THE PROBLEM OF BUFFON'S NEEDLE WITH A LONG NEEDLE

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TECHNICAL REPORT NO. 99
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I Introduction

The classical formulation of the Buffon needle problem [1;70] begins with a grid of parallel lines length \( d \) apart. A needle of length \( \ell < d \) is thrown onto the grid and one asks for the probability of an intersection. In case \( \ell > d \) there can be several intersections. The purpose of this note is to derive the density, and approximations to the moments, of the number of crossings.

This problem was suggested by Herbert Solomon from consideration of a problem of detection deployment. In Solomon's problem, the planar grid is a grid of detection lines (e.g. a light or laser shining to a photo electric cell) and the needle might be a stream of polluting material laid down at random by a ship or a plane.

The density of the number of crossings, given in Theorem 1, agrees with results stated without proof in Kendall and Moran (pg. 70-73). The new results in this note are the simple expression (2-1) for the distribution function; the approximate moments, valid for large values of the ratio \( \frac{\ell}{d} \) given in Theorem 2; and the limit theorem for the density, Theorem 3.

II The Density of the Number of Intersections

If a needle of length \( \ell \) is thrown at random onto a plane ruled by parallel lines of distance \( d, d < \ell \), what is the probability density \( P(\ell) \) of the number of intersections?

Theorem 1. The number of intersections can range between 0 and \( \left[ \frac{\ell}{d} \right] + 1 \leq M \). Let the angles \( \theta_i, 0 \leq \theta_i \leq \frac{\pi}{2} \), be determined by

\[
\cos \theta_i = \frac{d}{\ell}, \quad \text{let}
\]
\[ \Delta_{i} = \frac{2l}{\pi d} \sin \theta_{i} - \frac{2i\theta_{i}}{\pi}. \]

Then for \( \left\lfloor \frac{2}{d} \right\rfloor > 2 \):

\[ P(0) = \Delta_{i} + 1 - \frac{2l}{\pi d}, \]

\[ P(i) = \Delta_{i-1} + \Delta_{i+1} - 2\Delta_{i} \text{ for } 1 \leq i \leq M-2; \]

\[ P(M-1) = \Delta_{M-1} - 2\Delta_{M-2}; \quad P(M) = \Delta_{M-1} \]

For \( \left\lfloor \frac{2}{d} \right\rfloor = 1 \), \( M \) has value 2. The results for \( P(0) \) and \( P(M) \) above hold and

\[ P(i) = \frac{4\theta_{i}}{\pi} + \frac{2l\pi}{d} - \frac{4l}{\pi d} \sin \theta_{i}. \]

**Proof.** Let \( x \) be the distance from the midpoint of the needle to the nearest line and \( \theta \) the acute angle formed. Thus \( 0 \leq x \leq \frac{d}{2} \) and \( 0 \leq \theta \leq \frac{\pi}{2} \).

![Figure 1](image-url)

The \((x, \theta)\) plane is portioned into disjoint pieces, the \( i \)th piece leading to exactly \( i \) crossings.
1. **Zero Crossings** For fixed $x$ there is no crossing if $\theta = \frac{\theta}{2}$ and the first crossing becomes possible as $\theta$ is decreased to the point when the upper right end of the needle just touches the line above.

Figure 2

```
\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (1,0) node[below] {$x$};
\draw[->] (0,0) -- (0,1) node[left] {$\theta$};
\draw (0,0) -- (1,1);\end{tikzpicture}
\end{center}
```

Thus, there is no crossing iff $\theta_x \leq \theta < \frac{\pi}{2}$ where $\theta_x$ satisfies $\frac{x}{2} \cos \theta_x = x$. The smallest such $\theta_x$ occurs when $x = \frac{d}{2}$. That is, we always have $\cos \theta > \frac{d}{x}$ if there are no crossings. The upper righthand area in Figure 4 is the one just described.

2. **M Crossings** Here the problem breaks into two cases, as $M$ is even or odd. The cases lead to different geometry but the same area.

