NORMS AND INEQUALITIES FOR CONDITION NUMBERS, III

BY

ALBERT W. MARSHALL and INGRAM OLKIN

TECHNICAL REPORT NO. 53
FEBRUARY 15, 1971

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT GP 17172

Ingram Olkin, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
NORMS AND INEQUALITIES FOR CONDITION NUMBERS, III

by

Albert W. Marshall* and Ingram Olkin

TECHNICAL REPORT NO. 53
February 15, 1971

PREPARED UNDER THE AUSPICIES
OF
NATIONAL SCIENCE FOUNDATION GRANT 17172
Ingram Olkin, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

*Supported in part by Boeing Scientific Research Laboratories.
1. **Introduction.** The condition number $c_\varphi$ of a nonsingular matrix $A$ is defined by

$$c_\varphi(A) = \varphi(A) \varphi(A^{-1}),$$

where ordinarily $\varphi$ is a norm. This definition can be extended to include singular and rectangular matrices by substituting the pseudo-inverse $A^+$ for $A^{-1}$. Condition numbers arise in various contexts, and serve, e.g., as measures of the difficulty in solving a system of linear equations (see Forsythe and Moler, 1967).

For condition numbers based on norms that are unitarily invariant (i.e., $\varphi(A) = \varphi(AU) = \varphi(VA)$ for all unitary matrices $U$ and $V$ of appropriate order), we obtain the following comparisons:

**Proposition 1.** If $A : m \times q$ is of rank $q$, $(A,B) : m \times n$ is of rank $n$, then

$$(1.1) \quad c_\varphi(A) \leq c_\varphi(A,B).$$

---

1Supported in part by Boeing Scientific Research Laboratories.

2Supported in part by the National Science Foundation Grant 17172, Stanford University.
**Proposition 2.** If $A: m \times n$ is of rank $k$, $H: n \times q$ is column orthonormal ($H^*H = I_q$), and rank $(A^*, H) = \text{rank } A^*$, then

(1.2) \[ \phi(AH) \leq \phi(A). \]

**Proposition 3.** If $A: m \times n$, and $\varepsilon > 0$, then

(1.3) \[ \phi(A + \varepsilon A^*) \leq \phi(A). \]

The inequalities are known for the case that $\phi(A)$ is the maximum singular value of $A$. For this norm, (1.1) and (1.2) were obtained by Hanson and Lawson (1969); (1.3) was obtained by Klinger (1968) when $A$ is normal and nonsingular, and by Tewarson and Ramnath (1968) without normality.

Unitarily invariant norms $\phi$ are monotone in the sense that if $A$ and $B - A$ are positive definite then $\phi(A) \leq \phi(B)$. Marshall and Olkin (1965, 1969) show:

**Proposition 4.** If $A$ is nonsingular and $\phi$ is unitarily invariant, then

(1.4) \[ \phi(A) \leq \phi(AA^*). \]

**Proposition 5.** If $A, B$ are positive definite and $\phi$ is a monotone norm, then
\begin{equation}
\phi(A+B) \leq \max \{ \phi(A), \phi(B) \}.
\end{equation}

We extend (1.4) to the case that $A$ is singular or rectangular, and show that no such general extension is possible for (1.5).

2. Preliminaries. In writing an inequality like $\phi(A) \leq \phi(A,B)$, or even in defining $\phi(A) = \phi(A) \phi(A^\ast)$ we have acted as though $\phi$ is defined on matrices of various orders. If this is to be the case, we must be careful to specify what is meant by a norm because, e.g., the triangle inequality $\phi(A+B) \leq \phi(A) + \phi(B)$ makes no sense if $A$ and $B$ are of different orders. However, if we assume that augmentation of a matrix by blocks of zeros to the right and below does not change its norm, i.e.,

$$\phi(A) = \phi\left(\begin{array}{cc}
A & O \\
O & 0
\end{array}\right),$$

then we are free either to regard $\phi$ as defined on matrices of various orders, or to augment matrices by blocks of zeros to achieve a common order. To do this with impunity we must be sure that all norms $\phi$ defined on $m \times n$ matrices have, for all $i,j > 0$, the form

\begin{equation}
\phi(A) = \hat{\phi}\left(\begin{array}{cc}
A & 0 \\
0 & 0
\end{array}\right)
\end{equation}

for some norm $\hat{\phi}$ defined on the $m+i \times n+j$ matrices. This is not difficult to demonstrate. But since we are concerned exclusively with
unitarily invariant norms we want to be sure that \( \hat{\varphi} \) can be found which is unitarily invariant whenever \( \varphi \) is unitarily invariant.

