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Sojourn times, exit times, and jitter in multivariate Markov processes

J. Keilson

0. Introduction

The study of \textit{explicit} transient behavior for multivariate Markov processes is difficult. Descriptive distributions of interest are rarely available except in the presence of simplifying structural features such as reversibility in time [6], [13]. An understanding of transient behavior for more general systems is badly needed for reliability theory, inventory theory, congestion theory, and many other branches of applied probability.

In a typical system of interest, the state space $\Omega$ may be partitioned as $\Omega = G + B$, i.e., as the union of two subsets, the good states $G$ in which the system performance is regarded as satisfactory and the bad states $B$ in which system performance is unsatisfactory. The system alternates between the sets $G$ and $B$. Two random times associated with the set $G$ are of interest as candidates for the failure time of the system. The first is the sojourn time or visit time on the set $G$. The second is the remaining time to departure from the set $G$ at ergodicity which we will call the ergodic exit time from $G$. It will be shown that, for a broad class of processes of interest, these two times have distributions related in the same way as the lifetime and residual lifetime at ergodicity of ordinary renewal theory. This generalized renewal theory for the partitioned state space of a stationary process provides a framework for the study of system failure and the quantification of system
failure time for redundant repairable systems. The nature of the
two types of failure times, the structure of their distribution
functions, and the relative magnitude of their expectations will
be discussed.

A special case of interest is that for coherent [1],[2] systems
whose components are Markov, i.e., have exponentially distributed
failure times and replacement times. It is shown that for this
case both failure times have completely monotone densities.

When visits to the bad set B occur infrequently, the exponen-
tiability associated with the limit behavior [7],[8] may be exploited.
The expected failure time needed may be related to the expected
sojourn time on the set G, which is available from the ergodic
behavior. An important role is played by jitter, the clustering
of the entry epochs into the set G.
1. **Sojourn times and exit times**

Let \( \bar{x}(t) \) be a multivariate Markov process with state space \( \Omega = \{x\} \). For simplicity we suppose that the process is strictly stationary. The state space \( \Omega \) has the partition \( \Omega = G + B \), with \( G \) and \( B \) disjoint. Sample paths of the process alternate between the sets \( G \) and \( B \), and it is assumed that the time intervals in either set between entry and the next departure have positive duration almost surely. We will call these interval lengths for set \( G \) the sojourn times or visit times in \( G \), and similarly for \( B \). If the stationary process \( \bar{x}(t) \) is known to be in state \( x \) at time \( t \), the r.v.

\[
T_{xB} = \inf \{ \tau : \bar{x}(t + \tau) \in B | \bar{x}(t) = x \}; \quad x \in G
\]

will be called the exit time to \( B \) from \( x \), and its c.d.f. will be designated by

\[
S_{xB}(\tau) = P[T_{xB} \leq \tau]. \quad (1.1)
\]

If it is known only that \( \bar{x}(t) \in G \), the corresponding r.v. \( T_B \) will be called the ergodic exit time from \( G \), and its distribution will be denoted by

\[
S_E(\tau) = P[T_B \leq \tau | \bar{x}(t) \in G]. \quad (1.2)
\]
Clearly

\[
S_E(\tau) = \int_G S_{xB}(\tau) \mu_\infty(dx)/\mu_\infty(G)
\]

(1.3)

where \( \mu_\infty(dx) \) is the stationary measure of \( X(t) \).

