GEOMETRIC AND PSEUDO-GEOMETRIC GRAPHS \((q^2+1,q+1,1)\)

BY

R. C. BOSE and S. S. SHRIKHANDE

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GEOMETRIC AND PSEUDO-GEOMETRIC GRAPHS \((q^2+1, q+1, 1)\)

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Abstract

Let \(\theta_0\) be a particular vertex of a strongly regular graph \(G\) with parameters \(v, n_1, p^1_{ll}, p^2_{ll}\). Let \(A\) be the adjacency matrix of \(G\), and \(B\) the submatrix of \(A\) whose rows correspond to the vertices of \(G\) adjacent to \(\theta_0\) and whose columns correspond to the vertices of \(G\) non-adjacent to \(\theta_0\). Then the design \(\mathcal{Q}(\theta_0)\) with incidence matrix \(B\) has the parameters \(v' = n_1, b' = v - n_1 - 1, r' = n_1 p^1_{ll} - 1, k' = p^2_{ll}\). In this paper we study the connection between \(G\) and \(\mathcal{Q}(\theta_0)\) when the graph \(G\) is geometric or pseudo-geometric \((q^2+1, q+1, 1)\).

1. Introduction.

1. A regular undirected graph \(G\) with \(v\) vertices and valence \(n_1\), without loops or multiple edges, is defined to be a strongly regular graph with parameters

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if any two adjacent vertices are both adjacent to $p_{11}$ other vertices, and any two non-adjacent vertices are both adjacent to $p_{21}$ other vertices.

Two vertices of $G$ may be called first associates if they are adjacent and second associates if they are non-adjacent. Given two distinct vertices $\theta$ and $\phi$ of $G$ which are $i$-th associates, the number of vertices in $G$ which are $j$-th associates of $\theta$ and $k$-th associates of $\phi$ is denoted by $p^i_{jk}$ $(i,j,k=1,2)$. It is known (Bose (1965), Bose and Clatworthy (1955)) that the value of $p^i_{jk}$ is independent of the pair of vertices $\theta$ and $\phi$ (which are $i$-th associates). Let $n_2$ denote the number of second associates of any vertex. The parameters $v, n_1, p^i_{jk}$ are known to satisfy the following relations.

\begin{align}
(1.1) \quad v, n_1, p^1_{11}, p^2_{11} \\
\text{if any two adjacent vertices are both adjacent to $p^1_{11}$ other vertices, and any two non-adjacent vertices are both adjacent to $p^2_{11}$ other vertices.}
\end{align}

\begin{align}
(1.2) \quad n_1 + n_2 = v - 1, \quad p^1_{12} = p^1_{21}, \quad p^2_{12} = p^2_{21}. \\
(1.3) \quad p^1_{11} + p^1_{12} + 1 = n_1 = p^2_{11} + p^2_{12}. \\
(1.4) \quad p^2_{12} + p^2_{22} + 1 = n_2 = p^1_{12} + p^1_{22}. \\
(1.5) \quad n^1_{1} p^1_{12} = n^2_{1} p^2_{12}, \quad n^1_{1} p^1_{22} = n^2_{2} p^2_{12}. 
\end{align}
Though the parameters are all completely determined by \( v, n_1, p_{11}^1, p_{11}^2 \), it is sometimes useful to have their values at hand. In this case the parameters of \( G \) may be written as

\[
(1.6) \quad \begin{bmatrix}
    v, n_1, n_2 \\
    p_{11}^1 & p_{12}^1 \\
    p_{22}^1 & p_{22}^2
\end{bmatrix}, \quad (p_{jk}^1) = \begin{bmatrix}
    p_{11}^2 & p_{12}^2 \\
    p_{22}^2 & p_{22}^2
\end{bmatrix}.
\]

2. The concept of strongly regular graphs is isomorphic with the concept of association schemes (Bose and Mesner (1959), Bose and Shimamoto (1952)) of partially balanced incomplete block (PBIB) designs. Such a scheme may be defined as a scheme or relations between \( v \) objects such that

(i) any two objects are either first associates or second associates,

(ii) each object has \( n_1 \) \( i \)-th associates; \( i = 1, 2 \),

(iii) if two treatments are \( i \)-th associates, then the number of treatments common to the \( j \)-th associates of the first and \( k \)-th associates of the second is \( p_{jk}^1 \) and is independent of the pair of treatments with which we start. Also \( p_{jk}^1 = p_{kj}^1 \).

From Bose and Clatworthy (1955) it is unnecessary to assume the constancy of all the \( r_{jk}^1 \)'s. If we assume that \( v, n_1, p_{11}^1, p_{11}^2 \) are constant, then the constancy of the other parameters follows, and \( p_{12}^1 = p_{21}^1, p_{12}^2 = p_{21}^2 \).

If we now identify the \( v \) objects of the association scheme with the \( v \) vertices of a graph \( G \), and consider two vertices as adjacent
or non-adjacent according as the objects are first associates or second associates, then a strongly regular graph with parameters (1.6), (1.7) is isomorphic to an association scheme with the same parameters.

3. A partial geometry \((r,k,t)\) is a system of points and lines with an incidence relation satisfying the four axioms given below (Bose (1963)). To avoid cumbersome expression we may use standard geometric language. Thus a point incident with a line may be said to be on it, and the line may be said to pass through the point. If two points are incident with a line we say that they are joined by the line, and if two lines are incident with a point we say that they intersect. The axioms are as follows:

A1. Any two points are incident with at most one line.
A2. Each point is incident with \(r\) lines.
A3. Each line is incident with \(k\) points.
A4. If the point \(P\) is not incident with the line \(L\), there pass through \(P\) exactly \(t\) lines intersecting \(L\) where \(1 \leq t\).

It can then be shown that the number of points \(v\) and the number of lines \(b\) in a partial geometry \((r,k,t)\) is given by

\[
(1.8) \quad v = k[(r-1)(k-1)+t]/t ,
\]

\[
(1.9) \quad b = r[(r-1)(k-1)+t]/t .
\]

The graph \(G\) of a partial geometry is defined as the graph whose vertices are the points of the geometry, and in which two vertices of \(G\) are adjacent or non-adjacent according as the corresponding points are
incident or non-incident with a common line. The graph of a partial geometry \((r,k,t)\) will be called a geometric graph \((r,k,t)\). It can be shown that a geometric graph \((r,k,t)\) is strongly regular with parameters

\[(1.10) \quad v = k[(r-1)(k-1)+t]/t, \quad n_1 = r(k-1),\]

\[(1.11) \quad p_{11}^1 = (t-1)(r-1) + k - 2, \quad p_{11}^2 = rt.\]

A strongly regular graph which has the same parameters as a geometric graph \((r,k,t)\) is called a pseudo-geometric graph \((r,k,t)\).

In this paper we shall study geometric and pseudo-geometric graphs \((q^2+1,q+1,1)\). The values of \(v, n_1, p_{11}^1, p_{11}^2\) are obtained by putting \(r = q^2 + 1\) and \(k = q + 1\) in \((1.10)\) and \((1.11)\). The other parameters are obtained by using the identities \((1.2)-(1.5)\). They are explicitly given by \((2.1)\) and \((2.2)\).

