TRANSFER FUNCTIONS FOR TWO SEASONAL ADJUSTMENT FILTERS

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GEORGE R. HEXT

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Summary

Two methods for the seasonal adjustment of time series are considered in this paper. They are the one in use by the Bureau of Labour Statistics for the seasonal adjustment of national economic time series, and the other as proposed by Hannan and modified by Nerlove.

Linear "moving average" approximations to these methods are obtained, and their transfer functions evaluated and plotted. On this basis, the theoretical spectral density functions for the series before and after adjustment suggest that both methods over-adjust for the seasonal components. Comments are also made on spectral density functions obtained from series analysed by the two methods.

1. Introduction

The purpose of this paper is to investigate the properties of two filters in use for the seasonal adjustment of economic time series, and to discuss these properties.

In general, the filters most easily described and investigated are the linear time-invariant filters (often called linear filters, for short). With such, a reading in the filtered series is a linear sum of the raw observations, and the coefficients in the linear sum do not vary from

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one time to another. Symbolically, given the raw observations \( X(t) \),
each filtered series is given by

\[
Y(t) = \sum_{\tau = -\infty}^{\infty} a_{\tau} X(t - \tau).
\]

The properties of a linear time-invariant filter are summarized by
its transfer function. This gives the effect of the filter on a sine-
wave signal of known frequency \( \omega \), in particular whether a given signal
is amplified or suppressed, and how its phase is altered.

Thus if we write \( X(t) = A e^{i\omega t} \) the resulting output is

\[
Y(t) = \sum_{\tau = -\infty}^{\infty} a_{\tau} A e^{i\omega(t-\tau)}
\]

\[
= A e^{i\omega t} \sum_{\tau = -\infty}^{\infty} a_{\tau} e^{-i\omega \tau}.
\]

For input \( X(t) = A e^{i\omega t} \), the function \( Y(t)/X(t) \) is called the trans-
er function; thus at each frequency the transfer function

\[
L(\omega) = \sum_{\tau = -\infty}^{\infty} a_{\tau} e^{-i\omega \tau}
\]

The amplification and phase change produced by the filter at any given
frequency \( \omega \) are the amplitude and phase angle of \( L(\omega) \).

The importance of these filter properties lies in the fact that any
given time series may be considered as the sum of a finite or infinite
number of sine waves. The relative magnitude of the component waves is
commonly indicated by the spectral density function (or, more generally,
by the spectral distribution), which gives the power of the series in any
frequency band. Because the filter, as we have seen, has different effects on components of the signal at different frequencies, the spectral density function of the filtered series will not be the same as that of the original series; in fact

\[ f_Y(\omega) = |L(\omega)|^2 f_X(\omega) \].

The purpose of "seasonal adjustment" of economic time series is to reduce or suppress those components of the signal due to regular annual causes, and thus lay bare the remainder for use in analysis, forecasting, or policy formulation. The method used is some kind of filtering process, whether linear or non-linear, and time-invariant or time-dependent.

In this paper we consider two filters that have been suggested specifically for the seasonal adjustment problem, that used by the Bureau of Labour Statistics (BLS) and that proposed by Hannan and developed by Nerlove (Hannan-Nerlove). In each case we shall propose linear time-invariant filters which closely reflect the intent of the author's original method, and yet are susceptible of analysis through transfer functions. This method should in many instances be easily adapted to the investigation of other non-linear time-dependent filters.

2. The Bureau of Labour Statistics Method of Seasonal Adjustment

The BLS method has been described in a number of places. These analyses and comments are based on the description given by Rothman (1960). We assume that the data are taken at monthly intervals, and that the seasonal component has a basic period of one year.
The basic model considered is the multiplicative one,

\[ X^*(t) = S^*(t) \times T^*(t) \times I^*(t) \]

where

- \( X^*(t) \) are the original monthly observations,
- \( S^*(t) \) is the seasonal component
- \( T^*(t) \) is the trend or low-frequency component, commonly called the "trend-cycle"
- \( I^*(t) \) is the irregular component, and corresponds to the residual or error term in other statistical models.