**Case a** $M = 2k$. For fixed $x$, $2k$ crossings are possible iff they are possible for $\theta = 0$. We then require $\frac{x}{2} + x > kd$ or $kd - \frac{x}{2} < x < \frac{d}{2}$. $2k$ crossings remain possible for $0 \leq \theta \leq \theta_x$ where $\frac{x}{2} \cos \theta_x = kd - x$. The region of the $(x, \theta)$ plane just described is pictured in the lower righthand region in Figure 4.

**Case b** $M = 2k+1$. Again, for fixed $x$, $M$ crossings are possible iff they are possible for $\theta = 0$. Note when $x = \frac{d}{2}$, only $2k$ crossings are possible since the needle is centered. As $x$ becomes smaller, $2k+1$ crossings become possible for $0 < x < kd - \frac{d}{2}$. The maximal number of crossings remain possible for $0 \leq \theta \leq \theta_x$ where $\frac{x}{2} \cos \theta_x = kd + x$. The region of the $(x, \theta)$ is shown below.
3. **i Crossings, 1 \leq i \leq M-2**  Here, i crossings are possible for any value of \( x \), \( 0 \leq x \leq \frac{d}{2} \), for any \( i \), \( 1 \leq i \leq M-2 \). For fixed \( x \), reasoning as above, there will be exactly i crossings for \( \frac{d}{2} \cos \theta_i = [\frac{1}{2}] d + x(-1)^{i-1} \). We draw below the \((x, \theta)\) plane for \( \frac{d}{d} = 5 \).
4. M-1 Crossings. As Figure 4 indicates, the region of the \((x, \theta)\) plane where M-1 crossings occur is bounded above by the curve
\[
\frac{\theta}{2} \cos \theta = (k-1)d + x, \quad \text{and below by the \(x\) axis for } 0 \leq x \leq kd - \frac{d}{2} \text{ along with the curve } \frac{\theta}{2} \cos \theta = kd - x. \quad \text{Here } M = 2k - 1. \quad \text{When } M = 2k \text{ the curve above is } \frac{\theta}{2} \cos \theta = kd - x. \quad \text{The curves below are } \frac{\theta}{2} \cos \theta = kd + x \text{ for } 0 \leq x \leq \frac{\theta}{2} - kd \text{ and the \(x\) axis for } \frac{\theta}{2} - kd \leq x \leq \frac{d}{2}.
\]

5. Computation of the Probabilities. We assume "at random" means that both \(x\) and \(\theta\) are uniform. Thus we need only compute the areas bounded by the curves described above and divide by \(\frac{\pi d}{4}\). For any real \(t\), we easily compute
\[
\int_0^{\frac{\theta}{2}} \cos^{-1}\left(\frac{t^2 - 2x}{d}\right)dx = \frac{\theta}{2} \left\{ \sin x - x \cos x \right\}_0^{\frac{\theta}{2} - 1}
\]
for \(t > 1\) and
\[
\int_0^{\frac{\theta}{2}} \cos^{-1}\left(\frac{t^2 + 2x}{d}\right)dx = \frac{\theta}{2} \left\{ \sin x - x \cos x \right\}_0^{\frac{\theta}{2} + 1}
\]
for \(t > 0\) where \(\cos \theta = t \frac{d}{\lambda} \).

Using these integrals, straightforward integration leads to the answers given above. For example, to compute \(P(0)\) we compute the area under \(\theta = \cos^{-1}\frac{2x}{\lambda}\). This is
\[
\int_0^{\frac{\theta}{2}} \left\{ \sin x - x \cos x \right\}_0^{\frac{\theta}{2} - 1} = \frac{\theta}{2} - \frac{\theta}{2} \left\{ \sin \theta - \frac{\theta}{2} \frac{d}{\lambda}\right\}
\]
Division by \(\frac{\pi d}{4}\) gives \(1 - P(0) = \frac{2\theta}{d} - \Delta_1\) which gives \(P(0)\). The remaining results given are "casey" but absolutely straightforward. q.e.d.
Note that the probabilities given add to 1:

\[ (1 + \Delta_1 - \frac{2\lambda}{\pi d}) + \Delta_0 + \Delta_2 - 2\Delta_1 + (\Delta_1 + \Delta_3 - 2\Delta_2) + \ldots + (\Delta_{m-3} + \Delta_{m-1} - 2\Delta_{m-2}) + (\Delta_{m-2} - 2\Delta_{m-1}) + \Delta_{m-1} = 1 + \Delta_0 - \frac{2\lambda}{\pi d} = 1. \]

