THE TYPICAL SPECTRAL SHAPE OF AN ECONOMIC VARIABLE

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C.W.J. GRANGER

TECHNICAL REPORT NO. 11
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C. W. J. Granger

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I. Introduction

During the past four or five years a fairly large number of power spectra have been estimated using economic data. (1) It might thus be an appropriate time to review the results obtained and to ask if the advent of spectral methods has thrown any light on the basic characteristics of economic variables. The almost unanimous result of these investigations is that the vast majority of economic variables, after removal of any trend in mean and seasonal components, have similarly shaped power spectra, the typical shape being:

![Log of power spectra graph]

Figure 1 - Typical Spectral Shape.

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(1) In addition to his own work, the author is familiar with the calculations by J. Cunyngham, D. Fand, M. Godfrey, M. Hatanaka, M. Nerlove, E. Perzen, and M. Suzuki.
It is the purpose of this paper to illustrate this result and to briefly discuss its implications both to economic theory in general and to economic model building in particular.

It is not, of course, suggested that every economic time series produce such spectra nor that nothing else is discernable from the estimated spectra other than this simple shape. Nevertheless, the fact that such a shape arises in the majority of cases does suggest that there are certain general, overall implications for economics, and possibly, that the estimation of power spectra alone is unlikely to be a productive technique. Cross spectral methods which, in the author's opinion, are likely to prove more important and which attempt to discover and explain the relationships between economic variables, will not be considered in this paper.\(^{(2)}\)

Spectral methods are based upon the idea of the decomposition of a stochastic process into a (possibly non-finite) number of orthogonal components, each of which is associated with a 'frequency.' Such a decomposition is always possible for processes which are 'covariance stationary' - i.e. have variances and autocovariances independent of real time. The power spectrum records the contribution of the components belonging to a given frequency band to the total variance of the process. If a band contributes a large proportion of the total variance it may be considered to be important compared to a band which contributes a smaller amount to the variance. As the introduction, interpretation and estimation

\(^{(2)}\) For a description of cross-spectral methods and other generalizations see [1].
of power spectra have been dealt with in detail elsewhere\(^{(3)}\) it is not considered necessary to expand on these subjects further in this paper. It must be emphasized that spectral methods do not require the specification of a model but follow directly from the assumption of stationarity\(^{(4)}\).

2. **Examples of Estimated Spectra**

A number of power spectra of economic series displaying the 'typical' shape have been published. Nerlove [2] shows the estimated spectra for the Federal Reserve Board index of industrial production (U.S.A., monthly, March 1920 - September 1960, seasonally adjusted), federally inspected hog slaughter and cattle slaughter (monthly, January 1907 - December 1960) and various U.S. employment series (monthly, July 1947 or 1948 - December 1961), both original series and seasonally adjusted). Cunnyngham [3] has estimated spectra for the wholesale commodity price index (monthly, 1908-1960) and money supply (monthly, 1908-1960). It has been found that price series from the New York stock market produce the typical shape. Diagrams illustrating this may be found in Granger and Morgenstern [4] and in chapter 4 of [1].

There also exist many unpublished estimated spectra. In the course of his studies of business cycle indicators, Hatanaka ([1], chapter 12, and [5]) has estimated the spectra of twenty two important economic

\(^{(3)}\) See [1] or Nerlove [2].

\(^{(4)}\) In fact, spectral methods can provide useful results for wider classes of process but description of these processes lies outside the scope of this paper.
series, the large majority of which display the typical shaped. Unpublished studies by Godfrey, Parzen and Suzuki provide further examples.

It must be emphasized that many of these series contained important seasonal components so that the 'typical shape' had added to it peaks at frequencies corresponding to a period of 12 months and its harmonics.

