \(\mathcal{E}y_t^4 < \infty\) is needed because it is involved in the variance of \(c_0\).

**Theorem 2.** If \(\mathcal{E}y_t^2 < \infty\) and the \(y_t\)'s are iid,

\[
\sqrt{T} \left[ F_T(\lambda) - F(\lambda) \right] = \frac{2}{\pi c_0} \sqrt{T} \sum_{h=1}^{T-1} \frac{\sin \lambda h}{h} c_h \xrightarrow{w} Z(\lambda), \quad 0 \leq \lambda \leq \pi,
\]

where \(Z(\lambda)\) is a Gaussian process with \(\mathcal{E}Z(\lambda) = 0\) and

\[
\mathcal{E}Z(\lambda)Z(\nu) = 2 \left[ \frac{\min(\lambda, \nu)}{\pi} - \frac{\lambda \nu}{\pi^2} \right].
\]

**Proof.** Because the \(y_t\)'s are iid with \(\mathcal{E}y_t^2 = \sigma^2 < \infty, c_0 \xrightarrow{p} \sigma^2\). Then Theorem 2 follows from Lemma 1. \(\blacksquare\)

The condition that the \(y_t\)'s are iid can be relaxed. For Lemma 1 we only need that the \(y_t\)'s are independent and the \(\mathcal{E}y_t^2\)'s are bounded.
REFERENCES


