JACOBIANS OF MATRIX TRANSFORMATIONS AND FUNCTIONAL EQUATIONS

BY

I. OLKIN and A. R. SAMPSON

Stanford University

TECHNICAL REPORT NO. 40
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Jacobians of Matrix Transformations and Functional Equations

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1. Introduction. Jacobians of matrix transformations have been obtained by a variety of methods, the most notable being based on a chain of transformations for which each part can more easily be evaluated, e.g., by differential techniques, Deemer and Olkin (1951), Olkin (1953), Olkin and Roy (1954), Hua (1958), or by induction, Hsu (1953), Hua (1958), and Jack (1964-65). The differential techniques are relatively straightforward, but may involve detailed computations, which to a certain extent are avoided by induction. The latter was employed most extensively by Jack who provided simple new proofs for many Jacobian results. In the present paper we also consider a chain of transformations, from which a functional equation is induced on the relevant spaces. The solution of the functional equation (except for constants which may readily be evaluated from special cases), then yields the Jacobian. The method has its basis in a derivation of the Wishart distribution by Rasch (1948), and is also contained implicitly in some of the results of Hua and Jack.
2. Preliminaries. All matrices are real and of full rank; lower case letters represent vectors or scalars; capital letters denote matrices. By $A: p \times q$ we mean the general $p \times q$ matrix; $A'$ denotes the transpose of $A$, and $|A|$ its determinant.

Certain classes of matrices recur throughout and we reserve the following notation for these:
- $\bar{A}$ denotes a symmetric matrix,
- $\hat{A}$ denotes a skew-symmetric matrix,
- $\tilde{A}$ denotes a lower triangular matrix,
- $D_x = \text{diag}(x_1, \ldots, x_n)$ denotes the diagonal matrix with elements $x_1, \ldots, x_n$,
- $D_j(a) = \text{diag}(1, \ldots, 1, a, 1, \ldots, 1)$, where the $a$ occurs in the $j$-th position.

The Greek letters $\Gamma$ and $\Delta$ denote orthogonal matrices.

If a matrix $Y$ is a one-to-one function of a matrix $X$, $Y = f(X)$, the Jacobian $\partial(Y)/\partial(X)$ is denoted by $J(X \to Y)$. Because the ordering of the variables is arbitrary, the value of the Jacobian can vary in sign, and by $J(X \to Y)$ we mean the value of the Jacobian except for sign.

We make use of the following standard results. If $Y = f(X)$ and $Z = g(Y) = g(f(X))$, then

\[(2.1) \quad J(X \to Z) = J(X \to Y) J(Y \to Z);\]

and in particular, if $J(X \to Y) \neq 0$, 

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(2.2) \[ J(X \rightarrow Y) \cdot J(Y \rightarrow X) = 1. \]

If \( Y = f(X), \ Z = g(W) \), then

(2.3) \[ J(X, W \rightarrow Y, Z) = J(X \rightarrow Y) \cdot J(W \rightarrow Z). \]

For suitably smooth transformations

(2.4) \[ J(X \rightarrow Y) = J(dX \rightarrow dY), \]

so that the Jacobian of a nonlinear transformation may be obtained from the Jacobian of the linear transformation in the differentials. This will generally simplify the computations.

3. Induced functional equations. In this section we solve a number of functional equations which arise from linear transformations in the following manner. Consider the transformation

(3.1) \[ Y = AX, \]

where \( X \) and \( Y \) are \( p \times q \) matrices, and \( A \) is a \( p \times p \) non-singular matrix. To generate a functional equation for the Jacobian, let

\[ Z = BY, \]
so that

\[(3.2) \quad J(X \to Z) = J(X \to Y) \cdot J(Y \to Z).\]

As the transformation is linear, \(J(X \to Y)\) is a function of \(A\) alone, say \(h(A)\). Then from (3.2),

\[(3.3) \quad h(AB) = h(A) \cdot h(B),\]

for all non-singular \(p \times p\) matrices \(A\) and \(B\). The solution of (3.3) then yields (except for a constant) the Jacobian of the transformation (3.1). By considering other transformations we find that it is necessary to solve (3.3) over the class of diagonal, triangular, orthogonal, and symmetric matrices.

**Theorem 1.** Let \(\mathcal{D}_p\) be the space of non-singular \(p \times p\) diagonal matrices, and \(h: \mathcal{D}_p \to \mathbb{R}\) such that

1. \(h(D_a) \cdot h(D_b) = h(D_{ab})\),
2. \(h(\cdot)\) is continuous,

then \(h(D_a) = \prod_{1}^{p} \frac{c_i}{a_i}\).

**Proof.** If \(D_\epsilon = \text{diag}(\epsilon_1, \ldots, \epsilon_p)\), where \(\epsilon_i = \pm 1\), \(i = 1, \ldots, p\),
then \(h^2(D_\epsilon) = h(D_\epsilon^2) = h(I) = 1\). Hence by an appropriate choice of \(D_\epsilon\), we may assume that for all \(D_a, D_b\), \(a_i > 0, \ b_i > 0\), \(i = 1, \ldots, p\).
Letting \( a_i = \exp u_i, \ b_i = \exp v_i, \ i = 1, \ldots, p, \) and
\( g(u_1, \ldots, u_p) = \log h(D_a), \) we obtain the vector version of the Cauchy equation \( g(u) + g(v) = g(u + v) \) for which the solution is
\( g(u) = \sum c_i u_i \) (e.g., see Aczél (1966) p. 348), which yields the result. \( \| \)

**Theorem 2.** Let \( U_p \) be the space of \( p \times p \) lower triangular matrices with unit diagonals, and let \( h: U_p \rightarrow \mathbb{R} \) such that

(i) \( h(\hat{U}_1) h(\hat{U}_2) = h(\hat{U}_1 \hat{U}_2), \)

(ii) \( h(\cdot) \neq 0, \)

then \( h(\hat{U}) = 1. \)

**Proof.** Define \( U_{ij} \) to be the matrix with unit diagonal elements, \( u_{ij} \) in the \((i,j)\)-th position, and zeros elsewhere. Then

\[ \hat{U} = \hat{U}_1 \ldots \hat{U}_p, \]

where \( \hat{U}_j = \hat{U}_{jj} \ldots \hat{U}_{jj}. \) Note that

\[ \hat{U}_{ij}^2 = D_i(2) \hat{U}_{ij} D_i(\frac{1}{2}), \]

so that \( h(\hat{U}_{ij}^2) = h^2(\hat{U}_{ij}) = h(D_i(2)\hat{U}_{ij}D_i(\frac{1}{2})) = h(\hat{U}_{ij}). \) Consequently,
\( h(\hat{U}_{ij}) = 1, \) from which the result follows. \( \| \)

**Theorem 3.** Let \( \mathcal{G}_p \) be the space of non-singular \( p \times p \) lower triangular matrices, and let \( h: \mathcal{G}_p \rightarrow \mathbb{R} \) such that

\[ \]
(i) \( h(\tilde{T}_1) h(\tilde{T}_2) = h(\tilde{T}_1 \tilde{T}_2), \)