Figure 2 provides an example of a 'typical' spectral shape with peaks corresponding to the seasonal component. It shows the log. of the estimated power spectrum for United States bank clearing data (monthly, 1875-1958) from which an exponential linear trend (exp(a+bt)) has been estimated and subtracted. Despite this trend-removal procedure, considerable power remains at very low frequencies. In this example, roughly 99% of the total power lies in the frequency band \((0, 2\pi/40)\) which illustrates the extreme subtlety of the analysis one attempts in measuring the power spectrum at higher frequencies. The fact that it is usually possible to obtain a good estimate at all frequencies illustrates the considerable power of the available estimating procedures. The diagram shows the spectral estimates using the Parzen window and with 60, 80, and 99 lags. The consistent shape of the estimated spectrum using these different lags implies that the shape found may be accepted as being realistic with some confidence.

3. The Problem of Trend

The most noticeable characteristic of the typical spectral shape is the overpowering importance of the low frequency components. The majority of economic time series contain important trends in mean and it is known that such trends raise the value of the power spectrum at the low
frequencies. One must thus ask if the 'typical shape' is not largely
due to these trends.

Before inquiring further into this possibility one must first decide
upon a definition of trend. A definition which is generally acceptable
is not obvious, it being remembered that we have no reason to suppose
that the trend in mean can be well approximated by a simple polynomial
or exponential expression or even that it will be monotonic. It is clear
that a curve that would be considered 'trend' in a short series would
not be so considered if the series were longer. An example of this
would be temperature readings gathered every minute for four hours during
the day. The known daily fluctuation would appear as a trend in such
data but would not be considered as trend if the data were available for
a period of three months (in which case the annual fluctuation would
appear as 'trend'). Thus the definition is likely to depend on the
amount of data (n) available. In this paper trend in mean will be
defined as comprising all frequency components with frequency equal to
or less than $2\pi/n$, i.e. all components with wave-length equal to or
greater than the length of the series. Although, of course, this defini-
tion is to a certain extent arbitrary, it is precise and does appear to
include the intuitive meaning of trend given by the majority of users
of time series methods.

This definition of trend does not allow us to answer in the negative
the question stated in the first paragraph of this section. The reason
for this is that if the spectrum of a series containing a trend in mean
is estimated directly, the resulting shape is likely to be the 'typical'
shape due to a characteristic of the estimation procedure known as
leakage. If the true spectrum contains an important peak at some frequency not only will the value of the estimated spectrum at neighboring frequency bands be raised but up to 3% of the value of the peak may also be found in the estimated value at other nearby frequency bands. This leakage, although it is a small percentage of the large value at the peak, may nevertheless introduce overwhelming biases in the estimates of the spectrum at these frequencies. As the component we have called trend will usually contribute by far the largest proportion of the total variance of the series, the value of the spectrum at the zero frequency band \((0 \leq \omega \leq \pi/2m, \text{ if } m \text{ lags used in the spectral estimating procedure})\) will be large and the leakage effect will make it likely that the estimated spectrum will be quickly but smoothly decreasing for the next few frequency bands.

Thus, suppose for instance that a series is constructed as the sum of white noise\(^{(5)}\) and an important but very long cycle (with period longer than the length of the series). The theoretical spectrum will be as in figure 3 but the leakage effect will produce an estimated spectrum as shown in figure 4.

\[
\log f(\omega) \uparrow \quad \log \hat{f}(\omega) \downarrow
\]

\[
\pi \quad \omega
\]

**Figure 3**

\(^{(5)}\) A series is called white noise if the covariance between the value at time \(t\) and the value at time \(s\) is zero for all \(t \neq s\).
It follows from this that if one estimates the power spectrum of an economic series containing an important trend a 'typical shape' spectral estimate is likely to result. However, the important point about the typical shape is that it still appears even if trend in mean is removed. Suppose that the trend is removed by either a filter or regression method. It is known that such methods will both effectively remove a trend in mean and that the spectrum of the residual will be (almost) unbiased for all frequencies except for very low frequencies which, in general, will lose power. In the investigations cited above it is found that the power spectrum of the residual is basically of the form:

![Logarithm of power spectrum](image)

Figure 4

i.e. of the 'typical shape' apart from the loss of power at the low frequencies. Moreover, the same basic shape is found regardless of the length of data available, the size of the truncation point used in the estimation procedure, or the trend removal method used. For very long series (such as pig-iron production or monthly bank clearing data for the U.S., for which over a thousand recordings are available) the trend-removal techniques need affect only extremely low frequencies and the indication is consistently that the logarithm of the power spectrum.
increases monotonically as frequency decreases for frequency bands lower than that corresponding to a 12-month period.