4. Given a set of \(v\) treatments, arranged in \(b\) sets or blocks, such that each treatment occurs in \(r\) blocks, and each block contains \(k\) distinct treatments we have a design \(\mathcal{D}\) with parameters \(v, b, r, k\). The incidence matrix of \(\mathcal{D}\) is a \(v \times b\) matrix \((n_{ij})\) such that \(n_{ij} = 1\) or \(0\) according as the \(i\)-th treatment does or does not occur in the \(j\)-th block.

The design \(\mathcal{D}\) is said to be a balanced incomplete block (BIB) design (Bose (1938)) if it satisfies the additional condition that any pair of treatments occurs together in exactly \(\lambda\) blocks. The parameters of the design are then \(v, b, r, k, \lambda\).
Given a (two class) association scheme on \( v \) objects with parameters (1.6) and (1.7), a (two class) partially balanced incomplete block (PBIB) design based on this association scheme is a design whose treatments are the objects of the association scheme, and in which a pair of treatments occurs together in exactly \( \lambda_1 \) blocks if the treatments are first associates, and exactly \( \lambda_2 \) blocks if the treatments are second associates.

Two particular association schemes, viz. the group divisible and the negative Latin square association schemes described are of special importance for our purposes.

If \( v = mn \) objects are partitioned into \( m \) disjoint sets, each containing \( n \) objects, then we get a group divisible (GD) association scheme by taking two objects to be first associates if they belong to the same set and second associates if they do not belong to the same set. The parameters of the association scheme are

\[
(1.12) \quad v = mn, \quad n_1 = n - 1, \quad p_{11}^1 = n - 2, \quad p_{11}^2 = 0.
\]

A PBIB design based on a GD association scheme is said to be a group divisible or a GD design. Two treatments which belong to the same set occur together in \( \lambda_1 \) blocks, and two treatments not belonging to the same set occur together in \( \lambda_2 \) blocks. The parameters of a design are \( v, b, r, k, m, n, \lambda_1, \lambda_2 \).

The combinatorial properties of GD designs are given by Bose and Connor (1952). GD designs can be divided into three types, singular, semi-regular, or regular, according as (i) \( r = \lambda_1 \), (ii) \( r > \lambda_1 \), \( rk - \lambda_2 v = 0 \), (iii) \( r > \lambda_1 \), \( rk - \lambda_2 v > 0 \).
It can be shown that for a semi-regular group divisible (SRGD) design each block contains exactly $k/m$ treatments from each set.

Another important association scheme is the negative Latin square association scheme with parameters

\[(1.15) \quad v = k^2, \quad n_1 = r(k+1), \quad p_{11}^{1} = (r+1)(r+2) - (k+2), \quad p_{11}^{2} = r(r+1).\]

The corresponding graph may be called an $NL_r(k)$ graph. The combinatorial properties of these schemes were studied by Mesner (1967). They are also of importance in connection with strongly regular graphs for which $p_{11}^{1} = p_{11}^{2} = d$. (See Bose and Shrikhande (1970)).

5. The adjacency matrix $A$ of a graph $G$ is defined to be the matrix $A = (a_{ij})$ where $a_{ii} = 0$, and for $i \neq j$, $a_{ij} = 1$ or 0 according as the $i$-th vertex is adjacent or non-adjacent to the $j$-th vertex. It is easy to see that if $G$ is a strongly regular graph with parameters (1.1) then its adjacency matrix $A$ satisfies

\[(1.14) \quad A^2 = n_1 I_v + p_{11}^{1} A + p_{11}^{2} (J_v - I_v - A)\]

where $I_v$ is the identity matrix of order $v$ and $J_v$ is a matrix with all elements 1.

Let $\theta_0$ be a particular vertex of $G$. Let $B$ be the $n_1 \times n_2$ submatrix of $A$, whose rows correspond to the vertices $\theta_1, \theta_2, \ldots, \theta_{n_1}$ which are adjacent to $\theta_0$, and whose columns correspond to the vertices $\beta_1, \beta_2, \ldots, \beta_{n_2}$ which are non-adjacent to $\theta_0$. There are exactly $p_{12}^{1}$
unities in the row of $B$ corresponding to $\theta_1$, since there are $p_{12}^1$ vertices among $\beta_1, \beta_2, \ldots, \beta_{n_2}$ which are adjacent to both $\theta_0$ and $\theta_1$. Similarly the column of $B$ which corresponds to $\beta_j$ has exactly $p_{11}^2$ unities, since there are $p_{11}^2$ vertices among $\theta_1, \theta_2, \ldots, \theta_{n_1}$ which are adjacent to both $\theta_0$ and $\beta_j$. Hence $B$ is the incidence matrix of a design with parameters $v' = n_1, b' = n_2, r' = p_{12}^1, k' = p_{11}^2$. We say that this design corresponds to the vertex $\theta_0$, and we denote it by $\mathcal{D}(\theta_0)$.

In this paper we shall study the connection between a geometric or a pseudo-geometric graph $(q+1, q+1, 1)$ and the design $\mathcal{D}(\theta_0)$ corresponding to a particular vertex $\theta_0$.

2. Pseudo-geometric graphs.

1. Let $G$ be a pseudo-geometric graph $(r, k, 1)$. Then $G$ will be said to have the property $(P)$ with respect to the vertex $\theta_0$, if the $r(k-1)$ vertices adjacent to $\theta_0$ can be partitioned into $r$ disjoint sets $S_1, S_2, \ldots, S_r$ such that any two vertices belonging to the same set $S_i$ are adjacent. Then $K_1 = S_i \cup \theta_0$ is a clique of size $k$. Since $t = 1$, any vertex $\theta_{iu}$ which belongs to $S_i$ is non-adjacent to any vertex $\theta_{i'u'}$ which belongs to $S_{i'}$, $i \neq i'$.

We shall say that $G$ has the additional property $(P*)$ if any two different vertices $\beta_j, \beta_j'$, both non-adjacent to $\theta_0$, and both adjacent to $\theta_1$ and $\theta_1'$, are themselves non-adjacent, where $\theta_{ij}$ and $\theta_{i'j'}$ are any two vertices belonging to $S_i$ and $S_{i'}$, respectively, $i \neq i'$. The property $(P*)$ is equivalent to the non-existence of a subgraph
of $G$ with vertex $\theta_0$.

Let $\mathcal{D}(\theta_0)$ be the design corresponding to the vertex $\theta_0$ of a pseudo-geometric $(q^2+1, q+1, 1)$ graph $G$, which has the property (P) with respect to the vertex $\theta_0$. We show that $\mathcal{D}(\theta_0)$ is a semi-regular group divisible (SRGD) design with $q^2+1$ sets each of size $q$.

Now the parameters of $G$ are

\begin{equation}
(2.1) \quad v = (q+1)(q^3+1), \quad n_1 = q(q^2+1), \quad n_2 = q^4 ;
\end{equation}

\begin{equation}
(2.2) \quad (p_{jk}^1) = \begin{bmatrix} q-1 & q^3 \\ q^3(q-1) \end{bmatrix}, \quad (p_{jk}^2) = \begin{bmatrix} q^2+1 & (q^2+1)(q-1) \\ q(q-1)(q^2+1) \end{bmatrix}.
\end{equation}

By definition the treatments of $\mathcal{D}(\theta_0)$ correspond to the vertices of $G$ adjacent to $\theta_0$, and the blocks of $\mathcal{D}(\theta_0)$ correspond to the vertices of $G$ non-adjacent to $\theta_0$. Hence the number of treatments is $v' = n_1 = q(q^2+1)$, and the number of blocks is $b' = n_2 = q^4$. 