(ii) \( h(\cdot) \neq 0 \) is continuous,

then \( h(\tilde{T}) = \prod_{i=1}^{p} t_i^{c_i}. \)

Proof. Every matrix \( \tilde{T} \) may be written as \( \tilde{T} = \tilde{U} D_t \) where \( U \in \mathcal{U}_p, \)
\( D_t \in \mathcal{O}_p. \) Furthermore, (i) implies

\[ h(\tilde{U}_1) h(\tilde{U}_2) = h(\tilde{U}_1 \tilde{U}_2), \quad \tilde{U}_1, \tilde{U}_2 \in \mathcal{U}_p, \]

\[ h(D_a) h(D_b) = h(D_a D_b), \quad D_a, D_b \in \mathcal{O}_p. \]

By (i) and Theorems 1, 2, \( h(\tilde{T}) = h(\tilde{U}) h(D_t) = \prod_{i=1}^{c_i} t_i. \)

Theorem 4. Let \( \mathcal{O}_p \) be the space of \( p \times p \) orthogonal matrices, and let \( h: \mathcal{O}_p \rightarrow \mathbb{R} \) such that

(i) \( h(\Gamma_1) h(\Gamma_2) = h(\Gamma_1 \Gamma_2), \)

(ii) \( h(\cdot) \geq 0, \)

then \( h(\Gamma) = 1. \)

Proof. Every orthogonal matrix \( \Gamma \) may be written as a product
\( \Gamma = \prod \overline{\Gamma}_i, \) of symmetric orthogonal matrices, \( \overline{\Gamma}_i. \) But for any such
\( \overline{\Gamma}_i, \) \( h^2(\overline{\Gamma}_i) = h(\overline{\Gamma}_i^2) = h(I) = 1, \) from which the result follows. \|

Theorem 5. Let \( \mathcal{A}_p \) be the space of non-singular \( p \times p \) matrices, and let \( h: \mathcal{A}_p \rightarrow \mathbb{R} \) such that
(i) \( h(A) h(B) = h(AB) \),
(ii) \( h(\cdot) \geq 0 \) is continuous,
then \( h(A) = |A|^c \).

Proof. There are various ways to prove the result depending on the particular representation used. We here present several alternatives; these are discussed in MacDuffee (1946, p. 75-80).

Every matrix \( A \in \mathbb{A}_p \) may be represented as

\[(3.3) \quad A = \overline{S}_1 \overline{S}_2,\]

where \( \overline{S}_1, \overline{S}_2 \in \mathcal{S}_p \), the space of \( p \times p \) symmetric matrices;

\[(3.4) \quad A = \tilde{T} \tilde{U}' \quad \tilde{T} \in \mathcal{T}_p, \quad \tilde{U} \in \mathcal{U}_p;\]

an equivalent representation is

\[A = \tilde{U}_1 D_2 \tilde{U}_2', \quad \tilde{U}_1 \tilde{U}_2 \in \mathcal{U}_p, \quad D t \in O_p;\]

\[(3.5) \quad A = \tilde{T} \Gamma, \quad \tilde{T} \in \mathcal{T}_p, \quad \Gamma \in O_p;\]

\[(3.6) \quad A = \Gamma \tilde{T} \Gamma', \quad \tilde{T} \in \mathcal{T}_p, \quad \Gamma \in O_p;\]

\[(3.7) \quad A = \overline{S} \Gamma, \quad \overline{S} \in \mathcal{S}_p, \quad \Gamma \in O_p;\]

\[(3.8) \quad A = \Gamma D_\alpha \Delta, \quad \Gamma, \Delta \in O_p, \quad D_\alpha \in O_p.\]
To use (3.3), \( h(A) = h(\overline{S}_1) h(\overline{S}_2) \). Since \( \overline{S} = \Gamma D_\sigma \Gamma' \), \( \Gamma \in \mathcal{G}_p \), \( D_\sigma \in \mathcal{G}_p \), (i) implies

\[
h(\overline{S}_1) = h(\Gamma D_\sigma \Gamma') = h(D_\sigma).
\]

By Theorem 1, \( h(D_\sigma) = \Pi c_i^\sigma \). Because of invariance under permutations, i.e., \( h(D_\sigma) = h(P D_\sigma P') \), \( c_1 \ldots c_p = c_1 \ldots c_p \) for any permutation of the indices, it follows that \( c_j = c, \ j = 1, \ldots, p \).

To use (3.5), we have \( h(A) = h(\tilde{T}) h(\Gamma) \), so that

\[
h(\tilde{T}_1) h(\tilde{T}_2) = h(\tilde{T}_1 \tilde{T}_2), \quad \tilde{T}_1, \tilde{T}_2 \in \mathcal{R}_p.
\]

\[
h(\Gamma_1) h(\Gamma_2) = h(\Gamma_1 \Gamma_2), \quad \Gamma_1, \Gamma_2 \in \mathcal{G}_p.
\]

But these equations are solved in Theorems 3 and 4. Thus

\[
h(A) = \Pi_{i=1}^p t_{ii}^c.
\]

By invariance of diagonal matrices under permutations, i.e.,

\[
h(D) = h(PDP') \quad \text{for all permutation matrices,} \quad c_1 = \ldots = c_p = c.
\]

From this \( h(A) = \Pi_{i=1}^p t_{ii}^c = |\tilde{T}|^c = |A|^c \).

Proofs using the other representations may be constructed analogously.
Remark. In the above we have used the following procedure. If

\[ h(A_1 A_2) = h(A_1) h(A_2) \text{ for } A_1, A_2 \in \mathcal{A}, \]
and if for any \( A \in \mathcal{A} \), there exists a \( P \in \mathcal{P} \subset \mathcal{A} \), \( Q \in \mathcal{Q} \subset \mathcal{A} \) such that \( A = PQ \), then it is immediate that

\[ \begin{align*}
(i) & \quad h(P_1 P_2) = h(P_1) h(P_2) \text{ for } P_1, P_2 \in \mathcal{P}, \\
(ii) & \quad h(Q_1 Q_2) = h(Q_1) h(Q_2) \text{ for } Q_1, Q_2 \in \mathcal{Q}.
\end{align*} \]

Suppose \( h_1(P) \) and \( h_2(Q) \) constitute a class of solutions of (ii) and (iii), respectively, then for some \( h_1 \) and \( h_2 \), \( h^*(A) = h_1(P) h_2(Q) \) is a solution of (i).

An example of this procedure occurs in the proof of Theorem 3, in which \( \mathcal{P} = \mathcal{P} \) and \( h_1(\mathcal{P}) = \Pi^{c_1}_{i=1} \), \( \mathcal{Q} = \mathcal{Q} \) and \( h_2(\mathcal{Q}) = 1 \), so that \( h^*(A) = \Pi^{c_1}_{i=1} \). However, the solution of (i) is the case \( c_1 = c \).

The previous theorems show that any multiplicative scalar-valued function on matrices is essentially the determinant function. The next result is motivated by the linear scalar function which we show to characterize the trace function.