It is, of course, possible that the typical shape does reach an absolute peak at some low frequency but if this be so, the peak must be at an extremely low frequency as no evidence of such a turning point has been found in studies of long series.

Studies in which the pre-whitening technique have been used, which decreases the bias due to leakage, support this finding (e.g., Nerlove [2]) as do analyses of data which contain no apparent trend in mean (e.g., New York Commercial Paper Rate, monthly, 1876-1914, see [1] chapter 4, [4]).

It must thus be concluded that the typical spectral shape of an economic series is as illustrated in figure 1 and that any possible trend in mean will only accentuate this shape.

4. Interpretation: Business Cycles

The existence of a typical spectral shape suggests the following law (stated in nonrigorous but familiar terms):

The long-term fluctuations in economic variables, if decomposed into frequency components, are such that the amplitudes of the components decrease smoothly with decreasing period. (6)

One may care to associate the frequency components with various of

---

(6) It might be possible to restate the law as "events which affect the economy for a long period are more important than those which affect it only for a short time."
the 'cycles' suggested as existing in economic series by some economists prior to the Second World War, such as the Kondratieff long wave (40 to 60 years), Kuznets long wave (20 to 30 years), the building cycle (15 to 20 years), minor or Kitchin cycle (2 to 4 years) and so forth. In this case the law suggests that the amplitude of the Kondratieff long wave is greater than the amplitude of the Kuznet's long wave which is greater than the amplitude of the building cycle and so on.\(^7\) However, the law also indicates that the spectrum does not consist of a series of peaks of decreasing size corresponding to each of the above cycles but is rather a smooth, decreasing curve with no (significant) peaks corresponding to periods longer than 12 months.

It has, of course, long been realized that those 'cycles' are not strictly periodic fluctuations which would correspond to jumps in the spectral distribution function but that they should be interpreted as important frequency bands. The evidence of estimated power spectra is that the 'peaks' corresponding to these bands do, in fact, blend together to form a smooth, continuous curve. If this is the case it might well be asked why certain fluctuations have been picked out as being of special importance, particularly the minor business cycle of roughly 30-50 months duration. To discuss this we first need to consider the likely properties of a sample from a process possessing a spectrum of the 'typical shape.'

\(^7\) If this were not the case, it is doubtful if such fluctuations could have been 'seen.' Thus, the existence of this early work in business cycles can be interpreted as an early discovery of this part of the 'law.'
The most obvious property of such a sample would probably be a visual long-term fluctuation which would not be in any way periodic. The fluctuation need not be visible at all moments of time and would, of course, be somewhat blurred by the high frequency and seasonal components.\(^{(8)}\) The estimated power spectrum would have essentially the correct shape but would usually also contain a number of peaks and troughs about the true spectral shape. A different sample from the same process would produce a similar estimated spectrum but with the peaks and troughs at different frequencies. This 'sampling error' is due to the fact that the samples are only of finite length. The peaks found in the estimated spectra should not, of course, be significant. Suppose now that a sample is such that the most visible peak is found in the power spectrum at a frequency corresponding to 63 months, say. In this case it appears likely that if one looks at the data with the intention of finding a 'cycle,' the cycle one will 'find' is one with a period near 63 months. In a sense, the sample does contain such a 'cycle' but it will be only a transient feature as it is unlikely to occur in the next stretch of data from the process.