Because of the telescoping of the terms making up \( p_i \), we also have

\[ (2-1) \quad P(\text{# crossings } \leq i) \overset{d}{=} F(i) = 1 - (\Delta_i - \Delta_{i+1}) \text{ for } i = 0, 1, 2, \ldots, m-1. \]

\[ (2-1) \text{ shows that } \Delta_i \text{ is a decreasing function of } i. \]

III The Moments of the Distribution of Crossings

In this section approximations to the moments valid for large values of \( \frac{\lambda}{d} \) are derived. The approximations are used to prove a limit theorem for the density via the method of moments. Finally, a better error term in the approximation of the moments is derived.

With notation as above, let \( \{a_i\}_{i=1}^m \) be any real numbers. Summation by parts shows:

\[ (3-1) \quad \sum_{i=0}^m a_i p_i = a_0 (1 - \frac{2\lambda}{\pi d}) + a_1 \Delta_0 + \sum_{i=1}^{m-1} \Delta_i^2 (a_{i-1} - a_i). \]

Here \( \Delta(a_i) \overset{d}{=} a_{i+1} - a_i \) is the differencing operator. \( (3-1) \) easily yields

\[ \sum_{i=0}^m p_i = 1, \quad \sum_{j \leq i} p_j = 1 - (\Delta_i - \Delta_{i+1}) \text{ for } i = 0, 1, \ldots, m-1 \text{ as well as} \]

\[ \sum_{i=0}^m \sum_{j \leq i} p_{i-j} = 1 - (\Delta_i - \Delta_{i+1}) \text{ for } i = 0, 1, \ldots, m-1. \]
The expression for the mean is frequently derived using a symmetry argument \[3;253\]. For higher moments, some approximation is needed.

**Theorem 2** \[
\mu_k = \frac{d}{m} \sum_{k=0}^{m} k \mu_k = c_k a^k + O(a^{k-1}) \quad \text{for} \quad k \geq 1.
\]

\[
a = \frac{g}{d}, \quad c_k = \frac{s_k}{\sqrt{\pi} t_k^{k+2}}.
\]

**Proof**

The result is true with no error for \(k = 1\), so assume \(k \geq 2\) using (3-1):

\[
\mu_k = \Delta_0 + \sum_{i=1}^{m-1} \Delta^2 [(i-1)^k] \Delta_i.
\]

Since \(\Delta_0 = O(a)\) and it is easy to see that \(\Delta^2 [(i-1)^k] = k(k-1)i^{k-2} + O(i^{k-4})\),

\[
(3-2) \quad \mu_k = k(k-1) \sum_{i=1}^{m-1} i^{k-2} \Delta_i + 0(a) + O(\sum i^{-1} \Delta_i).
\]

For \(k = 2, 3\), \(\Delta^2 [(i-1)^k] = k(k-1)i^{k-2}\) and the second error term is absent. Thus, it only remains to estimate

\[
\sum_{i=1}^{m-1} i^{k-2} \Delta_i = \frac{2}{\pi} \sum i^{k-2} \left\{ a \left[ 1 - \left( \frac{i}{a} \right)^2 \right]^{\frac{3}{2}} - i \cos^{-1} \left( \frac{i}{a} \right) \right\}
\]

\[
= \frac{2}{\pi} a^k \sum \frac{1}{a} \left\{ \left( \frac{i}{a} \right)^{k-2} \left[ 1 - \left( \frac{i}{a} \right)^2 \right]^{\frac{3}{2}} - \left( \frac{i}{a} \right)^{k-1} \cos^{-1} \left( \frac{i}{a} \right) \right\}.
\]