When \( \varphi \) is unitarily invariant, \( \varphi(\mathbf{A}) \) depends on \( \mathbf{A} \) only through its singular values. More precisely, if \( \alpha_1^2, \ldots, \alpha_n^2 \) are the characteristic roots of \( \mathbf{A}^*\mathbf{A} \),

\[
(2.2) \quad \varphi(\mathbf{A}) = \varphi(\alpha_1, \ldots, \alpha_n)
\]

for some symmetric gauge function (SGF) \( \varphi \) (see von Neumann, 1937). If we define \( \hat{\varphi}(x_1, \ldots, x_{n+j}) = \varphi(x_1, \ldots, x_n) \) where \( |x_1| \geq \ldots \geq |x_{n+j}| \) are obtained by reordering \( x_1, \ldots, x_{n+j} \), then \( \hat{\varphi} \) is a SGF which gives rise to a unitarily invariant norm \( \varphi \) satisfying (2.1).

In view of these remarks, one sees immediately that because (1.4) holds for nonsingular matrices \( \mathbf{A} \), it must also hold for singular and rectangular matrices.

For any real number \( a \), we use the notation

\[
a^+ = a^{-1} \quad \text{if} \quad a \neq 0, \quad a^+ = 0 \quad \text{if} \quad a = 0.
\]

Suppose \( \mathbf{A} \) is an arbitrary matrix, \( \mathbf{U} = \mathbf{A}^*\mathbf{A} \) and \( \mathbf{V} = \mathbf{A}\mathbf{A}^* \). The non-zero singular values of \( \mathbf{A} \), \( \mathbf{U}^{\frac{1}{2}}, \mathbf{V}^{\frac{1}{2}} \) are identical. When \( \varphi \) is unitarily invariant, \( \varphi(\mathbf{A}) \) depends only on these singular values, so that

\[
(2.3) \quad c_\varphi(\mathbf{A}) = c_\varphi(\mathbf{U}^{\frac{1}{2}}) = c_\varphi(\mathbf{V}^{\frac{1}{2}}).
\]
3. **Augmented Matrices.** Using (2.3) and the notation $U = (A, B)^*(A, B)$, $U_{11} = A^*A$, we see that for any unitarily invariant norm $\varphi$, inequality (1.1) becomes

\begin{equation}
\frac{1}{\varphi(U_{11}^{\frac{1}{2}})} \leq \varphi(U^{\frac{1}{2}}).
\end{equation}

Here, $U$ is positive definite because rank $(A, B) = n$.

Inequality (1.2) can be similarly rewritten: Since $H^*H = I_q$, there exist orthogonal $\Gamma$ and $\Delta$ such that

$$H = \Gamma(\begin{array}{c} I_q \\ 0 \end{array}) \Delta.$$ 

Let $B = A^*\Gamma$, $U = B^*B$ and define $U_{11}$ by

$$B^*\begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix}B = (U_{11}^{\frac{1}{2}} 0).$$

With this notation, (1.2) also becomes (3.1), but now $U$ has rank $k$ and is not positive definite unless $k=n$. However,

$$\text{rank}(A^*, H) = \text{rank}(\Gamma^*(A^*, H)(\begin{array}{c} I \\ 0 \end{array})) = \text{rank}(B^*, (I_q))$$

so that the condition $\text{rank}(A^*, H) = \text{rank} A^*$ of (1.2) is equivalent to

\begin{equation}
\text{rank}(B^*, (I_q)) = \text{rank} B.
\end{equation}
Of course, this means \( q \leq k = \text{rank } B \).