In the course of estimating \( T^*(t) \), a weighted arithmetic mean of the 13 observations \( X^*(t-6) \) through \( X^*(t+6) \) is taken. A simple estimate of the trend-free component, \( S^*(t) \times I^*(t) \) would then be estimated by dividing \( X^*(t) \) by \( T^*(t) \). Likewise, in the course of estimating \( S^*(t) \), a weighted arithmetic average of \( X^*(t-24), X^*(t-12), X^*(t), X^*(t+12), X^*(t+24) \) is taken. This too would be removed from the observations by a division operation. Thus the whole filter is a mixture of additive and multiplicative operations.

For the analysis of this paper we propose that all averages be weighted geometric means. Then the whole procedure becomes multiplicative, and, on taking logarithms, the derived procedure becomes additive. We further assume that we have ample data ahead of the time in which we are interested, so that we can always use centered averages, and thus avoid certain problems associated with one-sided or asymmetric filters. Finally we do not take into account the special procedure used by the ELS to deal with extreme or unusual observations.
The taking of geometric rather than arithmetic averages certainly makes our analysis different in detail from the BLS method. Yet there does not appear to be any good reason why one should not use a geometric average rather than an arithmetic or any other reasonable average. By this and the other adaptations made here, the intent and spirit of the BLS method have hopefully been captured, while at the same time the method has been laid open to analysis by transfer functions.

In the remaining paragraphs of this section we describe in detail our adaptation of the BLS method and obtain the various transfer functions in the course of this analysis.

Let \( X(t) \) denote the logarithm of the original observations. Our model now is

\[
X(t) = S(t) + T(t) + I(t),
\]

and we seek to estimate \( S, T \) and \( I \). The BLS method is described in detail in Appendix A of Rothman (1960); here we describe our log-linearised version of the method. Each step in the process is denoted by the number of the corresponding Table in the BLS print-out, and the filtered series thus obtained by \( X_n(t) \). We also consider the special "observation" series \( X(t) = e^{-i\omega t} \), and obtain the corresponding filtered series \( Z_n(t) \).

The transfer function from the original series to the series obtained at the \( n \)-th stage is then

\[
L_n(\omega) = Z_n(t)/X(t) = Z_n(t) \cdot e^{-i\omega t}.
\]
Before starting the work, it is convenient to introduce these definitions:

\[
L_y(\omega) = \frac{1}{12} (1 + 2 \cos \omega + \ldots + 2 \cos 5\omega + \cos 6\omega)
\]

\[
L_s(\omega) = b_0 + 2b_{12} \cos 12\omega + 2b_{24} \cos 24\omega
\]

\[
L_u(\omega) = a_0 + 2a_1 \cos \omega + 2a_2 \cos 2\omega + 2a_3 \cos 3\omega
\]

Then our analysis proceeds through the following 15 steps.

1. Twelve-month centered moving average (Table 101).

As mentioned above, we do not attempt to include the special treatment of end-effects. Thus on averaging over the year centered at \( t \),

\[
Y_1(t) = \frac{1}{12} \left\{ \frac{1}{2} X(t - 6) + X(t - 5) + X(t - 4) + \ldots + X(t + 5) + \frac{1}{2} X(t + 6) \right\}
\]

and

\[
Z_1(t) = \frac{1}{12} \left\{ \frac{1}{2} e^{i(t-6)\omega} + e^{i(t-5)\omega} + e^{i(t-4)\omega} + \ldots + e^{i(t+5)\omega} + \frac{1}{2} e^{i(t+6)\omega} \right\}
\]

\[= e^{i\omega t} L_y(\omega) \]

whence

\[
L_1(\omega) = L_y(\omega)
\]
2. Seasonal and irregular, first approximation (Table 102).

\[ x_2(t) = X(t) - Y_1(t) \]

and

\[ z_2(t) = e^{i\omega t} - Z_1(t) \]

\[ = e^{i\omega t}(1 - L_y(\omega)) \]