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Let $S$ be the set of $q(q^2+1)$ vertices of $G$ adjacent to $\theta_0$ and $B$ be the set of $q^4$ vertices of $G$ non-adjacent to $\theta_0$. Since $G$ has the property (P) with respect to the vertex $\theta_0$, the vertices in $S$ can be partitioned into $q^2+1$ disjoint sets $S_1, S_2, \ldots, S_{q^2+1}$ each of size $q$, such that any two vertices belonging to the same set $S_i$ are adjacent, $i = 1, 2, \ldots, q^2+1$, and any two vertices belonging to different set $S_i$ and $S_{i'}$, $i \neq i'$, are non-adjacent. This gives a partition of the $q(q^2+1)$ treatments of $\mathcal{D}(\theta_0)$ into $q^2+1$ sets each of size $q$, two treatments belonging to the same set if and only if the corresponding vertices belong to the same set $S_i$. We shall denote the vertices in the set $S_i$ by $\theta_{i1}, \theta_{i2}, \ldots, \theta_{iq}$. The treatment of $\mathcal{D}(\theta_0)$ corresponding to the vertex $\theta_{iu}$ will also be denoted by $\theta_{iu}$. We shall denote the vertices of $G$ non-adjacent to $\theta_0$ (i.e., vertices belonging to the set $B$) by $\beta_1, \beta_2, \ldots, \beta_{q^4}$. The block of $\mathcal{D}(\theta_0)$ corresponding to the vertex $\beta_j$ will also be denoted by $\beta_j$, $j = 1, 2, \ldots, q^4$. By definition the treatment $\theta_{iu}$ is contained in the block $\beta_j$ if and only if the vertex $\theta_{iu}$ is adjacent to the vertex $\beta_j$. Since each vertex in $S$ is adjacent to $p_{12} = q^3$ vertices in $B$, each treatment of $\mathcal{D}(\theta_0)$ occurs in $r' = q^3$ blocks. Again each vertex in $B$ is adjacent to $p_{11}^2 = q^2+1$ vertices in $S$. Hence each block of $\mathcal{D}(\theta_0)$ contains $k' = q^2+1$ treatments.

Let $\theta_{iu}$ and $\theta_{iu'}$ be two distinct vertices of $G$ belonging to the same set $S_i$. Then $\theta_{iu}$ and $\theta_{iu'}$ are both adjacent to the other $q-2$ vertices in $S_i$, and also to $\theta_0$. Since $p_{11}^1 = q-1$, it follows that $\theta_{iu}$ and $\theta_{iu'}$ are not simultaneously adjacent to any other vertex of $G$.

In particular no vertex in $B$ is adjacent to both $\theta_{iu}$ and $\theta_{iu'}$. Hence
any two treatments $\theta_{iu}$ and $\theta_{iu'}$ of $\mathcal{D}(\theta_0)$, which belong to the same set, cannot occur together in a block.

Again let $\theta_{iu}$ and $\theta_{i'u'}$ be two vertices of $G$, both adjacent to $\theta_0$, but belonging to different sets $S_i$ and $S_i'$. Then $\theta_{iu}$ and $\theta_{i'u'}$ cannot be simultaneously adjacent to any vertex in $S$. They are simultaneously adjacent to $\theta_0$. Hence they are both adjacent to $p_{11}^2 - 1 = q^2$ vertices in $B$. Hence any two treatments of $\mathcal{D}(\theta_0)$ which belong to different sets occur together in $q^2$ blocks.

We have thus shown that $\mathcal{D}(\theta_0)$ is a group divisible design with parameters

\begin{equation}
(2.3) \quad v' = q(q^2 + 1), \quad b' = q^4, \quad r' = q^3, \quad k' = q^2 + 1, \quad m' = q^2 + 1, \quad n' = q + 1, \quad \lambda_1' = 0, \quad \lambda_2' = q^2.
\end{equation}

Since $\lambda_2'v' - r'k' = 0$, the design is semi-regular. Hence each block contains exactly one treatment from each set (Bose and Connor (1952)). Hence we can state:

**Theorem (2.1).** If $G$ is a pseudo-geometric graph $(q^2 + 1, q + 1, 1)$ having the property $(P)$ with respect to the vertex $\theta_0$, then the design $\mathcal{D}(\theta_0)$ corresponding to the vertex $\theta_0$ is a semi-regular group divisible (SRGD) design, with parameters $(2.3)$.

**Corollary (2.1).** Each vertex non-adjacent to $\theta_0$, is adjacent to exactly one vertex in each of the sets $S_1, S_2, \ldots, S_{q^2 + 1}$.
2. Two blocks of $\mathcal{D}(\theta_0)$ will be called first associates if they correspond to two vertices in $B$ which are adjacent, and second associates if they correspond to two vertices in $B$ which are non-adjacent. Since each vertex in $B$ is adjacent to $p_{12}^2$ other vertices in $B$, and non-adjacent to $p_{22}^2$ other vertices in $B$, each block of $\mathcal{D}(\theta_0)$ is first associate of $m_1'$ blocks and second associate of $m_2'$ blocks where

$$(2.4) \quad m_1' = (q^2+1)(q-1), \quad m_2' = q(q-1)(q^2+1).$$

Let $\theta_{iu}$ be a treatment occurring in the block $\beta_j$. Then the vertex $\theta_{iu}$ belonging to $S_i$ is adjacent to the vertex $\beta_j$ belonging to $B$. Now $\theta_{iu}$ and $\beta_j$ are both adjacent to $p_{11}^1 = q-1$ vertices. These must all belong to $B$, because $\beta_j$ is not adjacent to $\theta_0$ and to any vertex of $S_i$ other than $\theta_{iu}$, and $\theta_{iu}$ is not adjacent to any vertex in $S - S_i$. This shows that the treatment $\theta_{iu}$ occurs in exactly $q - 1$ blocks which are first associates of $\beta_j$. Since $\theta_{iu}$ must occur in $r' - 1$ blocks other than $\beta_j$, it occurs in exactly $q^3 - q$ blocks which are second associates of $\beta_j$.

Again let $\theta_{iu}'$ be a treatment not occurring in the block $\beta_j$. Then the vertex $\theta_{iu}'$ belonging to $S_i$ is not adjacent to the vertex $\beta_j$. Let $\theta_{iu}$ be the unique vertex of $S_i$ which is adjacent to $\beta_j$. Now $\theta_{iu}'$ and $\beta_j$ are both adjacent to $p_{11}^2 = q^2 + 1$ vertices of $G$. Of these all except $\theta_{iu}$ belong to $B$. This shows that the treatment $\theta_{iu}'$ occurs exactly $q^2$ times among the blocks which are first associates of $\beta_j$, and hence exactly $q^3 - q^2 - 1$ times among the blocks which are second associates.
of \( \beta_j \). Hence with the notation and hypothesis of Theorem (2.1), we can state:

**Corollary (2.2).** If a treatment occurs in a block \( \beta_j \) of \( \mathcal{R}(\theta_0) \), then it occurs \( q-1 \) times among the blocks which are first associates of \( \beta_j \), and \( q^2 - q \) times among the blocks which are second associates of \( \beta_j \). If a treatment does not occur in a block \( \beta_j \) of \( \mathcal{R}(\theta_0) \) then it occurs \( q^2 \) times among the blocks which are first associates of \( \beta_j \) and \( q^2 - q^2 - 1 \) times among the blocks which are second associates of \( \beta_j \).