A subclass of the processes will be Markov jump processes [5] governed by the transition measure

\[
\nu(x,A) = \frac{d}{ds} \left[ P[X(t+s) \in A | X(t) = x] \right]_{s=0}
\]

(1.4)

\( x \notin A \) and \( \nu(x,\{x\}) = 0 \). For each set \( A \) in \( G \) one will then have the ergodic flow rate measure

\[
h_{BG}(A) = \int_B \nu(x,A) \mu_\infty(dx), \quad A \subset G.
\]

(1.5)

In keeping with (1.3), we may then formally define the c.d.f.

\[
S_V(\tau) = \int_G S_{xB}(\tau) h_{BG}(dx)/h_{BG}(G)
\]

(1.6)

for the sojourn time \( T^G_V \) and this will be meaningful whenever \( h_{BG}(G) \) is finite. A simple probabilistic argument similar to that in [7], shows that

\[
h_{BG}(G) E[T_V] = P(G).
\]

(1.7)

The assumed positivity of \( E(T_V) \) then implies that \( h_{BG}(G) \), the stationary flow rate from \( B \) into \( G \), is finite whenever \( P(G) \) is positive and that \( h_{BG}(A) \) is a bounded measure. (For stationary processes
other than jump processes, the ergodic flow rate measure \( h_{BG} (A) \) may be defined as the mean number of traversals from \( B \) to \( G \) arriving in \( A \), the average being taken over an infinite time interval. The finiteness of \( h_{BG} (A) \) may be inferred under the same conditions via (1.7) and (1.6) may be used to define \( S_V (\tau) \).

The ergodic exit time \( T_E \) may also be called the residual sojourn time in \( G \) at ergodicity. When the process \( X(t) \) is an alternating renewal process [4] on the state space \( \Omega = \{(i,x); \ i = g,b, \ 0 \leq x < \infty \} \) one finds easily that the ergodic exit time distribution \( S_E (\tau) \) and sojourn time distribution \( S_V (\tau) \) on \( G = \{(g,x); \ 0 \leq x < \infty \} \) have the same simple structural relation as that between the residual lifetime at ergodicity and the renewal time distribution of ordinary renewal theory [5]. In particular one obtains

\[
S_E (\tau) = \frac{\int_0^\tau (1-S_V (\tau'))d\tau'}{\int_0^\infty \tau' dS_V (\tau')} . \tag{1.8}
\]

In the next section it will be shown that this structural relation is valid in a more general setting.
2. **Extension to Markov chains, Markov jump processes, and semi-Markov processes**

The relation (1.8) is valid for any stationary Markov jump process and any partition \( \Omega = \Omega + \Omega \) of its state space for which the ergodic probabilities and sojourn times on \( \Omega \) and \( \Omega \) are positive. For simplicity, we will first demonstrate this for finite Markov chains.

a. **Markov chains**

Suppose the stationary process \( \mathbf{X}(t) \) may be represented via a finite stationary Markov chain \( J(t) \) in continuous time. This will be possible, for example, for a reliability system consisting of \( K \) independent components for each of which the failure time and repair time distributions are Gamma of integral order (Erlang).

Other examples from congestion theory arise easily, e.g., the \( M/G/s \) system with finite waiting room, when the service times needed are Erlang [14]. For such processes the state space \( \Omega \) is finite and the partition \( \Omega = \Omega + \Omega \) classifies each state as being in the set \( \Omega \) or \( \Omega \). Let \( s_{mb}(\tau) \) denote the first passage time density from \( m \) to set \( \Omega \) or \( \Omega \). Let \( e_m = P[J(t) = m] \) be the stationary probability of state \( m \). Suppose that \( J(t) \) is governed by hazard rates \( \nu_{nm} \) for transition from state \( m \) to state \( n \neq m \). Then for \( m \in \Omega \), \( h_m = \sum_{n \in \Omega} e_n \nu_{nm} \) is the mean rate at which transitions occur from set \( \Omega \) to state \( m \). The sojourn time density on set \( \Omega \) is then given by

\[
s_{\nu}(\tau) = \frac{\sum_{m \in \Omega} h_m s_{mb}(\tau)}{\sum_{m \in \Omega} h_m},
\]  

(2.1)
and the ergodic exit time density by

\[ s_E(\tau) = \frac{\sum_G e_m s_{mB}(\tau)}{\sum_G e_m} \]  

(2.2)

The analogue of (1.8) is given in the following theorem. For convenience we write \( s_{mB}(\tau) = s_m(\tau) \) throughout.