**Theorem 6.** Let \( h: \mathcal{A} \rightarrow \mathbb{R} \) such that
\[ \begin{align*}
(i) & \quad h(A+B) = h(A) + h(B), \\
(ii) & \quad h(AB) = h(BA), \\
(iii) & \quad h(\cdot) \text{ is continuous},
\end{align*} \]
then
\[ h(A) = c \, \text{tr} \, A. \]
Proof. Every matrix $A$ can be written as $A = \tilde{T}T'$, where $\tilde{T} \in \mathcal{O}_p$, $T \in \mathcal{O}_p$. Because of (i), $h(A) = h(\tilde{T})$. Let $I_{i,j}$ be the matrix whose entries are zeros except for the $(i,j)$-th entry, which is one. Then

$$\tilde{T} = \sum_{i \geq j} t_{i,j} I_{i,j},$$

and

$$h(\tilde{T}) = \sum_{i \geq j} h(t_{i,j} I_{i,j}).$$

But, for the appropriate permutation matrix $P$,

$$h(I_{i,j}) = h(PI_{i,j}P') = h(I_{j,i}),$$

so that

$$h(\tilde{T} + \tilde{T}') = h(\tilde{T}) + h(\tilde{T}') = 2h(\tilde{T}).$$

Consequently, we need only treat the case for symmetric $\tilde{\sigma}$. Since $\tilde{A} = \Delta \sigma \Delta'$, for $\Delta \in \mathcal{O}_p$, $\sigma \in \mathcal{O}_p$, we may confine ourselves to diagonal $\tilde{A}$. For $h(A) = g(\sigma_1', \ldots, \sigma_p')$, and because of invariance under permutations, the result follows immediately, e.g., Aczel (1966) p. 348.

4. Jacobians of linear transformations. We now obtain the Jacobian results for which the previous functional equations were generated.

Lemma 1. If $Y = AXB$, where $X$ and $Y$ are $p \times q$ variable matrices, and $A$ and $B$ are constant matrices, then $J(X \rightarrow Y) = |A|^q |B|^p$.  

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Proof. The Jacobian of the transformation $Y = XB$ may be obtained from Theorem 5 by taking the transpose, namely, $Y' = B'X'$. The result then follows from (3.1), (3.2), (3.3) and Theorem 5.||

Suppose $\overline{X}$ and $\overline{Y}$ are symmetric $p \times p$ matrices and

$$\overline{Y} = AXA',$$

then by letting

$$\overline{Z} = BYB',$$

we obtain $J(\overline{X} \to \overline{Z}) = J(\overline{X} \to \overline{Y}) J(\overline{Y} \to \overline{Z})$, which again yields (by linearity) the functional equations $h(BA) = h(A) h(B)$. By Theorem 5, $h(A) = |A|^c$. To evaluate $c$, let $A = \text{diag} (a, 1, \ldots, 1)$, and we readily obtain $c = p + 1$.

If $\hat{X}$ and $\hat{Y}$ are skew-symmetric, and $\hat{Y} = AXA'$, we again generate the functional equation $h(BA) = h(A) h(B)$. However, the constant $c$ evaluated by letting $A = \text{diag}(a, 1, \ldots, 1)$ is seen to be $p - 1$.

Both results are combined in

Lemma 2. If $\overline{Y} = AXA'$, then $J(\overline{X} \to \overline{Y}) = |A|^{p+1}$; if $\hat{Y} = AXA'$, then $J(\hat{X} \to \hat{Y}) = |A|^{p-1}$, where all matrices are of order $p$.

We next consider several transformations involving triangular matrices. Let
\[ \tilde{U}_2 = \tilde{A} U_1, \quad \tilde{U}_3 = \tilde{B} U_2, \]

where \( \tilde{A}, \tilde{B}, \tilde{U}_1 \in \mathcal{U}_p \). Noting that the product of lower triangular matrices with unit diagonals is of the same type, we have by the chain rule (2.1) that

\[ h(\tilde{A}) h(\tilde{B}) = h(\tilde{A}\tilde{B}), \]

where \( h(\tilde{A}) = J(\tilde{U}_1 \rightarrow \tilde{U}_2) \). By Theorem 3, \( h(\tilde{A}) = 1 \). In a similar way, if \( \tilde{U}_2 = \tilde{U}_1 \tilde{A} \), then \( J(\tilde{U}_1 \rightarrow \tilde{U}_2) = 1 \).

If

\[ \tilde{T}_2 = \tilde{A} \tilde{T}_1, \quad \tilde{T}_3 = \tilde{B} \tilde{T}_2, \]

where \( \tilde{A}, \tilde{B}, \tilde{T}_1 \in \mathcal{R}_p \) then \( h(\tilde{A}) h(\tilde{B}) = h(\tilde{A}\tilde{B}) \), where \( h(\tilde{A}) = J(\tilde{T}_1 \rightarrow \tilde{T}_2) \). By Theorem 4, \( h(\tilde{A}) = \Pi a_{ii}^{c_i} \). To determine the \( c_j \), we may choose

\( \tilde{A} = \text{diag}(a_1, \ldots, a_p) \). Then with \( \tilde{T}_1 = (t_{ij}^{(1)}), \tilde{T}_2 = (t_{ij}^{(2)}), \)

\( (t_{jj}^{(2)}, \ldots, t_{jj}^{(2)}) = a_j (t_{jj}^{(1)}, \ldots, t_{jj}^{(1)}), \quad j = 1, \ldots, p, \) so that \( c_j = j \).

An analogous argument may be used to show that the Jacobian of the transformation

\[ \tilde{T}_2 = \tilde{T}_1 \tilde{A} \]

is \( \Pi a_{ii}^{p-i+1} \). The functional equation is the same, but the determination of the exponent is now based on \( (t_{jj}^{(2)}, \ldots, t_{jp}^{(2)}) = a_j (t_{jj}^{(1)}, \ldots, t_{jp}^{(1)}) \).
In summary, we have

Lemma 3. If \( \tilde{U}_2 = \tilde{A} \tilde{U}_1 \tilde{B} \), where \( \tilde{A}, \tilde{U}_1, \tilde{B} \in \mathcal{U}_p \), then \( J(\tilde{U}_1 \rightarrow \tilde{U}_2) = 1 \).
If \( \tilde{T}_2 = \tilde{A} \tilde{T}_1 \tilde{B} \), where \( \tilde{A}, \tilde{T}_1, \tilde{B} \in \mathcal{A}_p \), then \( J(\tilde{T}_1 \rightarrow \tilde{T}_2) = \Pi \ a_{ii}^1 b_{ii}^{p-i+1} \).

5. Inverse transformations. To determine the Jacobian of the transformation from \( X \) to \( X^{-1} \), we may let \( L = AX \) from which

\[
J(X^{-1} \rightarrow X) = J(X^{-1} \rightarrow L^{-1}) \quad J(L^{-1} \rightarrow L) \quad J(L \rightarrow X).
\]

Write \( h(X) \) for \( J(X^{-1} \rightarrow X) \), then

\[
h(X) = |A|^p h(L) \ |A|^p,
\]

so that

\[
h(X) \ |A|^{-2p} = h(AX),
\]

for all \( A \) and \( X \). Thus, with \( X = I \),

\[
h(A) = c |A|^{-2p}.
\]

By (3.8), we may consider the case \( X \) diagonal, for which \( c \) is readily determined to be unity.