This last sum is a Riemann sum for

\[
(3-3) \quad \int_0^1 \left[ x^{k-2} (1-x^2)^{\frac{3}{2}} - x^{k-1} \cos^{-1}(x) \right] dx
\]

\[
= \frac{1}{2} \beta \left( \frac{k-1}{2} , \frac{3}{2} \right) - \frac{1}{2k} \beta \left( \frac{k+1}{2} , \frac{1}{2} \right) \quad \text{for} \quad k \geq 3; \quad = \frac{\pi}{6} \quad \text{when} \quad k = 2.
\]
Here \( \beta(a, b) \) denotes the beta function.

It is well known [2; 49] that for functions with a bounded derivative, the error made in replacing the sum by the integral is \( O\left(\frac{1}{a}\right) \). Replacing the sum by the integral, using this bound for the sum in the error term in (3-2), and simplifying the beta factors leads to the statement of the theorem. q.e.d.

**Corollary 1**

\[
\text{Variance} = \left[ \frac{1}{2} - \left( \frac{2}{\pi} \right)^2 \right] a^2 + O(a) \quad \text{where} \quad a = \frac{d}{a}.
\]

Numerically \( \left[ \frac{1}{2} - \left( \frac{2}{\pi} \right)^2 \right] = 0.094715^+ \).

**Theorem 3**

Notation as above, let \( I \) be the number of crossings then

\[
\lim_{a \to \infty} \frac{I}{a} \quad \text{converges in distribution to an arc sin distribution with density}
\]

\[
x(x) = \frac{2}{\pi} \frac{1}{(1-x^2)^{1/2}} \quad \text{for} \quad 0 < x < 1
\]

\[
= 0 \quad \text{elsewhere.}
\]

**Proof**

As \( a \to \infty \) the moments of \( \frac{I}{a} \) converge to the numbers \( c_k \) of Theorem 2.

A straightforward computation shows the arc sin distribution has moments \( c_k \). Since all the distributions concerned are constrained to the unit interval, the method of moments is in force and yields the desired result. q.e.d.

**Remark**

It is also possible to give a geometric proof of Theorem 3. This has the advantages of showing why the arc sin distribution appears as well as yielding a rate of convergence. Once it has fallen, translations of the needle, which preserve its angle to the grid, can only change the number of intersections by one. The number of intersections for a fixed angle \( \theta \) is \( \lfloor a \sin \theta \rfloor \) where \( \lfloor \cdot \rfloor \) denotes greatest integer.
Thus, letting \( I \) be the number of intersections

\[
\frac{I}{a} - \frac{2}{a} \leq |\sin \theta| \leq \frac{I}{a} + \frac{2}{a}
\]

so, for \( 0 < t < 1 \),

\[
P(0 \leq \sin \theta < t - \frac{2}{a}) < P(\frac{I}{a} < t) < P(0 \leq \sin \theta < t + \frac{2}{a}).
\]

Now \( P(0 \leq \sin \theta < a) = \frac{\sin^{-1} x}{2\pi} \) by symmetry. Finally, for \( a \) so large that \( 0 < x - \frac{2}{a} < x + \frac{2}{a} < 1 \), \( P(\frac{I}{a} < x) = \frac{\sin^{-1} x}{\pi} + O(\frac{1}{a}) \) as \( a \to \infty \). The constant implicit in the error term may be chosen independent of \( x \) for \( x \) bounded away from 1.

A more careful analysis yields a better error term in Theorem 2.