Let us now assume that \( G \) has the additional property (P*). We shall show that under this hypothesis any two blocks of \( \mathcal{R}(\theta_0) \) intersect in one treatment if they are first associates and in \( q-1 \) treatments if they are second associates. Let the block \( \beta_j \) contain two treatments \( \theta_{iu} \) and \( \theta_{i'j'} \). They must belong to different sets. Hence the vertex \( \theta_{iu} \) belongs to \( S_1 \) and the vertex \( \theta_{i'j'} \) belongs to \( S_1', i \neq i' \). If the pair \( \theta_{iu}, \theta_{i'j'} \) occurs in a block \( \beta_j \), which is a first associate of \( \beta_j \), then \( G \) has a subgraph shown in Figure 1, which is a contradiction. Hence two blocks of \( \mathcal{R}(\theta_0) \) which are first associates cannot have more than one treatment in common. Since each treatment of \( \beta_j \) occurs \( q-1 \) times among the first associates of \( \beta_j \), the number of blocks which are first associates of \( \beta_j \) and contain exactly one treatment of \( \beta_j \) is \( (q^2 + 1)(q-1) \). But this is the total number of first associates of \( \beta_j \). Hence any two blocks of \( \mathcal{R}(\theta_0) \) which are first associates intersect in exactly one treatment.

Let us now consider the distribution of the \( k' = q^2 + 1 \) treatments belonging to the block \( \beta_j \) of \( \mathcal{R}(\theta_0) \) among the \( m_2 \) blocks which are
second associates of $\beta_j$, where $m_2'$ is given by (2.4). Let $x_i$ be the number of treatments in which $\beta_j$ intersects the $i$-th block which is a second associate of $\beta_j$, $i = 1, 2, \ldots, m_2'$. By Corollary (2.2) each treatment of $\beta_j$ occurs $q^3 - q$ times among the second associates of $\beta_j$. Again any pair of treatments belonging to $\beta_j$ must occur $\chi_i^2 - 1 = q^2 - 1$ times among the second associates of $\beta_j$, since it cannot occur among the first associates. Hence

$$
\sum_{i=1}^{m_2'} x_i = (q^2 + 1)(q^3 - q) ,
$$

$$
\sum_{i=1}^{m_2'} x_i (x_i - 1) = (q^2 + 1)q^2(q^2 - 1) .
$$

Let

$$
\overline{x} = \frac{\sum_{i=1}^{m_2'} x_i / m_2'}{m_2'} = q + 1 .
$$

Hence

$$
\sum_{i=1}^{m_2'} (x_i - \overline{x})^2 = 0 .
$$

This shows that $x_i - \overline{x} = 0$, i.e., any two blocks of $\mathcal{D}(\theta_0)$ which are second associates, intersect in exactly $q+1$ treatments.
Theorem (2.2). If the pseudo-geometric graph \((q^2 + 1, q + 1, 1)\) of Theorem (2.1) has the additional property \((P^*)\) with respect to the vertex \(\theta_0\), then the design \(\mathcal{D}(\theta_0)\) corresponding to the vertex \(\theta_0\) has the property that any two blocks which are first associates intersect in exactly one treatment and any two blocks which are second associates intersect in \(q + 1\) treatments.

3. We shall now show that if \(G\) is the graph satisfying the hypothesis of Theorem (2.2) and if \(G_2\) is the subgraph of \(G\) whose vertices are the set of those vertices of \(G\) which are non-adjacent to \(\theta_0\), then \(G_2\) is a negative Latin square graph \(NL_{q-1}(q^2)\).

Let \(\beta_j\) be any vertex of \(G\), non-adjacent to \(\theta_0\). We have shown that \(\beta_j\) is adjacent to exactly \(q^2 + 1\) vertices in \(S\). Hence \(\beta_j\) is adjacent to exactly \(n^* = n_1 - (q^2 + 1) = (q - 1)(q^2 + 1)\) vertices in \(B\).

Let \(\beta_j\) and \(\beta_j'\) be two adjacent vertices belonging to \(B\). Since the blocks \(\beta_j\) and \(\beta_j'\) of \(\mathcal{D}(\theta_0)\) intersect in exactly one treatment, the vertices \(\beta_j\) and \(\beta_j'\) are adjacent to exactly one vertex of \(G\) belonging to \(S\). Hence \(\beta_j\) and \(\beta_j'\) are both adjacent to \(p_{ll}^1 = p_{ll}^1 - 1 = q - 2\) vertices in \(B\).

Let \(\beta_j\) and \(\beta_j'\) be two non-adjacent vertices belonging to \(B\). Since the blocks \(\beta_j\) and \(\beta_j'\) of \(\mathcal{D}(\theta_0)\) intersect in exactly \(q + 1\) treatments, the vertices \(\beta_j\) and \(\beta_j'\) are both adjacent to exactly \(q + 1\) vertices of \(G\) belonging to \(S\). Hence \(\beta_j\) and \(\beta_j'\) are both adjacent to \(p_{ll}^2 = p_{ll}^2 - (q + 1) = q(q - 1)\) vertices in \(B\). Hence we have

Theorem (2.3). If \(G\) is a pseudo-geometric graph \((q^2 + 1, q + 1, 1)\) having the properties \((P)\) and \((P^*)\) with respect to the vertex \(\theta_0\),
then the subgraph $G_2$ of $G$, whose vertex set is the set of those vertices of $G$ which are non-adjacent to $\theta_0$, is a strongly regular graph with parameters

$$(2.5) \quad v^* = q^4, \quad n_1^* = (q-1)(q^2+1), \quad p_{11}^* = q - 2, \quad p_{11}^{2*} = q(q-1)$$

i.e., a negative Latin square graph $NL_{q-1}(q^2)$.

**Corollary (2.3).** The dual $\mathcal{D}^*(\theta_0)$ of $\mathcal{D}(\theta_0)$ is a partially balanced incomplete block (PBIB) design based on a negative Latin square association scheme $NL_{q-1}(q^2)$.

4. We shall next prove the converse of Theorem (2.2). Given an SRGD design $\mathcal{D}$ with parameters (2.3), and having the additional property (I) that any two blocks of $\mathcal{D}$ intersect in either 1 or $q+1$ treatments, we shall show that there exists a pseudo-geometric graph $G$, having the properties (P) and (P*), with respect to a vertex $\theta_0$ of $G$, such that $\mathcal{D}$ is isomorphic to $\mathcal{D}(\theta_0)$.