The estimated spectra for a number of economic series do, in fact, contain peaks at a frequency corresponding to roughly 40 months. Examples of such a series are certain pre-First World War short-term interest rates (illustrated in [1, chapter 4] and in [4]) and stock price indices (illustrated in [4]). However, as Hatanaka points out

\(^{(8)}\) It would be theoretically possible to find the mean length of the long-term fluctuation but it seems doubtful if this would prove to be a useful concept.
[5] similar small peaks may be found in the logarithm of other economic series such as bank clearings and pig iron production. The reason for the logarithm of the series being required to find the peaks is that these series usually contain an important trend in variance, the effect of which is to blur any small peak in the spectrum. Taking the logarithm of the series, providing it is positive, moves any trend in variance into a trend in mean, which may be removed by the usual methods. (9)

The peaks found at or near 40 months are never statistically significant and the evidence available indicates that they are more pronounced prior to 1914. The fact that the same peak occurs in several series might appear to add to the statistical significance but actually is only a reflection of the fact that most economic series follow the long-term fluctuations of the economy as a whole.

The fact that such a peak is found in long series could, of course, indicate that this is a true and permanent characteristic of the economy. However, the evidence is not sufficiently strong for the rejection of the simple hypothesis that the true underlying spectrum is smooth and so does not contain a peak near 40 months. In any case, the size of the peak found in practice indicates that it is of little importance compared to neighboring frequency bands in terms of its contribution of the total variance of the series and so no evidence has been found that particular attention should be paid to this, or any other, frequency band.

(9) It is proved in [1] that if the standard deviation of the series is compared to the mean, the spectrum of the original series is similar in shape to the spectrum of the logarithm of the series.
5. **Description; Model Fitting**

Consider a process having the 'typical' spectral shape. If we define a filter $F_\alpha(\cdot)$ to be such that an input of series $X_t$ is transformed into an output series $Y_t$ where $Y_t$ obeys the equation

$$Y_t - \alpha Y_{t-1} = X_t$$

it seems certain that a good representation of our process will be $F_\alpha(\epsilon_t)$ or $F_\alpha[F_{\alpha'}(\epsilon_t)]$ where $\epsilon_t$ is white noise and $\alpha$ and $\alpha'$ have values in the range $1/2 \leq \alpha, \alpha' \leq 1$. For example, first order autoregressive schemes were fitted to the following series with results as indicated:

<table>
<thead>
<tr>
<th></th>
<th>estimated $\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) various stock price series (see [4])</td>
<td>0.98</td>
</tr>
<tr>
<td>2) manufacturer's inventory (1924-1955)</td>
<td>0.98</td>
</tr>
<tr>
<td>3) bank clearings</td>
<td>0.95</td>
</tr>
<tr>
<td>log. of bank clearings</td>
<td>0.97</td>
</tr>
<tr>
<td>4) lay-off rate (1930-1958)</td>
<td>0.78</td>
</tr>
<tr>
<td>5) average work week (1932-1958)</td>
<td>0.97</td>
</tr>
</tbody>
</table>

(the first three series are trend-adjusted, the latter two were not).

The results by Ames and Reiter [6], Orcutt [7], and unpublished results by Parzen concerning the fitting of autoregressive schemes to economic data further confirm the above suggestion. The main implication is that economic processes are almost unstable.

However, it must also be emphasized that these results to not preclude the possibility that the processes are best fitted by an explosive autoregressive model.
Quenouille [8, page 58] points out that the simple explosive Markov process

\[ Y_t - \beta Y_{t-1} = \epsilon_t, \quad \beta > 1 \]

may be viewed as the sum of two components \( X_t \) and \( Z_t \), where

\[ Y_t = X_t + Z_t, \]

\[ X_t - \beta^{-1} X_{t-1} = \eta_t, \]

and

\[ Z_t = Z_0 \beta^t. \]

Here \( \epsilon_t \) and \( \eta_t \) are both white noises.