Alternatively, we may take differentials namely,
\[(dx^{-1}) = -x^{-1}(dx) x^{-1},\]

and hence from (2.4) and Lemma 1,

\[J(x^{-1} \rightarrow x) = J(dx^{-1} \rightarrow dx) = |x|^{-2p}.\]

Similarly, for symmetric matrices,

\[(dS^{-1}) = S^{-1}(dS) S^{-1},\]

so that from Lemma 2, \[J(dS^{-1} \rightarrow dS) = |S|^{-(p+1)}.\] To generate a functional equation for the Jacobian, let \[\overline{M} = ASA',\] so that

\[J(S^{-1} \rightarrow S) = J(S^{-1} \rightarrow M^{-1}) J(M^{-1} \rightarrow \overline{M}) J(\overline{M} \rightarrow S).\]

For triangular matrices,

\[(dT) = T^{-1}(dT) T^{-1},\]

so that

\[J(dT^{-1} \rightarrow dT) = \Pi t_{ii}^{-(p-i+1)} = \Pi t_{ii}^{p-1}.\]

As before, a functional equation may be generated from \[\hat{L} = \hat{A}T,\] so that

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\begin{align*}
J(\tilde{T}^{-1} \to \tilde{T}) = J(\tilde{T}^{-1} \to \tilde{L}^{-1}) \cdot J(\tilde{L}^{-1} \to \tilde{T}) \cdot J(\tilde{L} \to \tilde{T}).
\end{align*}

These results are included in

**Lemma 4.** If \( X \in \mathcal{A}_p, \quad S \in \mathcal{S}_p, \quad T \in \mathcal{X}_p \), then

\begin{align*}
J(X \to \tilde{X}^{-1}) &= |X|^{-2p}, \quad J(S \to \tilde{S}^{-1}) = |S|^{-1}, \quad J(T \to \tilde{T}^{-1}) = \Pi \cdot t^{p+1}.
\end{align*}

6. **Nonlinear transformations.** It is immediate that every non-singular \( \tilde{T} \in \mathcal{X}_p \) can be written as

\[
\tilde{T} = \tilde{U}_D V,
\]

where \( \tilde{U} \in \mathcal{U}_p, \quad D \in \mathcal{D}_p \). Although the Jacobian may be evaluated directly, we obtain the result from a functional equation. Define

\[
\tilde{U}^* = \tilde{A}U, \quad D_v^* = D_vD_v, \quad \tilde{A} \in \mathcal{U}_p,
\]

\[
\tilde{L} = \tilde{U}^*D_v^* = \tilde{ATD}_b.
\]

Then

\[
J(\tilde{T} \to \tilde{U}, D_v) = J(\tilde{T} \to \tilde{L}) \cdot J(\tilde{L} \to \tilde{U}^*, D_v^*) \cdot J(\tilde{U}^*, D_v^* \to \tilde{U}, D_v).
\]

Let \( h(\tilde{U}, D_v) = J(\tilde{T} \to \tilde{U}, D_v) \), then from Lemma 3,
\[ h(\tilde{U}, D_v) = (\Pi b_i^{p-1}) h(\tilde{U}^*, D_v^*) (\Pi b_i^{-1}) \]
\[ = (\Pi b_i^{p-1}) h(\tilde{A}U, D_b D_v). \]

The choice \( \tilde{A} = U^{-1}, \quad D_b = D_v^{-1} \) yields
\[ h(\tilde{U}, D_v) = c \prod v_i^{-(p-1)}. \]

The constant is easily evaluated to be unity by considering the case of a diagonal transformation.

An analogous argument yields a similar result for the transformation \( \tilde{T} = D_v \tilde{U} \), where \( \tilde{U} \in U_p \). In summary, we have

**Lemma 5.** If \( \tilde{U} \in U_p \), \( \tilde{T}_1, \tilde{T}_2 \in X_p \), and \( \tilde{T}_1 = \tilde{U} D_v \), \( \tilde{T}_2 = D_v \tilde{U} \), then
\[ J(\tilde{U}, v \to \tilde{T}_1) = \Pi v_i^{p-1}, \quad J(\tilde{U}, v \to \tilde{T}_2) = \Pi v_i^{1-1}. \]

Every matrix \( X \) has a representation in terms of triangular matrices, namely,
\[ X = \tilde{U} \tilde{T}, \quad \tilde{T} \in X_p, \quad \tilde{U} \in U_p. \]

To generate a functional equation for the Jacobian, let
\[ \tilde{T} = \tilde{A} \tilde{T}_1, \quad \tilde{U} = \tilde{U} \tilde{U}_1, \quad Y = \tilde{T}_1 \tilde{U}_1, \]

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where $\tilde{A}, \tilde{T}_1 \in \mathcal{A}_p$, $\tilde{B}, \tilde{U}_1 \in \mathcal{U}_p$. Then $X = \tilde{A}Y\tilde{B}'$, and

$$J(\tilde{T}, \tilde{U} \rightarrow X) = J(\tilde{T}, \tilde{U} \rightarrow \tilde{T}_1, \tilde{U}_1) J(\tilde{T}_1, \tilde{U}_1 \rightarrow Y) J(Y \rightarrow X).$$

Let $h(\tilde{T}, \tilde{U}) = J(\tilde{T}, \tilde{U} \rightarrow X)$, then

$$h(\tilde{T}, \tilde{U}) = (\Pi a_{ii}^{-1})h(\tilde{T}_1, \tilde{U}_1)(\Pi a_{ii}^{P}).$$

The choice $\tilde{U}_1 = I$, $\tilde{T}_1 = I$ yields

$$h(\tilde{A}, \tilde{B}) = c \Pi a_{ii}^{P-1}.$$

The constant is easily evaluated to be unity by considering the case of a diagonal transformation.

An analogous result is obtained for the transformation $X = \tilde{T}'\tilde{U}$.

Lemma 6. If $\tilde{U} \in \mathcal{U}_p$, $\tilde{T} \in \mathcal{A}_p$, $X_1, X_2 \in \mathcal{A}_p$, and

$$X_1 = \tilde{T}\tilde{U}', \quad X_2 = \tilde{T}'\tilde{U},$$

then $J(\tilde{T}, \tilde{U} \rightarrow X_1) = \Pi t_{ii}^{P^{-1}}$, $J(\tilde{T}, \tilde{U} \rightarrow X_2) = \Pi t_{ii}^{-1}$.  

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Every symmetric matrix may be represented in terms of a triangular matrix:

\[ \bar{S} = \bar{T}'\bar{T}, \quad \bar{T} \in \mathcal{G}_p. \]

To generate a functional equation for the Jacobian let \( \bar{T} = \tilde{A}\bar{T}_1 \), \( \bar{S}_1 = \bar{T}_1\bar{T}_1' \), where \( \tilde{A}\bar{T}_1 \in \mathcal{G}_p \), so that \( \bar{S} = \tilde{A}\bar{S}_1\tilde{A}' \). Then

\[ J(\bar{T} \to \bar{S}) = J(\bar{T} \to \bar{T}_1) J(\bar{T}_1 \to \bar{S}_1) J(\bar{S}_1 \to \bar{S}). \]

If we let \( J(\bar{T} \to \bar{S}) = h(\bar{T}) \), then

\[ h(\bar{T}) = (\Pi a_{ii}^{-1}) h(\bar{T}_1) (\Pi a_{ii}^{p+i}). \]

The choice \( \bar{T}_1 = I \) yields

\[ h(\tilde{A}) = c \Pi a_{ii}^{p-i+1}. \]

Since the result depends only on the diagonal elements, we may determine the constant from the diagonals, \( s_{ii} = t_{ii}^2, \ i = 1, \ldots, p \), to be \( 2^p \).