Let the treatments of $\mathcal{D}$ be denoted by $\theta_{iu} (i=1,2,...,q^2+1; u=1,2,...,q)$, treatments with the same first subscript belonging to the same set. We can so order the treatments that treatments with a smaller first subscript come before treatments with a larger first subscript. When two treatments have the same first subscript then the one with the smaller second subscript comes earlier. Let the blocks of $\mathcal{D}$ be denoted by $\beta_j, j=1,2,...,q^4$. Let $B$ be the incidence matrix of $\mathcal{D}$. Let us call two blocks of $\mathcal{D}$ first associates if they intersect in a single treatment, and second associates if they intersect in $q+1$ treatments. Let $E$ be the association
matrix of $\mathbf{D}$, i.e., $E = (e_{j_1j_2}^i)$ where $e_{j_1j_2}^i = 1$ if the blocks $\beta_j$ and $\beta_j$ are first associates and zero if $j_1 = j_2$ or $\beta_j$ and $\beta_j$ are second associates. Then

$$B^T B = (q^2+1) I_{n_2} + E + (q+1)(J - E - I_{n_2})$$

$$= -qE + (q^2-q) I_{n_2} + (q+1) J_{n_2},$$

where $n_2 = \frac{1}{4}$.

Since the $q^2+1$ treatments in any block $\beta_j$, intersect the first associates of $\beta_j$ in 1 treatment and the second associates in $q+1$ treatments, it follows that

$$m_1^i + m_2^i = q - 1, \quad m_1^i + m_2^i(q+1) = (q^2+1)(q^2-1)$$

where $m_1^i$ is the number of $i$-th associates of $\beta_j$, $i=1,2$. Hence

$$m_1^i = (q^2+1)(q-1), \quad m_2^i = q(q-1)(q^2+1).$$

Thus each row sum of $E$ is $m_1^i$. Hence $E$ has a simple characteristic root $(q^2+1)(q-1)$. Again from Bose and Connor (1952) the distinct characteristic roots of $BB^T$ and therefore of $B^T B$ are

$$r'k' = q^2(q^2+1), \quad r' - \lambda_1' = q^3, \quad r'k' - \lambda_2'v' = 0.$$
Hence from (2.6), the characteristic roots of $E$ other than the simple root are $-(q^2-q+1)$ and $q-1$. From Shrikhande and Bhagwandas (1965) it follows that $E$ is the adjacency matrix of a strongly regular graph with parameters (2.5), i.e., an $NL_{q-1}(q^2)$ graph, and that

$$(2.6) \quad E^2 = (p_{11}^* - p_{11} E + q^2 J_{n_2} + (m' - p_{11}^*) I_{n_2})$$

$$= -(q^2-2q+2)E + q(q-1)J_{n_2} + (q^3-2q^2+2q-1)I_{n_2}. $$

Any two treatments of $\beta_j$ belong to different sets and therefore occur together in $\lambda_2' = q^2$ blocks of $\mathcal{D}$. Hence the configuration of the $q^2+1$ treatments of $\beta_j$, occurring in the second associates of $\beta_j$, is a balanced incomplete block (BIB) design with parameters

$$v = q^2+1, \quad b = q(q-1)(q^2+1), \quad k = q+1, \quad \lambda = q^2-1.$$  

Hence $r = q(q^2-1)$, i.e., each treatment of $\beta_j$ occurs $q(q^2-1)$ times among the second associates of $\beta_j$, and in consequence $r' - r - 1 = q-1$ times among the first associates of $\beta_j$.

Now let $\theta_{iu}$ be a treatment of $\mathcal{D}$, not occurring in $\beta_j$. Suppose it occurs $y_1$ times among the first associates and $y_2$ times among the second associates of $\beta_j$. If $\theta_{iu}$ belongs to the $i$-th set then $\beta_j$ contains exactly one treatment, say $\theta_{iu}'$, from the $i$-th set. Hence $\theta_{iu}$ cannot occur together with $\theta_{iu}'$ in any block, but occurs together with every other treatment of $\beta_j$, $\lambda_2 = q^2$ times among the second associates of $\beta_j$. Hence
\[ y_1 + y_2 = r' = q^3, \quad y_1 + y_2(q+1) = \lambda_2(k'-1) = q^4. \]

Thus \( y_1 = q^2 \) and \( y_2 = q^3 - q^2 \), i.e., each treatment of \( \mathcal{O} \) which does not occur in \( \beta_j \) occurs \( q^2 \) times among the first associates of \( \beta_j \) and \( q^3 - q^2 \) times among the second associates of \( \beta_j \).

It now follows that

\[
(2.9) \quad BE = (q-1)B + q^2(J_{n_1', n_2} - B),
\]

where \( J_{n_1', n_2} \) is an \( n_1 \times n_2 \) matrix with each element unity and \( n_1 = v' = q(q^2+1), \quad n_2 = b' = q^4 \).

Let \( J_t \) denote a \( t \times 1 \) column vector all of whose elements unity, and let \( J_t' \) denote its transpose. Let \( O_u \) denote a \( u \times 1 \) column vector all of whose elements are zero, and let \( O_u' \) denote its transpose. Let \( K \) denote the \( n_1 \times n_2 \) matrix

\[
(2.10) \quad K = \text{diag}[J_q', J_q', \ldots, J_q'].
\]

Remembering that each block of \( \mathcal{O} \) contains exactly one treatment from each set

\[
(2.11) \quad KB = J_{n_1', n_2}.
\]

Consider the graph \( G \) with the adjacency matrix
\[ A = \begin{bmatrix}
0 & j'_{n_1} & 0'_{n_2} \\
j_{n_1} & K-I_{n_1} & B \\
0_{n_2} & B^T & E
\end{bmatrix} \]

Then
\[ A^2 = \begin{bmatrix}
j'_{n_1}j_{n_1} & j'_{n_1}(K-I_{n_1}) & j'_{n_1}B \\
(K-I_{n_1})j_{n_1} & j_{n_1}j_{n_1} + (K-I_{n_1})^2 + BB^T & (K-I_{n_1})B + BE \\
B^Tj_{n_1} & B^T(K-I_{n_1}) + EB^T & B^TB + E^2
\end{bmatrix} \]

Now
\[ j'_{n_1}j_{n_1} = n_{1}, \quad j'_{n_1}(K-I_{n_1}) = (q-1)j'_{n_1}, \quad j'_{n_1}B = (q^2+1)j'_{n_1}, \]
\[ j_{n_1}j_{n_1} = J_{n_1}, \quad (K-I_{n_1})^2 = K^2 - 2K + I_{n_1} = (q-2)K + I_{n_1}, \]
\[ BB^T = r'I_{n_1} + (K-I_{n_1})\lambda_1' + (J_{n_1} - K)\lambda_2', \]
\[ = q^2I_{n_1} - q^2K + q^2J_{n_1}. \]
Hence

\[
A^2 = \begin{bmatrix}
\begin{array}{ccc}
\underbrace{\begin{array}{c}
\underbrace{\begin{array}{c}
(q^{-1})j_{n_1}^T \vspace{1em}
\end{array}}
\end{array}} & \underbrace{\begin{array}{c}
\underbrace{\begin{array}{c}
(q^{-2})j_{n_1}^T \vspace{1em}
\end{array}}
\end{array}} & \underbrace{\begin{array}{c}
\underbrace{\begin{array}{c}
(q^{-3})j_{n_1}^T \vspace{1em}
\end{array}}
\end{array}}
\end{array}
\end{bmatrix}
\]

\[
(q^{-1})j_{n_1}^T \quad (q^{-2})j_{n_1}^T \quad (q^{-3})j_{n_1}^T
\]

\[
\begin{array}{ccc}
(q^{-1})j_{n_1}^T & -\frac{1}{2} \begin{bmatrix}
(q^{-2})j_{n_1}^T \\
(q^{-2})j_{n_2}^T
\end{bmatrix}
\end{array}
\]

\[
\begin{array}{ccc}
(q^{-1})j_{n_1}^T & (q^{-2})j_{n_1}^T \\
(q^{-2})j_{n_1}^T & (q^{-3})j_{n_1}^T
\end{array}
\]

\[
-\frac{1}{2} \begin{bmatrix}
(q^{-2})j_{n_1}^T \\
(q^{-2})j_{n_2}^T
\end{bmatrix} + (q^{-3})j_{n_2}^T
\]

\[
-\frac{1}{2} \begin{bmatrix}
q^{-2}j_{n_1}^T \\
q^{-2}j_{n_2}^T
\end{bmatrix} + (q^{-3})j_{n_2}^T
\]

\[
= -(q^{-2})A + (q^{-1})J_V + (q^{-3})q^{-1}I_V
\]

\[
= (\frac{1}{2} - \frac{1}{2})A + \frac{1}{2} J_V + (\frac{1}{2} - \frac{1}{2})I_V
\]

