STOCHASTIC ORDER IN RENEWAL THEORY AND 
TIME-REVERSIBLE CHAINS

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0. Introduction

Let \( X(t) \) be the stationary age process for a renewal process governed by the c.d.f. \( F_T(x) \) for its lifetime \( T \). The associated residual lifetime \( R \), i.e., the time until the next renewal epoch has the c.d.f. \([3]\).

\[
F_R(x) = \mu_T^{-1} \int_0^x F_T(x') \, dx',
\]

where \( \bar{F}_T(x) = 1 - F_T(x) \) and \( \mu_T = \int_0^\infty dF_T(x) \). For the classes of lifetimes whose failure rates are monotone, IFR and DFR \([1]\), one has the stochastic order relations \( R \sim T \) and \( R \succ T \) respectively, where these have the meaning

\[
R \sim T : \quad \bar{F}_R(x) \leq \bar{F}_T(x), \ x \geq 0;
\]  \hspace{1cm} (2a)

and

\[
R \succ T : \quad \bar{F}_R(x) \geq \bar{F}_T(x), \ x \geq 0.
\]  \hspace{1cm} (2b)

The order relation \( R \succ T \) for DFR lifetimes has some interest in the context of reliability theory, specifically for the transient behavior of complex repairable systems. More generally it is of interest for the transient behavior of time-reversible Markov chains, e.g. birth-death processes, tree-processes, and multivariate extensions.

For many IFR and DFR renewal processes \([\text{see eq. (9)}]\), the residual lifetime \( T_x \) beyond age \( x \) converges in distribution as \( x \to \infty \) to a
r.v. $T_\infty$ which is exponentially distributed. For DFR, one then has $T \prec R \prec T_\infty$. The r.v. $T_\infty$ also has a meaning for Markov chains and reliability theory, and is associated with a simple limit-theoretic behavior described below.
1. Renewal Theory

Theorem 1. Let $X$ be the age of a stationary renewal process with lifetime $T$ for which $F_T(x)$ has decreasing failure rate (DFR). Let $R = T - X$ be the time until the next renewal epoch; let $T_x = R$ given that $X = x$; let $R_x = R - x$ given that $R > x$. Then

a) $T_x \aleph T_y$, $x \leq y$,
b) $R_x \aleph R_y$, $x \leq y$,
c) $T \aleph R$.

If $F_T(x)$ is IFR, a), b), and c) are reversed.

Proof. The stochastic increase of $T_x$ with $x$ for DFR is equivalent to

$$\bar{F}_{T_x}(y) = \frac{F_T(x+y)}{\bar{F}_T(x)} = P[T > x + y \mid T > x] \uparrow_x, y \geq 0 \hspace{1cm} (4)$$

and this is implied by the definition of DFR. The survival function of $R_x$ is

$$\bar{F}_{R_x}(y) = P[R > x + y \mid R > x] = \frac{\bar{F}_R(x+y)}{\bar{F}_R(x)} \hspace{1cm} (5)$$

when $X(t)$ is stationary. The stationary p.d.f. of $X(t)$ and of the residual lifetime $R$ at stationarity is $\bar{F}_T(x)/\mu_T$. For DFR, $\bar{F}_T(x)$ is log-convex and its integral $\bar{F}_R(x)$ is then also log-convex (See appendix). It follows that $R_x$ increases stochastically with $x$. To see that $R \geq T$ we note from (4) that
\[ J \int_0^\infty \frac{\bar{F}_T(x+y)}{\bar{F}_T(x)} \, dy = \int_0^\infty \frac{\bar{F}_T(y)}{\bar{F}_T(x)} \, dy \quad \uparrow x \] (6)

so that

\[ \frac{\int_0^\infty \bar{F}_T(y) \, dy}{\bar{F}_T(0)} = \mu_T \leq \frac{\int_x^\infty \bar{F}_T(y) \, dy}{\bar{F}_T(x)} \] (7)

i.e. [from (1)]

\[ \bar{F}_T(x) \leq \bar{F}_R(x) \] (8)

This could also be seen from the fact that \( \bar{F}_R(x) \) is a mixture (over the age at ergodicity) of \( \bar{F}_T(x) \), that \( T_x \) increases stochastically with \( x \) and that \( T_0 = T \). The reasoning for the IHR case is identical.