Thus

\[ L_2(\omega) = 1 - L_y(\omega) \]

3. Unforced seasonal, first approximation (Table 103).

Using the centered weighted average throughout, with constants \( b_0, b_{12}, b_{24} \), we obtain

\[ y_3(t) = b_{24} Z_2(t - 24) + b_{12} Z_2(t - 12) + b_0 Y(t) \]

\[ + b_{12} Z_2(t + 12) + b_{24} Z_2(t + 24) \]

and

\[ z_3(t) = b_{24} Z_2(t - 24) + b_{12} Z_2(t - 12) + b_0 Z_2(t) \]

\[ + b_{12} Z_2(t + 12) + b_{24} Z_2(t + 24) \]

\[ = (1 - L_y(\omega)) \left\{ b_{24} e^{i(t-24)\omega} + b_{12} e^{i(t-12)\omega} + b_0 \right. \]

\[ + b_{12} e^{i(t+12)\omega} + b_{24} e^{i(t+24)\omega} \]

\[ = (1 - L_y(\omega)) L_y(\omega) e^{i\omega t} \]
whence
\[ I_{3}(\omega) = (1 - I_{y}(\omega)) L_{s}(\omega). \]

In the BLS method, \( b_0 = 0.26, \ b_{12} = 0.20, \ b_{24} = 0.17. \)

4. Forced seasonal, first approximation (Table 104).

In the BLS method, the forcing is done over the calendar year. Here, to keep the procedure symmetric, we have to "force" (i.e., correct for the mean) across the 13 months centered on the month of interest. Thus

\[ Y_{4}(t) = Y_{3}(t) - \frac{1}{12} \left\{ \frac{1}{2} Y_{3}(t - 6) + Y_{3}(t - 5) + Y_{3}(t - 4) + \ldots \right. \]
\[ \left. + Y_{3}(t + 5) + \frac{1}{2} Y_{3}(t + 6) \right\}. \]

The relation for \( Z_{4}(t) \) is similar, and gives

\[ I_{h}(\omega) = (1 - I_{y}(\omega)) L_{3}(\omega) \]
\[ = (1 - I_{y}(\omega))^2 L_{s}(\omega). \]

5. Irregular, first approximation (Table 105).

First, to estimate the seasonally adjusted series,

\[ Y_{\text{seas}}(t) = X(t) - Y_{4}(t) \]

whence

\[ Y_{5}(t) = Y_{\text{seas}}(t) - Y_{1}(t) \]
and so

\[ L_5(\omega) = 1 - L_4(\omega) - L_1(\omega) \]

\[ = (1 - L_y(\omega))(1 - L_s(\omega) + L_y(\omega)L_s(\omega)). \]

From now on, when the filtered series corresponds to those in one of the ELS tables it will be denoted by the table number. This is done in order to bring out the relation between corresponding tables in the 100, 200 and 300 series. Unlisted series will be denoted by appropriate letter suffixes. For simplicity of notation, we drop the argument \( \omega \) from each of the transfer functions \( L_i(\omega) \).

The procedure used to obtain the transfer functions should by now be evident from the above. Thus we can be brief in describing the remaining steps which, in any event, more or less repeat the analysis of the corresponding step in the 100 series, and, as can be seen from the graphs, provide but slight adjustments to the "first approximation" transfer functions.

6. Moving average, modified once (Table 201).

On averaging the above irregulars \( Y_5(t) \) (which could be written \( Y_{105}(t) \)), we obtain the residual trend-cycle in this series; thus

\[ Y_{rtc}(t) = a_2 Y_5(t-3) + a_2 Y_5(t-2) + a_1 Y_5(t-1) + a_0 Y_5(t) \]

\[ + a_1 Y_5(t+1) + a_2 Y_5(t+2) + a_3 Y_5(t+3) \]

whence
\[ Z_{rtc}(t) = L_a(\omega) Z_5(t) \]

and

\[ L_{rtc}(\omega) = L_a(1 - L_y)(1 - L_a + L_a L_s - L_a L_s L_y) . \]