Hence from Lemma 2 of Shrikhande and Bhagwandas (1965), \( A \) is the adjacency matrix of a strongly regular graph \( G \) with parameters given by (2.1) and (2.2). If \( \theta_0 \) is the first vertex of this graph then \( B \) is the incidence matrix of \( \bigcirc (\theta_0) \). Hence \( \bigcirc \) and \( \bigcirc (\theta_0) \) are isomorphic. The treatments of \( \bigcirc \) correspond to the vertices of \( G \) adjacent to \( \theta_0 \), and the blocks of \( \bigcirc \) correspond to the vertices of \( G \) non-adjacent to \( \theta_0 \). The structure of \( A \) shows that \( G \) has the property \((P)\) with respect to the vertex \( \theta_0 \). The property \((P^*)\) follows from the fact that two blocks of \( \bigcirc \) which are first associates intersect in a single treatment.

**Theorem (2.4).** Given an SRGD design \( \bigcirc \) with parameters (2.3), having the property \((I_1)\) that any two blocks of \( \bigcirc \) intersect in
either 1 or \( q+1 \) treatments, there exists a pseudo-geometric \((q^2+1,q+1,1)\) graph \( G \), and a vertex \( \theta_0 \) of \( G \) such that \( D \) is isomorphic with the design \( D(\theta_0) \) corresponding to \( \theta_0 \), and \( G \) has the properties \((P)\) and \((P^*)\) with respect to \( \theta_0 \).

**Corollary (2.4).** If a treatment occurs in any block \( \beta_j \) of \( D \), then it occurs among \( q-1 \) first associates of \( \beta_j \) and \( q^2 - q \) times among the second associates of \( \beta_j \). If a treatment does not occur in \( \beta_j \), then it occurs \( q^2 \) times among the first associates of \( \beta_j \) and \( q^2 - q^2 \) times among the second associates of \( \beta_j \).

3. **Partial geometries and geometric graphs \((q^2+1,q+1,1)\).**

1. Let \( \mathcal{P} \) be a partial geometry \((q^2+1,q+1,1)\). The points and lines of \( \mathcal{P} \) satisfy axioms A1-A4 of §1, with \( r = q^2+1, k = q+1, t = 1 \). Let \( G \) be the graph of the geometry, i.e., the vertices of \( G \) are the points of \( \mathcal{P} \) and two vertices are adjacent if and only if the corresponding points are incident with the same line of \( \mathcal{P} \). Then \( G \) is strongly regular with parameters given by (2.1) and (2.2). Two points of the geometry which are incident with the same line and are therefore adjacent in \( G \), may be called adjacent points. Similarly two points of the geometry may be said to be non-adjacent if there is no line of the geometry incident with both. They are non-adjacent in \( G \).

Let \( \theta_0 \) be a vertex of \( G \), i.e., a point of \( \mathcal{P} \). Let \( \ell_i (i=1,2,\ldots,q^2+1) \) be the lines of \( \mathcal{P} \) incident with \( \theta_0 \). Let \( S_i \) denote the set of \( q \) points (other than \( \theta_0 \)) incident with \( \ell_i \). Then \( S \) the set of points adjacent to \( \theta_0 \), is the union of the disjoint sets.
and G obviously has the property (P) with respect to \( \theta_0 \). Any two vertices belonging to the same set \( S_i \) are adjacent, but if \( i \neq i' \), then a vertex belonging to \( S_i \) is non-adjacent to a vertex belonging to \( S_i' \).

It follows from \( t=1 \) that there cannot exist a triangle in the geometry, i.e., if \( \theta_1, \theta_2, \theta_3 \) are any three distinct points which are pairwise adjacent, then they must be incident with the same line \( \ell \). From this it follows that G has the additional property \((P^*)\) with respect to \( \theta_0 \). For let \( \theta_{iu} \) and \( \theta_{i'u'} \) be any vertices belonging to \( S_i \) and \( S_i' \), respectively, \( i \neq i' \). Let \( \beta_j \) and \( \beta_j' \) be two vertices non-adjacent to \( \theta_0 \) and both adjacent to \( \theta_{iu} \) and \( \theta_{i'u'} \). If \( \beta_j \) and \( \beta_j' \) are adjacent let \( \ell \) be the line incident with them. Then since \( \beta_j, \beta_j', \) and \( \theta_{iu} \) are pairwise adjacent, \( \theta_{iu} \) is incident with \( \ell \). Similarly \( \theta_{i'u'} \) is incident with \( \ell \). Hence \( \theta_{iu} \) and \( \theta_{i'u'} \) are adjacent, which is a contradiction. Thus \( \beta_j \) and \( \beta_j' \) must be non-adjacent.

It now follows from Theorems (2.1) and (2.2) that the design \( \mathcal{D}(\theta_0) \) corresponding to the vertex \( \theta_0 \) is an SRGD design with parameters (2.3), and possessing the property \((I_1)\) that any two blocks of \( \mathcal{D}(\theta_0) \) intersect in either 1 or \( q+1 \) treatments. Also from Corollary (2.5), if \( \theta_{iu} \) is any treatment in a block \( \beta_j \) of \( \mathcal{D}(\theta_0) \), then there are exactly \( q-1 \) other blocks which are first associates of \( \beta_j \) and contain the treatment \( \theta_{iu} \). We shall now show that in the present case \( \mathcal{D}(\theta_0) \) has the additional property \((I_2)\), that these \( q-1 \) blocks are mutually first associates. Let these blocks be denoted by \( \beta_{jw} \), \( w=1,2,\ldots,q-1 \). Let \( \ell \) be the line of the partial geometry incident with the points \( \theta_{iu} \) and \( \beta_j \).
Since $\beta_{j,w}$ is adjacent to both $\theta_{i,u}$ and $\beta_j$, from the non-existence of a triangle the point $\beta_{j,w}$ is incident with $\ell$. Thus $\beta_{j,1}, \beta_{j,2}, \ldots, \beta_{j,q-1}$ are pairwise adjacent, and the corresponding blocks are mutually first associates, so that any two of them have only the treatment $\theta_{i,u}$ in common. Hence we have the following theorem:

**Theorem (3.1).** If $G$ is the graph of a partial geometry $(q^2+1, q+1, 1)$, then the design $\mathcal{D}(\theta_0)$ corresponding to any vertex $\theta_0$ is SRGD with parameters (2.3), and possesses the properties $(I_1)$ and $(I_2)$.