To complete the proof of (1.1) and (1.2), it remains to show that (3.1) holds. Denote the characteristic roots of \( U = (u_{ij})_{i,j=1}^{n} \) by \( \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0 \) and the characteristic roots of \( U_{n-1} = (u_{ij})_{i,j=1}^{n-1} \) by \( \gamma_1 \geq \ldots \geq \gamma_{n-1} \geq 0 \). According to the separation theorem of Sturm,

\[ \alpha_1 \geq \gamma_1 \geq \alpha_2 \geq \ldots \geq \gamma_{n-1} \geq \alpha_n. \]

This shows that \( \alpha_i \geq \gamma_i, i=1,2,\ldots,n-1, \) and iteration of the argument yields \( \alpha_i \geq \beta_i, i=1,2,\ldots,q, \) where \( \beta_1 \geq \ldots \geq \beta_q \geq 0 \) are the characteristic roots of \( U_{11} \). Providing that \( \text{rank } U = k > q = \text{rank } U_{11} \), it follows in a similar fashion that \( \alpha_{k-j} \leq \beta_{q-j}, j=0,1,\ldots,q-1. \) The monotonicity of the SGF \( \phi \) related to \( \varphi \) via (2.2), and \( \alpha_i \geq \beta_i > 0, i=1,2,\ldots,q, \) together yield

(3.3) \[ \phi(\alpha_1,\ldots,\alpha_n) \geq \phi(\alpha_1,\ldots,\alpha_q,0,\ldots,0) \geq \phi(\beta_1,\ldots,\beta_q,0,\ldots,0). \]

The same monotonicity and \( \beta_{q-j} \geq \alpha_{k-j} > 0, j=0,1,\ldots,q-1 \) give

(3.4) \[ \phi(\alpha_1^{-1},\ldots,\alpha_n^{-1}) \geq \phi(\alpha_1^{-1},\alpha_2^{-1},\ldots,\alpha_{k-q+1}^{-1},0,\ldots,0) \geq \phi(\beta_1^{-1},\beta_2^{-1},\ldots,\beta_q^{-1},0,\ldots,0). \]

The combination of (3.3) and (3.4) proves (3.1) under the condition that \( \text{rank } U > \text{rank } U_{11} \), as it is for (1.1) when \( U \) is positive definite.
It remains to show that (3.1) holds under the conditions (3.2) and 
k=q. With k=q, (3.2) implies that \( B = (B_1, 0) \) where \( B_1 : m \times q \). Consequently,

\[
U = \begin{pmatrix}
B_1^* B_1 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
U_{11} & 0 \\
0 & 0
\end{pmatrix}.
\]

Since the non-zero roots of \( U \) and \( U_{11} \) coincide, (3.1) is trivial.

4. **Sums of Matrices.** In considering the possibilities of extending (1.5) to matrices that are not positive definite, we begin with two simple counterexamples. The first of these shows that non-singularity is insufficient for (1.5); the second shows that positive semi-definiteness is insufficient.

I. Let \( A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \), \( B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Then \( AA^* = 4I \) and \( BB^* = I \), so that \( c_\varphi(A) = c_\varphi(B) = 1 \) whenever \( \varphi \) is unitarily invariant. On the other hand, \( c_\varphi(A+B) > 1 \), e.g., when \( \varphi \) is the spectral norm.

II. Let \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Again, \( c_\varphi(A) = c_\varphi(B) = 1 \) when \( \varphi \) is unitarily invariant, but \( c_\varphi(A+B) > 1 \), e.g., when \( \varphi \) is the spectral norm.

The proof of (1.5) given by Marshall and Olkin (1969) depends upon the convexity of the inverse function on the domain of positive definite matrices:
\[(\theta U_1 + (1-\theta)U_2)^{-1} \leq \theta U_1^{-1} + (1-\theta)U_2^{-1}\]

whenever \(0 \leq \theta \leq 1\), \(U_1\) and \(U_2\) are positive definite, and where \(A \leq B\) means \(B-A\) is positive semi-definite. However, the pseudo-inverse is not convex on the domain of positive semi-definite matrices. To see this, let \(U_1\) be positive definite, \(U_2 = 0\). Then for \(0 < \theta < 1\),

\[(\theta U_1 + (1-\theta)U_2)^+ = \theta^{-1}U_1^{-1} > \theta U_1^{-1} = \theta U_1^+ + (1-\theta)U_2^+ .\]

It is, however, possible to extend (1.5) in a rather trivial but useful way:

**Proposition 6.** If \(A, B\) are positive semi-definite, rank \(A = \text{rank } B = \text{rank } (A, B)\), and if \(\varphi\) is unitarily invariant, then (1.5) holds. To see this we take \(B = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}\), where \(D\) is diagonal without loss of generality because \(\varphi\) is unitarily invariant. Then the rank condition and symmetry of \(A\) imply that \(A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}\), where \(A_{11}\) and \(D\) are of the same size. The application of (1.5) to \(A_{11}\) and \(D\) completes the proof.