In the BLS method, the values used for the constants are
\[ a_0 = 0.200, \ a_1 = 0.183, \ a_2 = 0.127, \ a_3 = 0.090. \]

This series \( Y_{rtc}(t) \) is then added to the original moving average \( Y_1(t) \) to give the once-modified moving average. Thus:

\[ Y_{201}(t) = Y_1(t) + Y_{rtc}(t) \]

whence

\[ L_{201}(\omega) = 1 - (1 - L_y)(1 - L_a + L_a L_s - L_a L_s L_y) . \]

7. Seasonal and irregular, second approximation (Table 202).

\[ Y_{202}(t) = X(t) - Y_{201}(t) . \]

Thus

\[ L_{202}(\omega) = 1 - L_{201}(\omega) \]

\[ = (1 - L_y)(1 - L_a + L_a L_s - L_a L_s L_y) . \]

8. Unforced seasonal, second approximation (Table 203).

\[ L_{203}(\omega) = L_s(\omega) L_{202}(\omega) \]

\[ = L_s (1 - L_y)(1 - L_a + L_a L_s - L_a L_s L_y) . \]
9. Forced seasonal, second approximation (Table 204).

\[ L_{204}(\omega) = (1 - L_y(\omega)) L_{205}(\omega) \]
\[ = L_s(1 - L_y)^2 (1 - L_a + L_a L_s - L_a L_s L_y). \]

10. Irregular, second approximation (Table 205).

\[ L_{205}(\omega) = 1 - L_{204}(\omega) - L_{201}(\omega) \]
\[ = (1 - L_y)(1 - L_s L_y)(1 - L_a + L_a L_s - L_a L_s L_y). \]

11. Moving average, modified twice (Table 301).

This is the series that is considered to be the trend-cycle component of the original observations. Thus

\[ L_{tc}(\omega) = L_{301}(\omega) \]
\[ = L_{201}(\omega) + L_a(\omega)L_{205}(\omega) \]
\[ = 1 - (1 - L_y)(1 - L_a + L_a L_s - L_a L_s L_y)^2. \]

12. Seasonal and irregular, third approximation (Table 302).

\[ L_{302}(\omega) = 1 - L_{301}(\omega) \]
\[ = (1 - L_y)(1 - L_a + L_a L_s - L_a L_s L_y)^2. \]

13. Unforced seasonal, third approximation (Table 303).

\[ L_{303}(\omega) = L_s(\omega) L_{302}(\omega) \]
\[ = L_s(1 - L_y)(1 - L_a + L_a L_s - L_a L_s L_y)^2. \]
14. Forced seasonal, third approximation (Table 304).

This is the final estimate for the seasonal component.

Thus

\[ L_{fs}(\omega) = L_{304}(\omega) \]

\[ = \{1 - L_y(\omega)\} L_{303}(\omega) \]

\[ = L_s(1-L_y)^2 (1-L_a + L_{s_a}L_s - L_{a_s}L_y)^2 . \]

15. Irregular component, third approximation (Table 305).

This is the final estimate for the irregular component.

Thus

\[ L_{ir}(\omega) = L_{305}(\omega) \]

\[ = 1 - L_{304}(\omega) - L_{301}(\omega) \]

\[ = (1-L_y)(1-L_s + L_{s_y})(1-L_a + L_{a_s}L_s - L_{a_s}L_y)^2 . \]

From this detailed analysis of the linear approximation to the BLS method, we conclude that the three components of a given time series are obtained by linear filters whose transfer functions are the following functions of frequency:

(1) For the trend cycle, (from Table 301)

\[ L_{tc}(\omega) = 1 - (1-L_y)(1-L_a + L_{a_s}L_s - L_{a_s}L_y)^2 . \]