Consider any three points $\theta_0, \theta_1, \theta_2$ of the partial geometry $(q^2+1, q+1, 1)$ which are pairwise non-adjacent. Then $\theta_1$ and $\theta_2$ can be identified with blocks of $\mathcal{D}(\theta_0)$, which are second associates and therefore intersect in $q+1$ treatments $\phi_0, \phi_1, \ldots, \phi_q$. Since the treatments in any block belong to different sets, $\phi_i$ and $\phi_j$ are non-adjacent in $G$. We therefore have

**Corollary (3.1).** Given any three points $\theta_0, \theta_1, \theta_2$ of a partial geometry $(q^2+1, q+1, 1)$, which are pairwise non-adjacent we can find a set of $q+1$ points $\phi_0, \phi_1, \ldots, \phi_q$ which are pairwise non-adjacent and each of which is adjacent to $\theta_0, \theta_1, \text{ and } \theta_2$.

2. We shall now prove the converse of Theorem (3.1). Suppose there exists an SRGD design $\mathcal{D}$ with parameters (2.3) and possessing the properties $(I_1)$ and $(I_2)$. Then as in §2, paragraph 4, we can first construct a strongly regular graph $G$ with parameters given by (2.1) and (2.2), and possessing a vertex $\theta_0$ with respect to which $G$ has the properties $(P)$ and $(I^*)$, and such that $\mathcal{D} = \mathcal{D}(\theta_0)$. $G$ is pseudo-geometric.
\((q^2+1,q^2+1,1)\). We shall show that in consequence of the additional property \((I_2)\), \(G\) must be geometric.

Let \(\mathcal{P}(V, \mathcal{L}, \mathcal{J})\) be an incidence structure consisting of a set of points \(V\), a set of lines \(\mathcal{L}\) and an incidence relation \(\mathcal{J}\) defined as follows:

The points of \(V\) are the vertices of \(G\). The point corresponding to \(\theta_0\) will be said to be of type 0, the points corresponding to the vertices adjacent to \(\theta_0\), i.e., to the treatments of \(\bigcup (\theta_0')\) will be said to be of type 1, and the points corresponding to the vertices of \(G\) non-adjacent to \(\theta_0\), i.e., to the blocks of \(\bigcup (\theta_0')\) will be said to be type 2. Thus there are \(v\) points, of which one of type 0, \(n_1\) are of type 1 and \(n_2\) are of type 2, where \(v, n_1, n_2\) are given by (2.1).

The lines of \(\mathcal{L}\) are certain subsets of \(V\). The lines of type 1 are the subsets \(K_i = S_i \cup \theta_0', i = 1, 2, \ldots, q^2+1\). Again let \(\theta_{1u}\) be any treatment of \(\bigcup (\theta_0')\), i.e., a point of type 1. Let \(\beta_j = \beta_{j0}\) be a block containing \(\theta_{1u}\), then \(\theta_{1u}\) is contained in \(q-1\) first associates of \(\beta_j\) which we have denoted by \(\beta_{j1}, \beta_{j1}, \ldots, \beta_{j1}, q-1\) and which from property \((I_2)\) are mutually first associates. Then we shall call the set of points \(\theta_{1u}, \beta_{j0}, \beta_{j1}, \ldots, \beta_{j1}, q-1\) a line of type 2. The treatment \(\theta_{1u}\) occurs in each of the blocks \(\beta_{j0}, \beta_{j1}, \ldots, \beta_{j1}, q-1\) and is the only treatment common to any two of them. Hence in defining the line we could have started from \(\theta_{1u}\) and any one of the blocks \(\beta_{jw}, v = 0, 1, \ldots, q-1\).

The incidence relation \(\mathcal{J}\) is the containing contained relation. We have then to show that \(\mathcal{P}(V, \mathcal{L}, \mathcal{J})\) satisfies anxious Al-Al for a partial geometry.
By definition each line has \( q+1 \) points. Thus axiom A3 is satisfied.

Note that lines have been so defined that any two points incident with a line are adjacent points of \( \mathcal{G} \). Hence any two non-adjacent points cannot be both incident with the same line. If \( l \) is a line and \( \varphi_1 \) and \( \varphi_2 \) are two points incident with it the other \( q-1 \) points incident with \( l \) are adjacent to \( \varphi_1 \) and \( \varphi_2 \). Since \( p_{1ll} = q-1 \), there are no other points adjacent to both \( \varphi_1 \) and \( \varphi_2 \). From this it follows that there is no line other than \( l \) incident with both \( \varphi_1 \) and \( \varphi_2 \). Thus any two points whether adjacent or not cannot be simultaneously incident with more than one line. This proves axiom A1. We shall now verify that two adjacent points are both incident with one line which must necessarily be unique:

A point adjacent to \( \theta_0 \) must be of type 1. Let \( \theta_{iu} \) belonging to \( S_i \) be such a point. Then the line \( K_i = S_i \cup \theta_0 \) is incident with both.

If two points of type 1 are adjacent they belong to the same set \( S_i \). Then \( K_i = S_i \cup \theta_0 \) is incident with both.

We have already defined a line of type 2 incident with a point \( \theta_{iu} \) of type 1, and a point \( \beta_j \) of type 2 adjacent to \( \theta_{iu} \).

Finally let \( \beta_{jw} \) and \( \beta'_{jw} \) be two points of type 2, adjacent to each other. Then the blocks \( \beta_{jw} \) and \( \beta'_{jw} \) are first associates and intersect in a single treatment \( \theta_{iu} \). Then \( \theta_{iu} \) and \( \beta_{jw} \) define a line \( l \) of type 2 incident with both \( \theta_{iu} \) and \( \beta_{jw} \). The other points of this line are the set of \( p_{1ll} = q-1 \) points simultaneously adjacent to \( \theta_{iu} \) and \( \beta_{jw} \). Since \( \beta'_{jw} \) is adjacent to both \( \theta_{iu} \) and \( \beta_{jw} \), it is incident with \( l \).

We next prove axiom A2 that each point is incident with exactly \( q^2+1 \) lines.
The point \( \theta_0 \) is incident with the lines \( K_1, K_2, \ldots, K_{q^2+1} \) of type 1 and no line of type 2.

Let \( \theta_{1u} \) belonging to \( S_1 \) be a point of type 1. Then it is incident with the line \( K_i \) of type 1. The treatment \( \theta_{1u} \) is contained in \( r' = q^3 \) blocks of \( \mathcal{S}(\theta_0) \). Let \( B \) denote this set of blocks. Let \( \beta_{j0} \) belong to \( B \). Then \( \beta_{j0} \) defines a line of type 2 incident with \( \theta_{1u} \) and containing the points \( \beta_{j0}, \beta_{j1}, \ldots, \beta_{j,q-1} \) which accounts for \( q \) blocks of \( B \). Let \( \beta_{j0}' \) be a block of \( B \) distinct from these. Then we get another line incident with \( \theta_{1u} \) and \( \beta_{j0}' \) containing \( q \) other blocks of \( B \).

Proceeding in this manner we obtain \( q^2 \) lines of type 2 incident with \( \theta_{1u} \). These together with \( K_i \) provide the \( q^2+1 \) lines incident with \( \theta_{1u} \).