This result can be used to show (1.3) for unitarily invariant norms as follows. Let \(\gamma_1^2 \geq \ldots \geq \gamma_m^2\) be the characteristic roots of \((A + \varepsilon(A^*)^+) (A + \varepsilon(A^*)^+)^* = AA^* + 2\varepsilon AA^* + \varepsilon^2 (AA^*)^+ \). If we write

\[A = \Gamma \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \Delta,\]

where \(D = \text{diag}(\alpha_1, \ldots, \alpha_k)\) so that the \(\alpha_i\) are the roots of \((AA^*)^{1/2}\), then \(A^* = \Delta^* \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} \Gamma^*\), and \(\gamma_j^2\) are the roots of
\[
\Gamma \begin{pmatrix}
D^2 & 0 \\
0 & 0
\end{pmatrix} \Gamma^* + \varepsilon \Gamma \begin{pmatrix}
I & 0 \\
0 & 0
\end{pmatrix} \Gamma^* + \varepsilon^2 \begin{pmatrix}
D^{-2} & 0 \\
0 & 0
\end{pmatrix} \Gamma^* = \begin{pmatrix}
D^2 + 2 \varepsilon I + D^{-2} & 0 \\
0 & 0
\end{pmatrix}.
\]

Thus \( \gamma_j = \alpha_{i_j}^{-1} + \varepsilon \alpha_{i_j}^{-1} \), \( j = 1, 2, \ldots, k \), \( \gamma_j = 0 \), \( j = k+1, \ldots, m \), for some permutation \( i_j \). Consequently, with \( W = (AA^*)^{1/2} \) inequality (1.3) can be written as

\[
c_\varphi(W + \varepsilon W^+) \leq c_\varphi(W).
\]

But \( c_\varphi(W) = c_\varphi(\varepsilon W^+) \), so that this follows from the above generalization of (1.5).

As a very special case of (1.3), we have for positive definite matrices \( A \) that, for any \( u_{\perp} \), \( u_1 > 0 \),

\[(4.1) \quad c_\varphi(u_{\perp} A^{-1} + u_1 A) \leq c_\varphi(A).\]

It is of interest to compare this with the following (Marshall and Olkin, 1969): If \( A \) is positive definite, \( \varphi \) is unitarily invariant, \( 1 \leq v_1 \leq \ldots \leq v_\ell \) and \( u_i \geq 0 \), \( 0 \leq i \leq \ell \), then

\[
c_\varphi(A) \leq c_\varphi(u_0 A + \ldots + u_\ell A^{-v_\ell}) \leq c_\varphi(A^{-v_\ell}),
\]

\[
c_\varphi(A^{-1}) \leq c_\varphi(u_0 A^{-1} + \ldots + u_\ell A^{v_\ell}) \leq c_\varphi(A^{-v_\ell}).
\]

One might be tempted to conjecture that (4.1) can be extended as follows:
\[ c_\varphi(u_1A^{-1} + u_2A^{-2} + \ldots + u_{j-1}A^{-j} + u_jA^{j}) \leq c_\varphi(u_xA^{-k} + \ldots + u_{j-1}A^{-j} + u_jA^{j}) \]

This is false, as can be seen by taking \( A = \text{diag}(1,1/2) \) and \( \phi(x_1, x_2) = \max(|x_1|, |x_2|) \). Then

\[ c_\varphi(A^{-1} + A + A^2) = \frac{12}{11} < \frac{5}{4} = c_\varphi(A^{-1} + A) . \]

5. **An Application of Majorization.** We have shown in §3 that for any unitarily invariant norm \( \varphi \), and for positive definite and certain other matrices \( U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \),

\[ c_\varphi(U^{1/2}) \geq c_\varphi(U_{11}^{1/2}) . \]

Using different methods, we show here that

\[(5.1) \quad c_\varphi(U) \geq c_\varphi(\begin{pmatrix} U_{11} & 0 \\ 0 & U_{22} \end{pmatrix}) \geq c_\varphi(U_{11}) \]

whenever \( U \) is positive semi-definite and \( \varphi \) is unitarily invariant.