(2) For the seasonal (forced) (from Table 304)

\[ L_{fs}(\omega) = L_s(1-L_y)^2 (1+L_{a_s}L_s + L_{a_s}L_y)^2 . \]
(3) For the irregular component (from Table 305)

\[ L_{ir}(\omega) = (1 - L_y)(1 - L_s + L_s L_y)(1 - L_a + L_a L_s - L_a L_s L_y)^2. \]

We recall that \( L_a, L_s, \) and \( L_y \) are all functions of \( \omega \); they are given above, just before step 1. Note that because of the symmetry of each filter about time \( t \), each transfer function is real, and the filters described here introduce no phase shift. As we have remarked above, this filter represents the seasonal adjustment of historical data, rather than of present data. We believe, however, that this is sufficient for the purpose of this paper, which is to discuss some general characteristics of seasonal adjustment procedures in current use.

The ELS procedure is iterative. The first approximations to the three required series or their transfer functions are produced in steps 1 - 5. These are then adjusted by steps 6 - 10, and readjusted by steps 11 - 15, which repeat steps 6 - 10.

If the adjustment cycle of steps 6 - 10 is repeated \((p - 1)\) times in all (to produce what would be the \( p \)-th approximation), it can be shown by induction that the transfer functions are as follows:

1. For the trend cycle (from Table p01)

\[ L_{tc}(\omega) = 1 - (1 - L_y)(1 - L_a + L_a L_s - L_a L_s L_y)^{p-1}. \]

2. For the seasonal (forced), (from Table p04)

\[ L_{fs}(\omega) = L_s(1 - L_y)^2(1 - L_a + L_a L_s - L_a L_s L_y)^{p-1}. \]
For the irregular component (from Table p05)

\[ L_{ir}(\omega) = (1 - L_y)(1 - L_s + L_s L_y)(1 - L_a + L_a L_s - L_a L_s L_y)^{p-1}. \]

Note that for each \( p \),

\[ L_{tc}(\omega) + L_{fs}(\omega) + L_{ir}(\omega) = 1 \]

confirming that at each approximation we have divided the series completely into three components.

The BLS method itself thus corresponds to the case \( p = 3 \). \( p = 1 \) and \( p = 2 \) give intermediate stages of the BLS method.

Note too that to obtain a filtered series of \( N \) terms, each application of the yearly average filter, given by \( L_y(\omega) \), requires a further six terms at each end; and for the averaging and the seasonal filters, given by \( L_a(\omega) \) and \( L_s(\omega) \), the figures are 3 and 24 respectively. Thus from the last term in \( (1 - L_a + L_a L_s - L_a L_s L_y) \), each complete adjustment iteration requires almost three years of additional data at each end of the original observed series. (To some extent this has been compensated for by the process of "end correction", given in the BLS method but not discussed here.)

Using the given BLS values of the coefficients in \( L_a(\omega) \) and \( L_s(\omega) \), the three transfer functions given above have been plotted for \( p = 1, 2, \) and \( 3 \), in Figs. 1 - 9, thus showing the relative effects of the different iterations or adjustments in the BLS method. Also shown, in Fig. 10, is the "adjustment factor" \( (1 - L_a + L_a L_s - L_a L_s L_y) \), giving the relation between the transfer function, in successive approximations.
Transfer functions for first approximation to the log-linearised version of the ELS filter.
Figure 4

Figure 5

Figure 6
Transfer functions for second approximation to the log-linearised version of the ELS filter.
Transfer functions for third approximation to the log-linearised version of the BLS filter.
Figure 10

Adjustment factor used in obtaining successive approximations to the ELS filter.
3. **Comments on the BLS Method**

As analysed here, we see that in the successive iterations or adjustments of the BLS method, the transfer functions are all very nearly the same except near the origin. For these low frequencies, the second and third approximations do indeed separate out more completely the low frequency or "trend-cycle" component; the cost of this, however, is the greater passage of "sidebands" or higher frequency components with the successive iterations.