Theorem 2.1 Consider a finite irreducible stationary Markov chain \( J(t) \) on the state space \( \Omega \). Let \( s_V(\tau) \) be the sojourn time density on the proper subset \( G \) of \( \Omega \), and let \( s_E(\tau) \) be the exit time density from \( G \). Then

\[ s_E(\tau) = \frac{1}{\mu_V} \int_0^\infty s_V(\tau')d\tau' \]  

(2.3)

where \( \mu_V \) is the expected sojourn time.

The proof is based upon the following lemma.

Lemma 2.1. Let \( s_m(\tau) \) be the passage time density from state \( m \in G \) to the complement \( B \), and let \( \sigma_m(s) = \int e^{-s\tau} s_m(\tau)d\tau \) be its Laplace transform. Then, for all \( m \in G \),

\[ \sigma_m(s) = \frac{\nu_m}{\nu_m + s} \left\{ \sum_G \beta_{mn} \sigma_n(s) + \sum_B \beta_{mn} \right\} \]  

(2.4)

where \( \nu_m = \sum_\Omega \nu_{mn} \) and \( \nu_{mn} = \nu_m \beta_{mn} \), for all \( m,n \in \Omega \).

For (2.4) we need only note that the random time from state \( m \) to set \( B \) is the sum of the dwell time in state \( m \) and the time to set \( B \) from the next state. The dwell time is exponentially distributed and
the two times are independent. The lemma then follows.

Proof of Theorem 2.1. It follows from (2.4) that for all \( m \) in \( G \),

\[
(s + \nu_m) \sigma_m(s) = \sum_{G} \nabla_{mn} \sigma_n(s) + \sum_{B} \nabla_{mn} .
\]

(2.5)

The net stationary flow \( h_{GB}^m(B) \) from \( G \) to \( B \) must be balanced by the net stationary flow \( h_{BG}^n(G) \) from \( B \) to \( G \), and one has

\[
\sum_{m \in G} \sum_{n \in B} e_{mn} \nu_{mn} = \sum_{m \in B} \sum_{n \in G} e_{mn} \nu_{mn} .
\]

(2.6)

Also, balance between the flow from any state \( n \in \Omega \) and flow to state \( n \) requires that

\[
\nu_n e_n = \sum_{G} e_{mn} \nu_{mn} + \sum_{B} e_{mn} \nu_{mn} .
\]

(2.7)

Multiplying (2.5) by \( e_{mn} \) and summing over the states of \( G \), we have

\[
\sum_{G} (s + \nu_m) e_{mn} \sigma_m(s) = \sum_{m \in G} \sum_{n \in G} e_{mn} \nu_{mn} \sigma_n(s) + \sum_{m \in G} \sum_{n \in B} e_{mn} \nu_{mn} .
\]

If we substitute from (2.7) into the first summation term on the right of the equal sign and from (2.6) into the second term, we find

\[
s \sum_{G} e_{mn} \sigma_m(s) = \{ \sum_{G} \sum_{B} e_{mn} \nu_{mn} \} (1 - \alpha(s))
\]

(2.8)

where \( \alpha(s) = \sum_{G} \theta_n \sigma_n(s) \) and
\[ \theta_n = \frac{\sum_{m \in B} \sum_{n \in B} e_{mn} \nu_{mn}}{\sum_{m \in G} \sum_{n \in B} e_{mn} \nu_{mn}} = \frac{h_n}{\sum G h_n} \] (2.9)

We then identify \( \alpha(s) \) from (2.1) to be the transform of the sojourn time density, i.e., \( \alpha(s) = \sigma_V(s) \). If we divide (2.8) by \( s \) and pass to the limit \( s \to 0^+ \), we find that

\[ \sum e_m = \{ \sum_{G B} e_{mn} \nu_{mn} \} \mu_V. \] (2.10)