The second inequality of (5.1) is immediate from the fact that unitarily invariant norms are monotone (see §1). To prove the second inequality, let \( \Gamma = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix} \) be a unitary matrix such that

\[ \Gamma_1 U_{11} \Gamma_1^* = \text{diag}(\beta_1, \ldots, \beta_q) \equiv D_\beta, \quad \text{and} \quad \Gamma_2 U_{22} \Gamma_2^* = \text{diag}(\beta_1, \ldots, \beta_{n-q}) \equiv D_\delta . \]
Of course, the characteristic roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $U$ are the same as the characteristic roots of

$$\Gamma UT^* = \begin{pmatrix} D_\beta & \Gamma_{11} U_{12} \Gamma_{21}^* \\ \Gamma_{12} U_{12}^* \Gamma_{21}^* & D_\delta \end{pmatrix}.$$

According to a result of I. Schur, the vector $\mathbf{d} = (\beta, \delta) = (d_1, d_2, \ldots, d_n)$ of diagonal elements of a positive semi-definite matrix is majorized by the vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ of characteristic roots in the sense that, possibly after reordering components,

$$d_1 \geq \ldots \geq d_n, \quad \alpha_1 \geq \ldots \geq \alpha_n$$

$$\sum_{i=1}^k d_i \leq \sum_{i=1}^k \alpha_i, \quad k = 1,2,\ldots,n-1, \quad \sum_{i=1}^n d_i = \sum_{i=1}^n \alpha_i.$$

This means (Marshall and Olkin, 1965, Lemma 3.3) that if $k$ is defined by $d_k > 0, d_{k+1} = \ldots = d_n = 0$ (so also $\alpha_k > 0, \alpha_{k+1} = \ldots = \alpha_n = 0$),

and if $\Phi$ is the SGF which corresponds to $\varphi$ as in (2.2), then

$$\Phi(d_1, \ldots, d_k, 0, \ldots, 0) \leq \Phi(\alpha_1, \ldots, \alpha_k, 0, \ldots, 0),$$

$$\Phi(d_1^{-1}, \ldots, d_k^{-1}, 0, \ldots, 0) \leq \Phi(\alpha_1^{-1}, \ldots, \alpha_k^{-1}, 0, \ldots, 0).$$

These inequalities together prove the first inequality of (5.1).
From the fact that the characteristic roots of \[
\begin{pmatrix}
U_{11} & 0 \\
0 & U_{22}
\end{pmatrix}
\] are majorized by the characteristic roots of \(U\), one might conjecture that if
\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}, \quad \tilde{A} = \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{21} & A_{22} & A_{23} \\
0 & A_{32} & A_{33}
\end{bmatrix}
\]
are positive definite, then the characteristic roots of \(\tilde{A}\) are majorized by the characteristic roots of \(A\). That this is false can be seen from the choice
\[
A = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{pmatrix}, \quad |a| < \frac{1}{\sqrt{2}}.
\]
The characteristic roots \((1-a, 1-a, 1+2a)\) of \(A\), and the characteristic roots \((1-a\sqrt{2}, 1, 1+a\sqrt{2})\) of \(\tilde{A}\) are not ordered either way by majorization.

Since we have obtained \(c_\varphi(U_{11}) \leq c_\varphi(U)\) and \(c_\varphi(U_{11}^\frac{1}{2}) \leq c_\varphi(U_{11}^\frac{1}{2})\), it is natural to inquire if one of the inequalities
\[
(i) \ c_\varphi(A) \leq c_\varphi(B), \quad (ii) \ c_\varphi(A^\frac{1}{2}) \leq c_\varphi(B^\frac{1}{2})
\]
is implied by the other. If \(A = \text{diag}(625, 25, 1)\) and \(B = \text{diag}(325, 325, 1)\) then it is easily checked that with \(\varphi(A) = (\text{tr } AA^*)^\frac{1}{2}\), (i) is violated but (ii) holds. On the other hand, interchanging these special \(A\)
and $B$ shows that (i) can hold when (ii) is violated.

A comparison of (3.1) and (5.1) suggests the possibility that

$$c \varphi(U^{\frac{1}{2}}) \geq c \varphi \begin{pmatrix} U^{\frac{1}{2}} & 0 \\ \frac{1}{11} & 0 \end{pmatrix} U^{\frac{1}{2}}$$

when $U$ is positive definite. Whether or not this is true remains an open question.
REFERENCES