A similar derivation may be used to obtain the Jacobian of the transformation \( \bar{S} = \bar{T}'\hat{T} \). Both results are combined in

**Lemma 7.** If \( \bar{S}_1 = \bar{T}'\bar{T}, \ \bar{S}_2 = \hat{T}'\hat{T} \), where \( \hat{T} \in \mathcal{G}_p \), then

\[ J(\bar{T} \to \bar{S}_1) = 2^p \Pi t_{ii}^i. \]
From the two representations \( S = \tilde{T}T', \tilde{T} = \tilde{U}D, \tilde{T} \in A_p \), \( \tilde{U} \in U \), we have the representation \( \tilde{S} = \tilde{U}D\tilde{U}', D' = D_V^2 \). The Jacobian may be evaluated from

\[
J(\tilde{U}, D_w \to \tilde{S}) = J(\tilde{U}, D_w \to \tilde{U}, D_v) J(\tilde{U}, D_v \to \tilde{T}) J(\tilde{T} \to \tilde{S})
\]

using Lemmas 5 and 7 to yield

**Lemma 8.** If \( \tilde{S} = \tilde{U}D\tilde{U}', \tilde{U} \in U \), \( D_w \in A_p \), then \( J(\tilde{U}, D_w \to \tilde{S}) = \Pi w_{i}^{p-1} \).

The representation \( \tilde{S} = \tilde{T}T' \), \( \tilde{T} \in A_p \) permits a transformation from \( \tilde{S} \) to a nonsymmetric square root. However, the transformation to a symmetric square root is also of interest. Define the matrix

\[
0(S) = (\text{tr} \, S^{i+j-2}), \quad i, j = 1, \ldots, p,
\]

then

**Lemma 9.** \( J(S \to S^2) = 2^p \left| S \right| \left| 0(S^2) \right| \left| 0(S) \right|^{\frac{1}{2}} = \Pi (\theta_i + \theta_j), \) where \( \theta_1 > \ldots > \theta_p \) are the characteristic roots of \( \tilde{S} \).

**Proof.** Taking differentials in \( \tilde{U} = S^2 \), and writing \( \tilde{S} = \Gamma D \Gamma' \), we have

\[
d\tilde{U} = (\Gamma D \Gamma') (d\tilde{S}) + (d\tilde{S}) (\Gamma D \Gamma').
\]
Let \( \overline{V} = \Gamma'(dU) \Gamma, \overline{W} = \Gamma'(dS) \Gamma \), then

\[
\overline{V} = \mathcal{D}_\theta \overline{W} + \overline{W} \mathcal{D}_\theta,
\]

and

\[
J(\overline{S} \to \overline{U}) = J(d\overline{S} \to d\overline{U}) = J(d\overline{S} \to \overline{V}) \cdot J(\overline{V} \to \overline{W}) \cdot J(\overline{W} \to d\overline{S}).
\]

From Lemma 1, \( J(d\overline{S} \to \overline{V}) = J(\overline{W} \to d\overline{S}) = 1 \), and from \( v_{ij} = (\theta_i + \theta_j) w_{ij} \), \( 1 \leq i, j \leq p \), it follows that \( J(\overline{W} \to \overline{V}) = \prod_{i=1}^{p} \theta_i \prod_{i<j} (\theta_i + \theta_j) \).

Let \( V(x) = V(x_1, \ldots, x_n) \) denote the Vandermonde matrix

\[
V(x) = \begin{bmatrix}
1 & \cdots & 1 \\
x_1 & \cdots & x_n \\
\vdots & & \vdots \\
x_{n-1} & \cdots & x_{n-1}
\end{bmatrix},
\]

then from the well-known result

\[
|V(x)| = \prod_{i<j} (x_i - x_j),
\]

we have

\[
\prod_{i<j} (\theta_i + \theta_j) = \frac{\prod_{i<j} (\theta_i^2 - \theta_j^2)}{\prod_{i<j} (\theta_i - \theta_j)} = \frac{V(\theta_1^2, \ldots, \theta_p^2)}{V(\theta_1, \ldots, \theta_p)}.
\]

However,
\[
V(x) V'(x) = (\sigma_{i+j-2}^n), \quad i, j = 1, \ldots, p,
\]

where \( \sigma_k = \sum_{i=1}^{n} x_i^k \). The result follows by noting that \( \Sigma \theta_i^k = \text{tr} S_i^k \).

As a consequence of Lemma 9, we have a more general version given by

**Lemma 10.** If \( \bar{V} = \bar{S} \bar{A} \bar{S} \), where \( \bar{S}, \bar{A} \in \mathcal{S}_p \) and \( \bar{A} \) is a positive definite constant matrix, then \( J(\bar{S} \rightarrow \bar{V}) = \Pi_{i \leq j} (\lambda_i + \lambda_j) \), where \( \lambda_1, \ldots, \lambda_p \) are the characteristic roots of \( \bar{S} \bar{A} \).

**Proof.** Let \( \bar{Z} = \bar{A}^{\frac{1}{2}} \bar{V} \bar{A}^{\frac{1}{2}} = (\bar{A}^{\frac{1}{2}} \bar{S} \bar{A}^{\frac{1}{2}})^2 = \bar{W}^2 \), then

\[
J(\bar{V} \rightarrow \bar{S}) = J(\bar{V} \rightarrow \bar{Z}) J(\bar{Z} \rightarrow \bar{W}) J(\bar{W} \rightarrow \bar{S}) = |A|^{(p+1)} \Pi_{i \leq j} (\lambda_i + \lambda_j) |A|^{-(p+1)}.
\]

We now consider transformations similar to that of Lemma 9 for triangular and general matrices. For a triangular matrix \( \tilde{T} \), if \( \tilde{M} = \tilde{T}^2 \), then

\[(6.1) \quad (d\tilde{M}) = \tilde{T}(dT) + (dT)\tilde{T}.\]

Using the representation \( \tilde{T} = U^{-1}D\theta U \), \( \tilde{U} \in \mathcal{U}_p \), \( D\theta \in \mathcal{D}_p \), \( (6.1) \) becomes

\[(6.2) \quad \tilde{M} = D\theta \tilde{T}^* + \tilde{T}^* D\theta,\]

where \( \tilde{M} = \tilde{U}(d\tilde{M})\tilde{U}^{-1} \), \( \tilde{T}^* = \tilde{U}(dT)\tilde{U}^{-1} \).
Hence

\[ J(\tilde{T} \to \tilde{M}) = J(d\tilde{T} \to \tilde{T}^*) J(\tilde{T}^* \to \tilde{M}^*) J(\tilde{M}^* \to d\tilde{M}). \]