Hence from (2.2) and (2.8) we have

\[ \sigma_E(s) = \frac{1 - \sigma_V(s)}{\nu_V s} \]

and (2.3) follows \( \square \).

b. Markov jump processes

The demonstration of the validity of (1.8) for Markov jump processes follows that of Theorem 2.1 with little change. In place of (2.5) we have, for example, in the notation of Section 1,

\[ s \sigma(x, s) + \nu(x, \Omega) \sigma(x, s) = \]

\[ = \int_G \nu(x, dx') \sigma(x', s) + \nu(x, B) \]

where \( \sigma(x, s) \) is the Laplace transform of \( s_{xB}(t) \). Similarly (2.8) becomes
\[ s \int_G u_\sigma(dx)s(x,s) = \{ \int_G u_\nu(dx)\nu(x,B) \} \{ 1 - \alpha(s) \} \]

where \( \alpha(s) = \int_G h_{BG}(dx)\sigma(x,s) / h_{BG}(G) \), and (1.8) follows quickly.

c. Semi-Markov processes

When the subsets \( G \) and \( B \) for \( X(t) \) have only a finite number of states at which they are entered, the theory of semi-Markov processes may be used to demonstrate the validity of (1.8). A similar demonstration is available when \( G \) has a finite number of entry states. The procedure is comparable to that of Cinlar [3] for the derivation of his equation (8.8). Details will not be given.

It seems clear that the relation (1.8) will be valid for any stationary multivariate Markov process when exit times and sojourn times are meaningful. A general proof in a completely broad setting should be a matter of notational breadth and measure theoretic technique.
3. **Consequences of the renewal relation**

When the renewal relation (1.8) is valid, the exit time distribution has simple properties of interest. One first observes from (1.8) that:

(a) $S_E(\tau)$ is absolutely continuous and that its density function $s_E(\tau)$ is monotone decreasing. The survival function $\bar{S}_E(\tau) = 1 - S_E(\tau)$ is therefore convex.

(b) It follows from the form of (1.8) and from (1.7) that

$$E[T_E] \geq \frac{1}{2} E[T_V] = \frac{1}{2} \frac{P(G)}{h_{BG}}$$

(3.1)

and this inequality is tight. As proof we note from an integration by parts and simple algebra that

$$\frac{E[T_E]}{E[T_V]} = \frac{1}{2} \left( 1 + \frac{\sigma^2}{\mu^2} \frac{1}{T_V} \right)$$

where $\sigma^2 / \mu^2$ is the squared coefficient of variation and this must be nonnegative. When $T_V$ has a fixed value, equality is assumed.

(c) The inequality (3.1) is helpful for the estimation of the expected exit time in that it permits a lower bound in terms of the ergodic probabilities, and these are generally more accessible than the transient behavior of the process. This will be further discussed in Section 7.
4. Reversibility in time

When the stationary Markov chain of Section 2 is reversible in
time [6], the sojourn time densities and ergodic exit time densities
have the very simple structure stated in the following theorem.

**Theorem 4.1** Let $J(t)$ and $G$ be as stated in Theorem 2.1.
The sojourn time density $s_V(\tau)$ and ergodic exit time density $s_E(\tau)$
are completely monotone.

**Proof:** In the notation of Section 2, we have

$$s_E(\tau) = -\frac{d}{d\tau} [1 - S_E(\tau)] = -\frac{d}{d\tau} \left\{ \sum_{G} \sum_{G} \frac{e_{m} p_{mn}^*(\tau)}{\sum_{G} e_{m}} \right\}$$

(4.1)

where $p_{mn}^*(\tau) = P[J(t_0 + t') = n \in G, 0 \leq t' \leq \tau | J(t_0) = m \in G]$. The
transition probabilities $p_{mn}^*(\tau)$, by virtue of the reversibility of $J(t)$
in time have the spectral representation [6]

$$p_{mn}^*(\tau) = e_n \sum_{m} \psi_{m}(j) \psi_{n}(j) e^{-w_j \tau}$$

where $\psi_{m}(j)$ is real, $w_j$ is positive, and the summation is finite.