It may be noted that when \( F_T(x) \) is absolutely continuous and its p.d.f. \( f_T(x) \) is log-concave or strongly unimodal [4], [6], [8], then \( F_T(x) \) is IFR (see Appendix). When \( F_T(x) \) and \( f_T(x) \) are completely monotone, one has the DFR case. The latter case is of special interest for time-reversibility [5], [9] and reliability, as will be discussed below.

When \( \bar{F}_T(x) \sim Ke^{-\eta x} \) as \( x \to \infty \)

then

\[ \frac{\bar{F}_T(x+y)}{\bar{F}_T(x)} = \bar{F}_T(x) \to e^{-\eta y} \] (9)

and \( T_x \) converges in distribution to \( T_\infty \) of exponential distribution.

By virtue of the stochastic monotonicity of Theorem 1, one then has

\[ T \prec R \prec T_\infty \] (10)
It is easily verified by L'Hospital's rule that $R_x$ converges to $T_\infty$ as well.
2. Time reversible chains and reliability theory

Let \( J(t) \) be any finite Markov chain in continuous time, which is irreducible and stationary. Let its state space \( \mathcal{F} \) be partitioned into two disjoint subsets \( \mathcal{F} = G + B \). A sample path alternately visits sets \( A \) and \( B \). It has been shown elsewhere [7] that the times spent visiting the set \( G \) have a p.d.f. \( s_v(T) \), and the exit time from \( G \) to \( B \) at ergodicity has a p.d.f. \( s_E(T) \) for which the relation

\[
s_E(T) = \frac{s_v(T)}{\lambda_v}
\]

with \( \lambda_v = \int_0^\infty s_v(T) \, dt \) is valid. It has also been shown that [7] when the chain is reversible in time, both \( s_E(T) \) and \( s_v(T) \) are completely monotone. The structural identity of (11) with (1) and Theorem 1 then imply that

\[
T_v < T_E < T_\infty
\]

i.e., that

\[
\bar{s}_v(T) \leq \bar{s}_E(T) \leq e^{-\eta T} = e^{-\tau/\varepsilon T_\infty}
\]

where \( \eta \) is the asymptotic decay rate of \( \bar{s}_v(\tau) \) or \( \bar{s}_E(\tau) \). (This decay rate is associated with the Frobenius eigenvalue of a stochastic matrix simply related to the transient chain on \( G \).)
3. Application to complex repairable systems

Certain complex repairable systems may be modeled usefully by time reversible ergodic chains [7]. In such models the set G corresponds to the "good" states for which the system functions and B the bad states for which it does not. The inequalities $T_E > T_V$ and $S_E(\tau) > S_V(\tau)$ are of interest for the two generalized failure times $T_E$ and $T_V$ natural for such systems. The time $T_E$ is the exit time to set B given that the system is in G, i.e., is working. The time $T_V$ is the duration of visits to set G.

The time $T_\infty$ also has meaning for such systems. Consider the probability

$$
\bar{F}_\omega(x) = \lim_{y \to \infty} P\{N(y+t) \in G, 0 \leq t \leq x \mid N(t') \in G, 0 \leq t' \leq y\}.
$$

(14)

The function $\bar{F}_\omega(x)$ is the survival function for the set G for a time exceeding x given that one has already been in set G for an "infinite" period of time. Under the conditions stated it can be shown that

$$
\bar{F}_\omega(x) = \exp\{-x/\mu_V^*\}
$$

(15)

where $\mu_V^*$ is the reciprocal of the final decay rate for $s_V(\tau)$ [or $S_E(\tau)$]. Equivalently $\mu_V^* = \varepsilon[T_V(\omega)]$. 