**Theorem 4** Notation as in Theorem 2:

\[
\mu_k = c_k \cdot k^3 + O\left(\frac{k^3}{a}\right).
\]

**Proof**

Proceeding as in Theorem 2, we need only evaluate \( m-1 \sum_{i=1}^{m-1} f \left( \frac{i}{a} \right) \).

where \( f(x) = x^{-2}(1-x^2)^{\frac{1}{3}} - x^{-1}\cos^{-1} x \). The Euler Maclaurin formula yields, for any twice differentiable \( g \),
\[
\sum_{i=1}^{m-1} g(i) = \int_0^{m-1} g(t) dt + \frac{1}{2} \left[ g(m-1) - g(0) \right] + \frac{1}{12} \left[ g'(m-1) - g'(0) \right]
\]

\[-\int_0^{m-1} p_2(x) g''(x) dx
\]

where \( p_2(x) = \frac{x^2}{2} - \frac{x}{2} + \frac{1}{12} \) \( 0 \leq x \leq 1 \) has period 1 on \((-\infty, \infty)\). Taking

\[
g(x) = f\left(\frac{x}{a}\right), \quad g'(x) = \frac{1}{a} \left\{ (k-2) \left(\frac{x}{a}\right)^{k-3} \left[ 1 - \left(\frac{x}{a}\right)^2 \right]^{1/2} - (k-1) \left(\frac{x}{a}\right)^{k-2} \cos^{-1} \left(\frac{x}{a}\right) \right\};
\]

\[
g''(x) = \frac{1}{a^2} \left\{ (k-2)(k-3) \left(\frac{x}{a}\right)^{k-4} \left[ 1 - \left(\frac{x}{a}\right)^2 \right]^{1/2} + \left(\frac{x}{a}\right)^{k-2} \left[ 1 - \left(\frac{x}{a}\right)^2 \right]^{1/2} - (k-1)(k-2) \left(\frac{x}{a}\right)^{k-3} \cos^{-1} \left(\frac{x}{a}\right) \right\}
\]

These are valid for \( k > 3 \). For \( k = 2 \),

\[
g'(x) = \frac{1}{a} \cos^{-1} \left(\frac{x}{a}\right), \quad g''(x) = \frac{1}{a^2} \left[ \frac{1}{1 - \left(\frac{x}{a}\right)^2} \right],
\]

for \( k = 3 \), \( g'(x) \) is as given above,

\[
g''(x) = \frac{1}{a^2} \left\{ \frac{x}{a} \left[ 1 - \left(\frac{x}{a}\right)^2 \right]^{1/2} - 2 \cos^{-1} \left(\frac{x}{a}\right) \right\}.
\]

Making the substitutions leads to:

\[
\sum_{i=1}^{m-1} f\left(\frac{i}{a}\right) = \int_0^{m-1} f\left(\frac{x}{a}\right) dt + \frac{1}{2} f\left(\frac{m-1}{a}\right) - f(0) - \frac{1}{12} f\left(\frac{m-1}{a}\right) - f'(0)
\]
\[
+ \frac{1}{a^2} \int_0^{m-1} p_2(x) f'' \left( \frac{x}{a} \right) dx.
\]

(3-4) \quad \int_0^1 f(y) dy - a \int_{m-1}^1 f(t) dt + O \left( \frac{1}{a} \right)^{\frac{3}{2}}

where we have used the easily verified fact that \( f(t) = 0 \) \((1-t)^{\frac{3}{2}}\) as \( t \to 1 \).

Using this last bound in the second integral in (3-4) leads to

\[
\frac{m-1}{\sum_{i=1}^m f\left( \frac{i}{a} \right)} = a \int_0^1 f(t) dt + O \left( \frac{1}{a} \right)^{\frac{3}{2}}.
\]

Valid for \( k \geq 3 \). For \( k = 2 \), a similar analysis leads to:

\[
\frac{m-1}{\sum_{i=1}^m f\left( \frac{i}{a} \right)} = a \int_0^1 f(t) dt - \frac{1}{2} + O \left( \frac{1}{a} \right)^{\frac{3}{2}}.
\]

Using these estimates and recalling the values of the integrals given in (3-3) completes the proof. \( \text{q.e.d.} \)

A numerical example: \( \frac{a}{d} = 15.5 \)

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<th>2</th>
<th>3</th>
<th>4</th>
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<td>0.042</td>
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<td>0.099</td>
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<td>0.054</td>
</tr>
</tbody>
</table>

Here \( \sum_{i=0}^{16} i^2 p_i = 120.3 \) while the value given by the approximation of Theorem 2 is 120.1.
References