Hence

$$\sum_{G} \sum_{G} e_{m} p_{mn}^*(\tau) = \sum_{j} e^{-w_j \tau} \{ \sum_{m} e_{m} \psi_{m}(j) \}^2$$

and $1 - S_E(\tau)$ and $s_E(\tau)$ are completely monotone. The complete mono-
tonicity of the sojourn time density $s_V(\tau)$ follows now from (1.8).
5. Application to system reliability and communication nets

The results of Section 4 have immediate application to reliability theory and communication networks.

For reliability theory, one may consider a system consisting of \( K \) independent repairable components, each of which has exponentially distributed failure times and repair times. Let the sets of working components of the system be enumerated as \( A_1, A_2, \ldots, A_K \), where \( N_0 = 2^K \) is the number of distinct subsets chosen from \( K \) elements. Let \( N(t) \) be the index of the working subset. When the reliability structure is coherent \([1],[2]\), the process \( N(t) \) is irreducible. For the stationary process, there will be detailed balance, i.e., one will have

\[
e_n \nu_n = e_m \nu_m \quad \text{where} \quad e_n = P[N(t) = n],
\]

as may be readily seen, and consequently the process \( N(t) \) is reversible in time. The above formalism is therefore applicable. The ergodic exit time density for the set of working states is well suited to the role of failure time and this density is completely monotone. An expanded discussion of such systems has been given in \([11]\).

For communication nets, one may consider nets whose links are either open or closed, with all links independent and having exponentially distributed opening and closing times. The net is assumed to be connected in the sense that every node may be reached from every other node when all the links are closed. Suppose one asks for the time until the net separates, i.e., until some pair of nodes cannot
communicate. The mathematical structure of this system is identical with that above. The states of the net are the sets of communicating links. If we take G to be the set of states for which all pairs of nodes are connected, we see that the ergodic exit time from G has completely monotone density.
6. **Monotonicity of expected exit times and sojourn times**

Consider the sequence of partitions of the state space
\[ \Omega = G_N + B_N \quad (N=1,2,\ldots) \] for which \( G_N \subset G_{N+1} \). The ergodic exit times \( T^{(N)}_E \) from \( G_N \) will have expectations \( \mu_{EN} = \mathbb{E} T^{(N)}_E \) which increase monotonically with \( N \) in the sense that \( P(G_N)^\mu_{EN} \) increases with \( N \) and \( P(G_N) \) goes to 1. This follows directly from (see (2.2))

\[
P(G_N)^\mu_{EN} = \sum_{G_N} e_m \mathbb{E} T^{mB_N}_{mB_N} \tag{6.1}
\]

and the monotonicity of \( \mathbb{E} T^{mB_N}_{mB_N} \) with \( N \). The sojourn times \( T^{(N)}_V \) will have expectations \( \mu_{VN} \) which need not increase when \( G_N \) increases and can even fluctuate with \( N \). This may be seen most readily in the context of birth-death processes.

Consider the stationary birth-death process \( N(t) \) governed by upward and downward transition rates \( \lambda_n, \eta_n \) with \( \lambda_n > 0 \) for all \( n \) and \( \eta_n > 0, n \geq 1, \eta_0 = 0 \). The corresponding ergodic probabilities are given by

\[
e_n = e_0 \left( \frac{\lambda_0}{\eta_1} \right) \left( \frac{\lambda_1}{\eta_2} \right) \cdots \left( \frac{\lambda_{n-1}}{\eta_n} \right) \tag{6.2}
\]

with \( e_0 \) found from the normalization \( \sum e_n = 1 \). Let \( G_N = \{n=0,1,\ldots,N\} \). Then \( \mu_{VN} \) for \( G_N \) is given by

\[
\eta_{N+1} e_{N+1} \mu_{VN} = \sum_{0}^{N} e_j \tag{6.3}
\]
as may be seen from (1.7) and the process structure. Let 
\((K_n)_{n=0}^{\infty}\) be a sequence of positive numbers. Suppose now the process
\(N(t)\) is altered to a new birth-death process \(N^*(t)\) by choosing new
parameters.