In effect we have a limit theorem which states that the longer the system has been working the closer the residual working time to a fixed exponential distribution. The proof hinges on the idea of quasistationarity discussed for example by Darroch and Seneta [2]. The limit theorem here described was formulated with E. Arjas in a more general setting.

It may be of interest to note that the familiar equality of the distributions for the age and residual time implies that for complex repairable systems the history of the sample (or ensemble) provides an empirical distribution for the distribution $F_E(x)$ of the failure time $T_E$. 
4. Appendix

Some properties of log-concave and log-convex functions will be given. Let \( f(x) \) be a nonnegative (integrable) function on \([0, \infty)\), continuous in the interior of its interval of support, and write \( \tilde{F}(x) = \int_x^\infty f(u) \, du \). The following conditions are equivalent to log-concavity (log-convexity) of \( f(x) \).

(i) \( f^2(u) \geq (\leq) \, f(u+t) \, f(u-t) \quad 0 \leq t \leq u \)

(ii) \( f(pu +qv) \geq (\leq) \, f^p(u) \, f^q(v) \quad 0 \leq p = 1 - q \leq 1, \, 0 \leq u,v \)

(iii) \( f(v) \, f(u+t) \geq (\leq) \, f(u) \, f(v+t) \quad 0 \leq u \leq v \), \( t \geq 0 \)

(iv) \( \frac{\tilde{F}(x+a) - \tilde{F}(x+a+b)}{\tilde{F}(x) - \tilde{F}(x+b)} \) decreases (increases)

in \( x \) for \( x \geq 0 \) and all \( a,b > 0 \) for which the quotient is defined. (For lattice analogues of (i) and (iii), see [8].)

Conditions (i) and (ii) are well known expressions obtained from the concavity (convexity) of \( \log f(x) \). See [10, section 1.4].

Condition (iii) is derived from (ii) as follows: Take \( p = \frac{v - u}{v - u + t} \). Then \( f(u+t) = f(pu + (1-p) \, (v+t)) \geq f^p(u) \, f^{1-p}(v+t) \), and

\[
\begin{align*}
f(v) &= f(1-p)u + p(v+t)) \geq f^{1-p}(u) \, f^p(v+t)
\end{align*}
\]

Multiplying these gives (iii). By taking \( u = v - t \) in (iii) one obtains (i). Condition (iv) is found by integrating (iii) as below:

\[
(*) \int_{x+a}^{x+a+b} f(v) \, dv \cdot \int_x^{x+b} f(u+t) \, du \geq (\leq) \int_{x+a}^{x+a+b} f(v+t) \, dv \cdot \int_x^{x+b} f(u) \, du
\]
The validity of (*) is obvious for \( x \geq 0 \), \( a \geq b > 0 \). When the intervals \([x, x+b]\) and \([x+a, x+a+b]\) overlap \(*\) is proven by adding inequalities for disjoint intervals and cancellation of the common part in both sides of (*) . From (*) the monotonicity of (iv) is easily found. Conversely, monotonicity of (iv) implies (*) , from which (iii) is retrieved by dividing through by \( x^2 \) and letting \( x, a, b \to 0 \).

Condition (iv) is of interest because \( b \to \infty \) proves that \( f(x) \) log-concave (log-convex) implies that \( \frac{F(x+a)}{F(x)} \) decreases (increases) in \( x \) for all \( a > 0 \). In particular, \( F^2(x) \geq (\leq) F(x+a) F(x-a) \).

It follows from (i) that \( F(x) \) is log-concave (log-convex). When \( F(x) \) is a c.d.f. \( \overline{F}(x) \) log-concave (log-convex) is equivalent to \( \overline{F}(x) \) IFR (DFR).

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