But \( J(d\tilde{T} \to \tilde{T}^*) = J(M^* \to d\tilde{M}) = 1 \) and \( J(\tilde{T}^* \to \tilde{M}^*) = \Pi (\theta_i + \theta_j). \)

For a general matrix \( X \), if \( L = X^2 \), then

(6.3) \[ (dL) = X(dx) + (dx)X. \]

Using the representation \( X = \Gamma \tilde{T} \Gamma' \), \( \Gamma \in \theta_p \), \( \tilde{T} \in \tilde{\alpha}_p \), and \( \tilde{T} = \tilde{U}^{-1} D_\theta \tilde{U} \), (6.3) becomes

\[ L^* = D_\theta X^* + X^* D_\theta, \]

where \( L^* = \tilde{U} \Gamma'(dL) \Gamma \tilde{U}^{-1}, \ X^* = \tilde{U} \Gamma'(dx) \Gamma \tilde{U}^{-1}. \) Hence

\[ J(X \to L) = J(dx \to X^*) J(X^* \to L^*) J(L^* \to dL). \]

But \( J(dx \to X^*) = 1 \), \( J(L^* \to dL) = 1 \), and \( J(X^* \to L^*) = 2^p \Pi \theta_i \Pi (\theta_i + \theta_j)^2. \)

Combining the above we have

**Lemma 11.** If \( \tilde{T} \in \tilde{\alpha}_p \), \( X \in \alpha_p \) then

\[ J(\tilde{T} \to \tilde{T}^2) = \Pi (t_{ii} + t_{jj}), \]

\[ J(X \to X^2) = 2^p \Pi \theta_i \Pi (\theta_i + \theta_j)^2, \]

where \( \theta_1 > \ldots > \theta_p \) are the characteristic roots of \( X. \)
There are surprisingly few "natural" transformations for more than one matrix. One of these is the simultaneous decomposition of symmetric matrices:

\[(6.4)\]
\[
\overline{U}_1 = \overline{W}D_\theta W', \quad \overline{U}_1 + \overline{U}_2 = WW',
\]

where \(|W| \neq 0, \quad 0 < \theta_i < 1, \quad i = 1, \ldots, p, \quad \text{and} \quad \theta_1 > \ldots > \theta_p\) are the roots of the determinantal equation \(|\overline{U}_1 - \theta(\overline{U}_1 + \overline{U}_2)| = 0\). Various representations equivalent to (6.4) may be used, e.g., \(\overline{U}_1 = \overline{W}D_\alpha W', \quad \overline{U}_2 = \overline{W}D_\beta W', \quad \alpha \beta = 1, \quad \text{or} \quad \overline{U}_1 = \overline{W}D_\mu W', \quad \overline{U}_2 = WW'.\) However, we find (6.4) to simplify later computations.

To evaluate the Jacobian from \(\overline{S}_1 = \overline{U}_1, \overline{S}_2 = \overline{U}_1 + \overline{U}_2\) to \(W, \theta,\) let

\[
\overline{S}_1^* = AS_1 A', \quad \overline{S}_2^* = AS_2 A', \quad Y = AW,
\]

so that

\[
\overline{S}_1^* = YD_\theta Y', \quad \overline{S}_2^* = YY'.
\]

Then

\[
J(\overline{S}_1, \overline{S}_2 \rightarrow W, D_\theta) = J(\overline{S}_1, \overline{S}_2 \rightarrow \overline{S}_1^*, \overline{S}_2^*) \cdot J(\overline{S}_1^*, \overline{S}_2^* \rightarrow Y, D_\theta) \cdot J(Y, D_\theta \rightarrow W, D_\theta).
\]

If we let \(h(W, D_\theta) = J(\overline{S}_1, \overline{S}_2 \rightarrow W, D_\theta),\) then

\[
h(W, D_\theta) = |A|^{(p+1)} h(Y, D_\theta) |A|^{-p}.
\]

The choice \(W = I\) yields

\[
h(A, D_\theta) = |A|^{-(p+2)} h(I, D_\theta),
\]

so that the Jacobian \(J(\overline{S}_1, \overline{S}_2 \rightarrow W, D_\theta)\) factors into a product of \(|W|^{-(p+2)}\) and a function of \(\theta\) alone, \(g(\theta) = h(I, D_\theta).\) To evaluate \(g(\theta),\) let

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\[ \overline{R} = \overline{S}_1 \overline{S}_2^{-1} \overline{S}_1 = WD_\varphi W', \quad D_\varphi = D_\theta^2, \]

so that

\[ J(\overline{S}_1, \overline{S}_2 \rightarrow W, D_\theta) = J(\overline{S}_1, \overline{S}_2 \rightarrow \overline{R}, \overline{S}_2) J(\overline{R}, \overline{S}_2 \rightarrow W, D_\varphi) J(W, D_\varphi \rightarrow W, D_\theta). \]

Using Corollary 10, (6.5) becomes

\[ |W|^{-(p+2)} g(\theta) = \left( \prod_{1 \leq i < j} \lambda_i + \lambda_j \right) \left( \prod_{1 \leq i \leq j} |W|^{-(p+2)} g(\varphi) \right) \prod_{1 \leq i < j} (\theta_i - \theta_j)^{-1}, \]

where \( \lambda_1, \ldots, \lambda_p \) are the characteristic roots of \( \overline{S}_1 \overline{S}_2^{-1} \). But the roots of \( \overline{S}_1 \overline{S}_2^{-1} \) are \( \theta_1, \ldots, \theta_p \), and hence from (6.6),

\[ g(\theta_1, \ldots, \theta_p) = \prod_{1 \leq i < j} (\theta_i + \theta_j) \prod_{1 \leq i \leq j} g(\theta_1^2, \ldots, \theta_p^2). \]

Multiply both sides of (6.7) by \( \prod_{1 \leq i < j} (\theta_i - \theta_j) \), and let

\[ t(\theta) = t(\theta_1, \ldots, \theta_p) = g(\theta_1, \ldots, \theta_p) \prod_{1 \leq i < j} (\theta_i - \theta_j). \]

Then (6.7) becomes \( t(\theta) = t(\theta^2) \). By iteration, \( t(\theta) = t(\theta^{2^n}) \) for all \( n \), and since \( 0 < \theta_i < 1 \), \( i = 1, \ldots, p \), it follows that \( t(\theta) = c \). The constant may be evaluated from the case of a diagonal transformation.

**Lemma 12.** If \( \overline{S}_1 = WD_\theta W', \overline{S}_2 = WW' \), then

\[ J(W, \theta \rightarrow \overline{S}_1, \overline{S}_2) = 2^p |W|^{p+2} \prod_{1 \leq i < j} (\theta_i - \theta_j). \]
7. Orthogonal Transformations. It is generally more troublesome to determine the Jacobian of a transformation involving an orthogonal matrix. This is due, in part, to the fact that the usual procedures require additional arguments when dealing with constraints. However, by a reparameterization, the Jacobian of the transformation to the reparametrized variables may be evaluated as before. This means that we will be integrating over a different space.