\[
\lambda_n^* = K_n \lambda_n, \quad \eta_{n+1}^* = K_n \eta_{n+1}
\]  

(6.4)

so that

\[
\frac{\lambda_n^*}{\eta_{n+1}^*} = \frac{\lambda_n}{\eta_{n+1}}, \quad n \geq 0.
\]  

(6.5)

Then the ergodic probabilities \(e_n^*\) for the new process coincide with \(e_n\)
from (6.2) and normalization. Moreover, from (6.3)

\[
\frac{\eta_{n+1}^* \mu_{VN}}{\eta_{n+1} \mu_{VN}^*} = K_n \frac{\mu_{VN}}{\mu_{VN}} = 1.
\]  

(6.6)

If we choose \(K_N\) as needed out to some \(K_{N_0}\), and set \(K_N = 1, N \geq N_0\),
the values of \(\lambda_n\) and \(\mu_n\) will be unchanged beyond \(n = N_0\) and the ergo-
dicity of the process \(N(t)\) will be unaffected. But the sequence \(\mu_{VN}\)
can have any desired behavior to any desired length, for example, fluct-
tuating. We note, however, for the sets \(G_N\) and \(\overline{G_N} = B_N\), one has

\[
\sum_{G_N} e_n = \frac{E T_V(G_N)}{E T_V(G_N) + E T_V(B_N)}
\]  

(6.7)

so that

\[
\frac{E T_V(B_N)}{E T_V(G_N)} \to 0, \quad N \to \infty
\]  

(6.8)

and the decrease is strictly monotone.
Such monotonicity arguments help show that the ergodic exit time is better suited to the role of system failure time than the sojourn time.
7. Jitter, exponentiality, and asymptotics

The lack of monotonicity of the sequence of sojourn times \( \mu_{VN} \) for birth-death processes observed in Section 6 is closely tied to a phenomenon we may call "jitter", conveying the clustering of entry epochs into a set of interest.

Consider a particular stationary birth-death process \( N(t) \), and a sequence \( (K_n)_{n=0}^{\infty} \) for which \( K_n = 1, \ n \neq N, \ K_N = 100 \). By virtue of (6.6), the sojourn times \( \mu_{VN} \) coincide with \( \mu_{VN}^* \) for all \( n \neq N \), and \( \mu_{VN}^* = \mu_{VN}/100 \). The two processes differ only in that the transition rates \( \lambda_N^* \) and \( \eta_{N+1}^* \) between the states \( N \) and \( N+1 \) are greater than \( \lambda_N \) and \( \eta_{N+1} \) by the factor 100 and this increase generates a clustering of entry epochs into \( G_N \). The jitter introduced at the level \( N \) by raising \( K_N \) reduces \( \mu_{VN}^* \) because many of the sojourns on \( G_N = \{0,1,2, \ldots, N\} \) then consist of transitions from \( N+1 \) to \( N \) and immediate returns to \( N+1 \). The brief frequent visits to \( G_N \) then reduce the mean sojourn time.