From Fig. 10, we see that the factor \( \left\{ 1 - L_a(\omega) + L_a(\omega)L_s(\omega) - L_a(\omega)L_s(\omega)L_y(\omega) \right\} \), which accounts for the adjustment between successive approximations, differs from 1 only in the first one-third of the frequency range. Comparison of the seasonal component transfer functions for \( p = 0, 1, 2 \) shows that these functions are almost identical throughout their range, including the first one-third. Therefore we conclude that the **effect of the successive approximations on the seasonal component is negligible**. These approximations do, however, affect the separation of the seasonally adjusted series into the trend-cycle and irregular components, as we have already indicated.

The worth of these further approximations \( (p = 2, 3) \) must be weighed by the analyst in any given situation. He will have to bear in mind (1) the increasing high frequency content of the trend-cycle component as more approximations are made, and (2) the greater number of original observations and calculations needed to obtain these adjustments.

The analyst, too, would be advised to study the overall conclusions given at the end of this paper, before embarking on any analysis using the BLS method as it now stands.
4. **The Hannan-Nerlove Method of Seasonal Adjustment**

The second procedure examined in this paper is due to Hannan, as adapted by Nerlove. In essence the method is exceedingly simple, but the analysis becomes more complicated when we attempt to remove the "trend" or low frequency component from the analysis. A full description of the process is given by Nerlove (1964) -- it is developed from the general methods proposed by Hannan (1960 and 1963).

To summarise the method: Nerlove assumes a linear model, and first removes much of the trend by a special "pre-whitening" process that he calls "quasi-differencing". From the "trend-free" observations, following Hannan, he estimates the seasonal component $S^*(t)$ for each month by removing an overall mean, and then computing the five-year unweighted moving average of the observations for that month. He then "recolours" these seasonal values to obtain the seasonal component of the original series.

There are fewer stages in this computation than in the BLS method as described above, but they are slightly more complicated to analyse since the "quasi-differencing" and subsequent "recolouring" processes are not symmetric. We shall introduce such symmetry as we are able without losing the spirit of the Hannan-Nerlove method by redefining

$$
\hat{\alpha}_k^j(t) = \frac{1}{6} \sum_{j=-6}^{6} \varepsilon_j \cos 2\pi \left( \frac{k}{12} \right) j \bar{Y}(t-j), \quad k = 1, 2, \ldots, 5.
$$

where

$$
\varepsilon_j = \begin{cases} 
\frac{1}{2} & \text{for } j = -6, 0 \\
1 & \text{for all other } j \text{ in the sum.}
\end{cases}
$$

The new definitions for $\hat{\alpha}_0^j$ and for $\hat{\beta}_k^j$ follow similarly. We shall also
assume that we are analysing "historic" data, so that we have all the
data ahead of the time of interest that we might wish to bring into the
analysis. Finally we make the necessary changes to ensure that the fil-
ter is time-invariant.

For our analysis, it is convenient to define

\[ H(\omega) = (1 - \gamma e^{-i\omega}) \]

with complex conjugate \( \tilde{H}(\omega) \), and

\[ J(\omega) = \frac{1}{5} (1 + 2 \cos 12\omega + 2 \cos 24\omega) \]

\( J(\omega) \), of course, is a special case of \( L_6(\omega) \) introduced in describing
the BLS method. Also

\[ C_k(\omega) = \frac{1}{6} \left[ 1 + 2 \sum_{j=1}^{5} \cos \frac{2\pi k}{12} j \cos \omega j + (-1)^k \cos 6\omega \right] \], \( k=1,2,\ldots,5 \)

\[ C_6(\omega) = \frac{1}{12} \left[ 1 + 2 \sum_{j=1}^{5} (-1)^j \cos \omega j + \cos 6\omega \right] \]

\[ D_k(\omega) = \frac{1}{6} \left[ 2 \sum_{j=1}^{5} \sin \frac{2\pi k}{12} j \sin \omega j \right] \], \( k=1,2,\ldots,5 \)

and

\[ A_k = \sum_{r=0}^{p} \binom{p}{r} (-\gamma)^r \cos \frac{2\pi k}{12} \]

\[ B_k = \sum_{r=0}^{p} \binom{p}{r} (-\gamma)^r \sin \frac{2\pi k}{12} \]

These last are identical with Nerlove's definitions (Equation 3.11).