Two reparameterizations have been used. Hsu (1953) makes use of the automorphic between orthogonal matrices \( \Gamma \) with \( |\Gamma + I| \neq 0 \) and skew-symmetric matrices \( \hat{X} \):

\begin{equation}
\Gamma = (I + \hat{X})^{-1} (I - \hat{X}), \quad \hat{X} = (I + \Gamma)^{-1} (I - \Gamma).
\end{equation}

An alternative expression for (7.1) which is more convenient is

\begin{equation}
\Gamma = \Gamma(\hat{X}) = 2(I + \hat{X})^{-1} - I.
\end{equation}

Note that \( d\Gamma = -2(I + \hat{X})^{-1} dX(I + \hat{X})^{-1} \), and by (7.2)

\begin{equation}
\Gamma'(d\Gamma) = -\frac{1}{2}(I + \Gamma') (d\hat{X}) (I + \Gamma).
\end{equation}

A parameterization which appears to be simple, namely,

\[ \Gamma = \exp \hat{Y}, \]

leads to difficulty in that \( d(\exp \hat{Y}) \) is not easily expressed in terms of \( d\hat{Y} \).
Tumura (1965) uses the representation of an orthogonal matrix as a product of \( p(p-1)/2 \) orthogonal matrices, each of which depends on only one angle. More explicitly, define

\[
R_y(\alpha) = \text{diag}(I_{y-1}, \Delta(\alpha), I_{p-y-1}),
\]

\[
\Delta(\alpha) = \begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix},
\]

then

\[
(7.4) \quad \Gamma = \Gamma(\alpha) = \prod_{k=1}^{p-1} R_k(\alpha_{p-k}) = \prod_{k=1}^{p-1} R_k(\alpha_{p-k}).
\]

Jack and Macbeath (1959) and Jack (1964-65) deal directly with the orthogonal space, rather than with a space generated by the parameterization.

We first consider a canonical transformation

\[
(7.5) \quad Z = \Gamma_1 W \Gamma_2, \quad \Gamma_1, \Gamma_2 \in O(p),
\]

and where \( Z \) and \( W \) may have some special properties to make the transformation 1-1. If \( \Delta_1, \Delta_2 \in O(p) \) and we let

\[
Z^* = \Delta_1 Z \Delta_2 = \Delta_1 \Gamma_1 W \Gamma_2 \Delta_2 = \psi_1 W \psi_2,
\]

then

\[
J(Z \to W_1, \Gamma_1, \Gamma_2) = J(Z \to Z^*) J(Z^* \to W_1, \psi_1, \psi_2) J(W_1, \psi_1, \psi_2 \to W_1, \Gamma_1, \Gamma_2).
\]
Let \( h(W, \Gamma_1, \Gamma_2) = J(Z \to W, \Gamma_1, \Gamma_2) \) and assume \( J(Z \to Z^*) = 1 \), then

\[
(7.6) \quad h(W, \Gamma_1, \Gamma_2) = h(W, \psi_1, \psi_2) \ g(\Delta_1, \Gamma_1) \ g(\Delta_2, \Gamma_2),
\]

where \( g(\Delta, \Gamma) = J(\Delta \Gamma \to \Gamma) \). The choice \( \Gamma_1 = \Gamma_2 = I \) yields

\[
(7.7) \quad h(W, \Delta_1, \Delta_2) = \frac{h(W, I, I)}{g(\Delta_1, I)g(\Delta_2, I)} = h_1(W) \ h_2(\Delta_1) \ h_2(\Delta_2),
\]

so that the Jacobian of the transformation (7.5) factors into functions of \( W, \Delta_1 \), and \( \Delta_2 \) separately. This permits us to more easily evaluate each of the terms.

To evaluate \( g(\Delta, I) \), consider the transformation \( \psi = \Delta \Gamma \). In terms of the parameterization (7.1),

\[
\psi(\hat{\mathbf{y}}) = \Delta(\hat{\mathbf{A}}) \ \Gamma(\hat{\mathbf{x}}).
\]

We have from (7.3) that

\[
(I+\psi') (d\hat{\mathbf{y}}) (I+\psi) = (I+\Gamma') (d\hat{\mathbf{x}}) (I+\Gamma),
\]

so that

\[
J(d\hat{\mathbf{x}} \to d\hat{\mathbf{y}}) = |(I+\Gamma)(I+\Delta \Gamma)^{-1}|^{P-1} = g(\Delta, \Gamma).
\]

Consequently,
\begin{align}
(7.8) \quad g(\Delta, I) &= z^P|I+\Delta|^{-1} = |I+\Delta|^{-1}.
\end{align}

Tumura (1965) shows that the parameterization \((7.4)\) yields the term

\begin{align}
(7.9) \quad g(\Delta(\alpha), I) &= \prod_{i=1}^{p-1} \prod_{j=1}^{p-2} \sin^{p-j-1} \alpha_{ij}.
\end{align}

Jack (1959) shows that \(g(\Delta, I)\) is an absolute constant.

We may now consider special transformations for which we obtain the corresponding functions \(h_1(W)\).

Let

\begin{align}
(7.10) \quad \vec{S} &= \Gamma D_\theta \Gamma', \quad \Gamma \in \mathcal{G}_p, \quad D_\theta \in \mathcal{D}_p,
\end{align}

where \(\theta_1 > \ldots > \theta_p\) are the characteristic roots of \(\vec{S}\), and let

\begin{align*}
\vec{V} &= \vec{S}^2 = \Gamma D_\mu \Gamma', \quad D_\mu = D_\theta^2.
\end{align*}

Then from

\begin{align*}
J(\vec{S} \to \Gamma, \theta) &= J(\vec{S} \to \vec{V}) J(\vec{V} \to \Gamma, \mu) J(\Gamma, \mu \to \Gamma, \theta),
\end{align*}

and Lemma 9, we have

\begin{align*}
h_1(\theta) h_2(\Gamma) &= \prod_{1 \leq i < j} (\theta_i + \theta_j) h_1(\mu) h_2(\Gamma) (z^P \mu)\iota_1^{-1}.
\end{align*}
Consequently,

\[ h_1(\theta) = \prod_{i<j} (\theta_i + \theta_j) h_1(\theta^2), \]

which is identical to (6.7) for which we have shown the solution to be

\[ h_1(\theta) = c \prod_{i<j} (\theta_i - \theta_j)^{-1}. \]

In summary, we have

**Lemma 1.3.** If \( \bar{S} = \Gamma D_\theta \Gamma' \) as defined by (7.10), and \( \Gamma(\hat{x}), \Gamma(\alpha) \) are defined by (7.2) and (7.4), respectively, then

\[
\begin{align*}
J(\Gamma, \theta \to \bar{S}) &= k_1 \prod_{i<j} (\theta_i - \theta_j), \\
J(\hat{x}, \theta \to \bar{S}) &= k_2(\hat{x}) \prod_{i<j} (\theta_i - \theta_j), \\
J(\alpha, \theta \to \bar{S}) &= k_3(\alpha) \prod_{i<j} (\theta_i - \theta_j),
\end{align*}
\]

where

\[
(7.11) \quad k_1 = 2^{-p(p-1)/4}, \quad k_2(\hat{x}) = 2^{p(p-1)/2} |I^{\hat{x}}|^{-(p-1)},
\]

\[
k_3(\alpha) = \prod_{i=1}^{p-2} \prod_{i=1}^{p-2} \sin^{p-1} \alpha_{ij}.
\]