In the study of the reliability of complex repairable systems [2], one may deal with a system modeled by a stationary multivariate Markov process \( \mathbf{X}(t) \) having many degrees of freedom. The state space \( \Omega \) consists of the working states \( G \) and system failure states \( B \), with visits to the set \( B \) infrequent. It is then attractive to use the ergodic exit time for the set \( G \) as the failure time for the system and to try to exploit the exponential limit theorems
associated with the rarity of visits to set $B$ [7],[8]. Such limit theorems suggest the value of the approximation

$$\overline{S}_E(\tau) = P[T_E > \tau] \approx \exp(-\tau/E[T_E]). \quad (7.1)$$

It has been shown previously [7],[10] that for any birth-death process $N(t)$ for which $\lambda_n/\eta_n \to 0$ as $n \to \infty$, one has asymptotic equality between the expected sojourn time $E[T_{VN}]$ and expected exit time $E[T_{EN}]$ for $G_N = \{0,1,2,\ldots,N\}$ as $N \to \infty$, i.e., one has

$$\lim_{N \to \infty} \frac{E[T_{VN}]}{E[T_{EN}]} = 1. \quad (7.2)$$

Such asymptotic equality of $E[T_{VN}]$ and $E[T_{EN}]$, when present, is particularly helpful when the ergodic probabilities of the states of the state space $\Omega$ are known. For then one has, by the same argument responsible for (6.3)

$$E[T_{VN}] = P[X(t) \in G_N]/i_{GB} \quad (7.3)$$

where $i_{GB}$ is the stationary flow rate from set $G$ to $B$ (or $B$ to $G$, since the two flows must be in balance). For a chain one has $i_{GB}$ from the ergodic probabilities via (2.6).

Even when the asymptotic equality of (7.2) is not known to be present, one may use the bound
\[ E[T_E] \geq \frac{1}{2} E[T_V] \] 

which is always present when (1.8) is available. For from (1.8) we have as in renewal theory \[ E[T_E] = \frac{1}{2} E[T_V^2]/E[T_V] \] so that

\[ \frac{E[T_E]}{E[T_V]} = \frac{1}{2} \left[ 1 + \frac{\sigma^2}{\mu^2} T_V \right] \] 

and (7.4) follows. Consequently from the monotonicity of \( e^{-x/y} \) with \( y \), one has for (7.1)

\[ P[T_E > \tau] \geq \exp(-\tau/E[T_E]) \geq \exp(-2\tau/E[T_V]) . \] 

From (7.3) we then have a lower bound for the survival probability in terms of the ergodic probabilities of the system states. We have in Theorem 4.1 seen that for processes reversible in time, the sojourn time densities are completely monotone. One then has ([9],[12]),

\[ (\sigma^2/\mu^2)_{T_V} \geq 1 \] 

so that for such processes

\[ E[T_E] \geq E[T_V] \] 

and this may be used to improve the bound in (7.6).

The asymptotic exponentialiality of the ergodic exit time distribution is altered for the sojourn time by the presence of jitter, the change consisting of the appearance of positive support at time zero in the asymptotic distributions. Thus let \( \mathbf{X}(t) \) be a stationary multivariate Markov process on \( \Omega \), let \( \Omega = G_N + B_N \) be a sequence of partitions
of the state space, and let $S_{EN}(\tau)$ be the ergodic exit time c.d.f. from $G_N$. Suppose that for some positive sequence $(\gamma_N)_1^\infty$ one has

$$1 - S_{EN}(\gamma_N^{-1}\tau) \rightarrow e^{-\tau}.$$ When the renewal relation (1.8) is valid, one has for the corresponding sojourn time distributions $S_{VN}(\tau)$ and their Laplace-Stieltjes transforms

$$\sigma_{VN}(\gamma_N s) = 1 - \mu_{VN}\gamma_N s \sigma_{EN}(\gamma_N s). \tag{7.8}$$

If $\gamma_N^{\mu_{VN}} + q, 0 \leq q < 1$ as $N \rightarrow \infty$, one has

$$1 - S_{VN}(\gamma_N^{-1}\tau) \rightarrow q e^{-\tau}. \tag{7.9}$$

When $\mu_{VN}\gamma_N = q_N$ oscillates one has for $N$ large

$$1 - S_{VN}(\gamma_N^{-1}\tau) = q_N e^{-\tau}. \tag{7.10}$$
REFERENCES