Thus, suppose the true model is an explosive autoregressive scheme with parameter \( \beta = 1.05 \), say. Then after trend removal the residual may be represented as the output of the filter \( F_\alpha(\eta_t) \) with \( \alpha = 0.95 \). The process of first removing the trend in mean and then analyzing the residual is essentially based upon the assumption that the trend is being generated by a different mechanism than is generating the residual.

As, however, many economic series have an exponential trend in mean and a typical spectral shape, it is not clear that the two mechanisms are unconnected. It is quite possible, in fact, that the trend contains useful information about the rest of the process and in such a case this information should be used.

It has not been felt worth-while attempting to fit such schemes to a large number of economic variables owing to the fact that one first requires to remove efficiently any seasonal component and also due to the important inaccuracies contained in the majority of economic data which will be inclined to introduce possibly important biases into the
estimates. Nerlove [2] has discussed the first of these problems and has indicated that some of the methods currently used would certainly spoil the estimate. For a complete description of the second problem see Morgenstern [9].

6. Implications for Model Building

Let us first investigate the very simple accelerator multiplier model:

\[
C_t = c Y_{t-1} + \epsilon_t', \\
I_t = \nu (Y_t - Y_{t-1}) + \eta_t', \\
Y_t = C_t + I_t
\]

where \(C\) is consumption, \(Y\) is national income, \(I\) is investment and it is assumed that \(\epsilon_t'\) and \(\eta_t'\) are both white noise processes.

Rearranging, we have

\[
Y_t - \alpha Y_{t-1} = \epsilon_t \\
C_t - \alpha C_{t-1} = \eta_t \\
I_t - \alpha I_{t-1} = \epsilon_t - \eta_t
\]

where \((1-\nu)\epsilon_t = \epsilon_t' + \eta_t'\), \(\eta_t = \epsilon_t' + \epsilon_{t-1}' + \frac{c}{1-\nu} (\epsilon_{t-1}' + \eta_t')\) and \(\alpha = \frac{\nu-c}{\nu-1}\).

\(Y_t, C_t, \) and \(I_t\) may each be considered as outputs of the filter \(F_{\alpha}( )\) but with different inputs. The power spectra of \(Y_t\) and \(C_t\) are
\[ f_Y(\omega) = \frac{f_\varepsilon(\omega)}{2(1 - \alpha \cos \omega)} \]

\[ f_C(\omega) = \frac{f(\omega)}{2(1 - \alpha \cos \omega)} \]

It should be noted that if \( \alpha \) is near one, the term \((1 - \alpha \cos \omega)\) will be dominating the values of \( f_Y(\omega) \) and \( f_C(\omega) \) for low frequencies and thus the 'typical shape' will arise. It is, perhaps, interesting to record the value of \( \alpha \) for various suggested 'realistic' values for the accelerator and the multiplier.

Table

Values of \( \alpha = \frac{v-c}{v-1} \) for 'realistic' combinations of \( c \) and \( v \)

<table>
<thead>
<tr>
<th>( c )</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>---</td>
<td>1.33</td>
<td>1.22</td>
</tr>
<tr>
<td>1/2</td>
<td>---</td>
<td>1.25</td>
<td>1.17</td>
</tr>
<tr>
<td>2/3</td>
<td>1.33</td>
<td>1.17</td>
<td>1.11</td>
</tr>
<tr>
<td>3/4</td>
<td>1.25</td>
<td>1.12</td>
<td>1.08</td>
</tr>
<tr>
<td>4/5</td>
<td>1.2</td>
<td>1.10</td>
<td>1.07</td>
</tr>
</tbody>
</table>

Thus, this simple model suggests that an explosive autoregressive scheme is appropriate and spectra of 'typical shape' will certainly be forthcoming.