Let \( \beta_{j0} \) be a point of type 2. Then the block \( \beta_{j0} \) contains \( q^2+1 \) treatments each belonging to a different set. Any one of these treatments and \( \beta_{j0} \) determine a unique line of type 2 incident with both. We thus obtain \( q^2+1 \) lines incident with \( \beta_{j0} \).

We have finally to prove axiom A4. Given a line \( \ell \) and a point \( \phi \) not incident with \( \ell \), we note that there cannot exist two different lines incident with \( \phi \), and intersecting \( \ell \) in \( \phi_1 \) and \( \phi_2 \) say. For in this case \( \phi \) being adjacent to \( \phi_1 \) and \( \phi_2 \) would be incident with \( \ell \) which is a contradiction. Each point \( \phi_i \) incident with \( \ell(i=1,2,\ldots,q^2+1) \) is incident with \( q^2 \) lines other than \( \ell \). Hence we obtain a set \( \mathcal{L}^* \) of \( q^2(q^2+1) \) lines other than \( \ell \) which are incident with a point of \( \ell \). Each line of \( \mathcal{L}^* \) contains \( q \) points not incident with \( \ell \), and no two of these lines can intersect in a point not incident with \( \ell \). We thus obtain a set of \( q^3(q^2+1) \) distinct points not incident \( \ell \), and each
incident with some line of \( \mathcal{L}^* \). Since \( v-(q^2+1) = q^3(q^2+1) \), each point \( \mathcal{Q} \), not incident with \( \mathcal{L} \), is incident with exactly one line of \( \mathcal{L}^* \). This proves axiom A4. Hence the incidence structure \( \mathcal{P}(V,\mathcal{L},\mathcal{Q}) \) is a partial geometry, and \( G \) is the graph of this geometry. Thus \( G \) is a geometric graph. We thus have

**Theorem (3.2).** Given an SRGD design with parameters \((2,3)\) and possessing the properties \((I_1)\) and \((I_2)\), there exists a partial geometry \((q^2+1,q^3+1,1)\), such that \( \mathcal{D} = \mathcal{Q}(\theta_0) \) where \( \mathcal{Q}(\theta_0) \) is the design corresponding to some vertex of the graph \( G \) of the partial geometry.

3. The existence of a partial geometry \((q^2+1,q^3+1,1)\) is known for any \( q \) which is a prime or a prime power. Primrose (1951) and Ray Chaudhuri (1962a) have considered quadrics in \( \text{PG}(n,q) \) the projective space of dimension \( n \) based on the field \( \text{GF}(q) \). If \( Q \) is an elliptic quadric in \( \text{PG}(5,q) \) then \( Q \) contains \( v = (q^2+1)(q^3+1) \) points and through each point of \( Q \) there pass \( r = q^2+1 \) generating lines lying completely in \( Q \). By considering the configuration of points and generators in \( Q \), Bose (1963) and Ray Chaudhuri (1962b) have shown that one gets a partial geometry \((q^2+1,q^3+1,1)\).

Another way of realizing the same partial geometry is due to Bose and Chakravarti (1966). Consider the Hermitian variety \( V \) with equation \( x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0 \), in a finite projective space \( \text{PG}(3,q^2) \) of three dimensions based on the finite field \( \text{GF}(q^2) \). Then \( V \) has \( b = (q^2+1)(q^3+1) \) points and contains \( v = (q^2+1)(q^3+1) \) generating lines. Through each point of \( V \) there pass \( q^2+1 \) generating lines and each
generating line has $q^2 + 1$ points. We get a partial geometry $(q^2 + 1, q^2 + 1, 1)$ by taking the generating lines of $V$ as the points, and the points of $V$ as the lines of a partial geometry.

**Corollary (3.2).** If $q$ is a prime or a prime power, an SRGD design with parameters $(2, 3)$ and possessing the properties $(I_1)$ and $(I_2)$ exists. Also an $NL_{q^2 + 1} (q^2)$ association scheme or graph exists.

Higman has shown (unpublished) that the two realizations of the partial geometry $(q^2 + 1, q^2 + 1, 1)$ described above are isomorphic.

4. We shall conclude this section by stating some unsolved problems.

Let $\theta_0$, $\theta_1$, $\theta_2$ be any three pairwise non-adjacent points of a partial geometry. Then we have shown in Corollary (3.1), that there exists a set of $q + 1$ points $\Phi_0, \Phi_1, \ldots, \Phi_q$ which are pairwise non-adjacent and each of which is adjacent to $\theta_0$, $\theta_1$, $\theta_2$. Now starting with the non-adjacent points $\Phi_0$, $\Phi_1$, $\Phi_2$ we may likewise obtain a set $q + 1$ points $\Phi_0$, $\Phi_1$, $\Phi_2$, $\ldots$, $\Phi_q$ which are pairwise non-adjacent and each of which is adjacent to $\Phi_0$, $\Phi_1$, $\Phi_2$. We may conjecture that $\theta_i$ is adjacent to $\Phi_j$ for all $i, j = 0, 1, 2, \ldots, q$.

Again one may ask whether a partial geometry $(q^2 + 1, q^2 + 1, 1)$ exists when $q$ is not a prime power. Even if such a geometry does not exist for a given $q$, a pseudo-geometric graph $(q^2 + 1, q^2 + 1, 1)$ might exist.

If $q$ is a prime power one may ask whether all partial geometries $(q^2 + 1, q^2 + 1, 1)$ are isomorphic. If the answer to this question is in the affirmative, then our conjecture stated above is certainly true since it is easily proved by geometrical considerations for the partial geometry derived from the elliptic quadric $Q_1$ in $PG(5, q)$. But the conjecture
could still be true even if there exist non-isomorphic partial geometries \((q^{2}+1,q+1,1)\).

Again let \(l_0\) and \(l_1\) be two non-intersecting lines of a partial geometry \((q^{2}+1,q+1,1)\). Let \(\theta_{00}, \theta_{01}, \theta_{02}, \ldots, \theta_{0q}\) be the points of \(l_0\). Through \(\theta_{0j}\) there passes a unique line \(m_j\) meeting \(l_1\) in a point which we may denote by \(\theta_{1j}(j=0,1,2,\ldots,q)\). The condition \(t=1\) shows that the points \(\theta_{10}, \theta_{11}, \ldots, \theta_{1q}\) are all distinct. Let the points of \(m_0\) be \(\theta_{00}, \theta_{10}, \ldots, \theta_{q0}\). Through any point \(\theta_{i0}\) of \(m_0\) there passes a unique line \(l_i\) meeting \(m_1\) in a point which we may denote by \(\theta_{i1}\), \((i=0,1,2,\ldots,q)\). The points \(\theta_{01}, \theta_{11}, \ldots, \theta_{q1}\) are all distinct. We may conjecture that \(l_i\) and \(m_j\) intersect in a point \(\theta_{ij}(i,j=0,1,\ldots,q)\).

Again this conjecture is certainly true if a partial geometry \((q^{2}+1,q+1,1)\) is always isomorphic to the partial geometry derived from the elliptic quadric \(Q\) (as can be proved by geometrical considerations) but it could still be true even if non-isomorphic partial geometries \((q^{2}+1,q+1,1)\) exist.
REFERENCES