Note that as a consequence of Lemma 1.3, we may obtain the Jacobian of the transformation from \( \Gamma \) to \( \hat{x} \) or \( \alpha \).
If
\[ Z = \tilde{T} \Gamma, \quad \tilde{T} \in \mathcal{P}, \quad \Gamma \in \mathcal{O}, \]
and we let
\[ Z^* = \tilde{A} Z = \tilde{A} \tilde{T} \Gamma = \tilde{U} \Gamma, \]
then
\[ J(Z \to \tilde{T}, \Gamma) = J(Z \to Z^*) J(Z^* \to \tilde{U}, \Gamma) J(\tilde{U}, \Gamma \rightarrow \tilde{T}, \Gamma). \]

By (7.6), if we write \( h(\tilde{T}, \Gamma) = J(Z \to \tilde{T}, \Gamma) \), we obtain
\[ h_1(\tilde{T}) h_2(\Gamma) = |\tilde{A}|^P h_1(\tilde{A} \tilde{T}) h_2(\Gamma) \Pi a_{ii}^{-1}. \]

The choice \( \tilde{T} = I \) yields
\[ h_1(\tilde{A}) = c \Pi a_{ii}^{-(p-1)}. \]

Consequently, we obtain

Lemma 14. If \( Z = \tilde{T} \Gamma, \quad \tilde{T} \in \mathcal{P}, \quad \Gamma \in \mathcal{O}, \) and \( \Gamma(\tilde{x}), \Gamma(\alpha) \) are defined by (7.2) and (7.4), respectively, then
\[ J(\tilde{\tau}, \Gamma \to Z) = k_1 \Pi t_{ii}^{P-i}, \]
\[ J(\tilde{\tau}, \hat{\chi} \to Z) = k_2(\hat{x}) \Pi t_{ii}^{P-i}, \]
\[ J(\tilde{\tau}, \alpha \to Z) = k_3(\alpha) \Pi t_{ii}^{P-i}, \]

where \( k_1, k_2, k_3 \) are defined by (7.11).

If

\[ Z = \tilde{s} \Delta, \quad \tilde{s} \in \mathcal{S}_p, \quad \Delta \in \mathcal{O}_p, \quad \Delta = \Delta(\hat{x}), \]

then \( (dZ) = (d\tilde{s})\Delta + \tilde{s}(d\Delta) \). Letting \( \tilde{s} = \Gamma D_\theta \Gamma' \), and simplifying, we obtain

\[ Z^* = \tilde{s}^* + D_\theta \hat{H}, \]

where \( Z^* = \Gamma'(dZ)\Delta'T, \quad \tilde{s}^* = \Gamma'(d\tilde{s})\Gamma, \)

\[ \hat{H} = \Gamma'(d\Delta')\Delta'T = -\frac{1}{2}\Gamma'(I+\Delta) (d\hat{x}) (I+\Delta') \Gamma. \]

Consequently,

\[ J(Z \to \tilde{s}, \Delta) = J(Z \to Z^*) J(Z^* \to \tilde{s}^*, \hat{H}) J(\tilde{s}^*, \hat{H} \to (d\tilde{s}), (d\hat{x})) \equiv J_1 J_2 J_3. \]

But \( J_1 = 1, \quad J_2 = |\sqrt{2}(I+\Delta)|_{P-1}^{-1}, \) and \( J_3 = \Pi_{i<j}(\theta_i - \theta_j). \) Hence, we have
Lemma 15. The Jacobian of (7.12) is

\[ J(Z \to \bar{S}, \Delta) = k \prod_{i < j} (\theta_i - \theta_j), \]

where \( \theta_1 > \ldots > \theta_p \) are the roots of \( \bar{S} \).

This representation permits an alternative derivation of the Jacobian of the transformation

\[ Z = \Gamma D_{\theta} \Delta, \]

(Olkin (1951), Jack and Macbeath (1959)). We use the sequence of transformations

\[ Z = \bar{S} \Sigma = (\Gamma D_{\theta} \Gamma') \Sigma = \Gamma D_{\theta} \Delta, \]

so that

\[ J(Z \to \Gamma, \theta, \Delta) = J(Z \to \bar{S}, \Sigma) J(\bar{S}, \Sigma \to \Gamma, \theta, \Sigma) J(\Sigma, \Sigma \to \Gamma, \theta, \Delta). \]

By Lemmas 15, 13, and (7.6), we obtain

Lemma 16. If \( Z = \Gamma D_{\theta} \Delta, \quad \Gamma, \Delta \in \mathcal{O}_p, \quad D_{\theta} \in \mathcal{D}_p, \quad \theta_1 > \ldots > \theta_p \) are the characteristic roots of \( ZZ' \), \( \Gamma(\hat{X}), \Delta(\hat{Y}) \) are defined by (7.4), \( \Gamma(\alpha), \Delta(\beta) \) are defined by (7.5), then
\[ J(\Gamma, \Delta, \theta \to Z) = k_1^2 \prod_{i < j} (\theta_i - \theta_j), \]
\[ J(\hat{X}, \hat{Y}, \theta \to Z) = k_2(\hat{X}) k_2(\hat{Y}) \prod_{i < j} (\theta_i - \theta_j), \]
\[ J(\alpha, \beta, \theta \to Z) = k_3(\alpha) k_3(\beta) \prod_{i < j} (\theta_i - \theta_j). \]

To obtain the Jacobian of the transformation

\[ X = \Gamma \tilde{T} \Gamma', \quad \Gamma \in \mathcal{O}_p, \quad \tilde{T} \in \mathcal{R}_p, \]

we again note, as before, that

\[ J(\chi \to \tilde{T}, \Gamma) = h_1(\tilde{V}) h_2(\Gamma). \]

To determine \( h_1(\tilde{V}) \), we let \( L = X^2 = \Gamma \tilde{T}^2 \Gamma' = \Gamma \tilde{V} \Gamma' \) so that

\[ J(\chi \to \tilde{T}, \Gamma) = J(\chi \to L) J(L \to \tilde{V}, \Gamma) J(\tilde{V}, \Gamma \to \tilde{T}, \Gamma). \]

Hence

\[ h_1(\tilde{V}) h_2(\Gamma) = 2^p \prod_{i} \prod_{i < j} (\theta_i + \theta_j)^2 h_1(\tilde{V}) h_2(\Gamma) \left[ \prod_{i < j} (t_{ii} + t_{jj}) \right]^{-1}. \]

Since the roots of \( X \) are the same as the roots of \( T \), i.e., \( \theta_i = t_{ii}, i = 1, \ldots, p \),

\[ h_1(\tilde{T}) = \prod_{i < j} (t_{ii} + t_{jj}) h_1(\tilde{T}^2). \]
As for (6.7), we obtain

\[ h_1(\tilde{r}) = c\Pi(t_{ij} - t_{jj}). \]

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References