Similar reasoning to the above may be used to obtain a more general result suggesting a reason for the 'typical' shape being obtained in practice. Let \( U \) be the shift operator such that \( UX_t = X_{t-1} \). Consider the vector of exogenous variables \( X' = (X_1, X_2, \ldots, X_n) \)
and suppose that $X_t$ obeys the matrix autoregressive scheme:

$$P(U) X_t = Q(U) Y_t + e_t$$

where $Y_t$ is a $1 \times q$ vector ($q \leq m$) of endogenous variables, $e_t$ is a $1 \times m$ white noise vector and $P(U), Q(U)$ are $m \times m$ and $m \times q$ matrices respectively, each element of which is a polynomial in $U$.

Thus, we may write (providing the inverse of $P$ exists)

$$X_t = P^{-1}(U) Q(U) Y_t + P^{-1} e_t$$

$$= \frac{L(U)}{|P(U)|} Y_t + \frac{K(U)}{|P(U)|} e_t$$

The power spectrum for each component of $X_t$ may be derived from this formula, but it should be noted that each of the power spectra will have a denominator term which is the square of the modulus of the determinant $|P(e^{i\omega})|$. Thus, if the determinant $|P(z)|$ has a root (or roots) near $z = 1$ (corresponding to $\omega = 0$), this denominator is likely to dominate the value of the power spectra at low frequencies. If the modulus of the determinant is not small for other values of $\omega$, the resulting spectrum is likely to be 'typical' i.e. high valued at low frequencies and lower values at other frequencies. The spectra of the components of $X_t$ (possibly after trend removal) may vary in details but their basic shape will be similar.

An implication of the existence of a typical spectral shape is that if a model is constructed involving economic variables, one aspect of deciding if it is reasonable is to inquire whether the power spectra derived from the model has the typical shape. An example of a simple
model which does have this property is Klein's model [10]. As noted by Theil and Boot [11], "this model describes the American economy as a system which is close to instability and, in fact, leaves the question of stability open."

7. Implications for Control

It has been suggested above that it might be possible to describe many economic variables in terms of one of either of two simple models:\(^{(10)}\)

(i) an explosive scheme

\[ X_t - \alpha X_{t-1} = \varepsilon_t, \]

\( \alpha \) greater than but near 1 (say 1 < \( \alpha \leq 1.4 \)) or

(ii) \( X_t = \) trend in mean + \( Y_t \)

where

\[ Y_t - \alpha Y_{t-1} = \varepsilon_t, \]

\( \alpha \) less than but near 1 (say 0.7 < \( \alpha < 1 \)).

It was pointed out that spectral methods alone are unable to distinguish between these two models. The essential difference between them is that in (i) both the trend in mean and the long-term fluctuations of the variables are generated by the same mechanism, whereas in (ii) they need not be.

One of the possible aspects of controlling an economy might be stated as an attempt to smooth out the long-term fluctuations without

\(^{(10)}\) The models are stated in terms of first-order autoregressive schemes. It is easily seen that the arguments still follow if higher-order schemes had been used.
affecting the (upward) trend. In spectral terms this could be described diagrammatically by moving from a variable with spectral shape A to one with spectral shape B:

![Diagram A and B](image)

Figure 5

i.e. removing power in the "business-cycle" band without removing power at zero frequency.

If model (i) were the correct representation it might be difficult to achieve this change without slowing down the trend. If model (ii) were correct the effect of applying such control might be to make the variable less stable, resulting in more violent very-long term fluctuations. Either of these two affects would probably be considered undesirable. It is seen that the effect of such a control policy will be to make the process become nearer to being a random walk. The implication of this would seem to be that for a method of control to be really appropriate for use it must be capable of controlling the whole of the low-frequency component of the variable and not just one section of this component. Thus, for instance, it can be argued that one should not use a "stop-go" policy similar to that used by the British government in recent years without at the same time implementing long-term policies which help to control the 'trend' element. It is clear that more
precise statements cannot be made on this subject using this approach without further information about the true inter-relationship between the trend and long-fluctuation elements of important economic variables.
REFERENCES