Finally we let
Tr[S] denote the transfer function of the linear filter S(t); and

Am[S] and Ph[S] denote the amplitude and phase angle of Tr[S]. All these three are functions of ω.

Then to obtain the transfer functions, we suppose that

\[ X(t) = e^{iωt} \]

The series of quasi-differences becomes

\[ Y_1(t) = X(t) - γ X(t-1) = e^{iωt} (1 - γ e^{-iω}) \]
\[ Y_2(t) = Y_1(t) - γ Y_1(t-1) = e^{iωt} (1 - γ e^{-iω})^2 \]
\[ \vdots \]
\[ Y_p(t) = Y_{p-1}(t) - γ Y_{p-1}(t-1) = e^{iωt} (1 - γ e^{-iω})^p \]
\[ = e^{iωt} H^p(ω). \]

As in Nerlove's paper, we denote the (symmetric) five-year mean for a given month by a bar. Then

\[ \bar{Y}_p(t) = \frac{1}{5} \left[ Y_p(t-24) + Y_p(t-12) + Y_p(t) + Y_p(t+12) + Y_p(t+24) \right] \]
\[ = e^{iωt} \left( 1 - γ e^{-iω} \right)^p \frac{1}{5} (1 + 2 \cos 12ω + 2 \cos 24ω) \]
\[ = e^{iωt} H^p(ω) J(ω). \]

Next, following Nerlove, we compute the Fourier coefficients of the \( \bar{Y}_p(t) \) series. In his paper, Nerlove (1964, p. 19) says that the mean should be removed from this series; however each of the coefficients that he computes does not depend on the mean value of the series, but only on
the deviations from this mean, or indeed from any other constant value.

Thus there is no need to make this correction, and we may proceed directly to obtain \( \hat{q}'(t) \) and \( \hat{\varphi}'(t) \). Thus for \( k = 1, 2, \ldots, 5 \),

\[
\hat{q}'(t) = \frac{1}{6} \sum_{j=-6}^{6} \varepsilon_j \cos \frac{2\pi nk}{12} j \bar{Y}_p(t+j)
\]

\[
= e^{i\omega t} H_p(\omega) J(\omega) \frac{1}{6} \sum_{j=-6}^{6} \varepsilon_j \cos \frac{2\pi nk}{12} j (\cos \omega j + i \sin \omega j)
\]

\[
= e^{i\omega t} H_p(\omega) J(\omega) \frac{1}{6} \left[ 1 + 2 \sum_{j=1}^{6} \varepsilon_j \cos \frac{2\pi nk}{12} j \cos \omega j \right]
\]

\[
= e^{i\omega t} H_p(\omega) J(\omega) C_k(\omega),
\]

while

\[
\hat{\varphi}'(t) = \frac{1}{12} \sum_{j=-6}^{6} \varepsilon_j \cos \pi j \bar{Y}_p(t+j)
\]

\[
= e^{i\omega t} H_p(\omega) J(\omega) \frac{1}{12} \left[ 1 + 2 \sum_{j=1}^{6} \varepsilon_j (-1)^j \cos \omega j \right]
\]

\[
= e^{i\omega t} H_p(\omega) J(\omega) C_6(\omega),
\]

Similarly, for \( k = 1, 2, \ldots, 5 \),

\[
\hat{\varphi}'(t) = \frac{1}{6} \sum_{j=-6}^{6} \varepsilon_j \sin \frac{2\pi nk}{12} j \bar{Y}_p(t+j)
\]

\[
= i e^{i\omega t} H_p(\omega) J(\omega) \frac{1}{6} \left[ 2 \sum_{j=1}^{5} \sin \frac{2\pi nk}{12} j \sin \omega j \right]
\]

\[
= i e^{i\omega t} H_p(\omega) J(\omega) D_k(\omega),
\]
while
\[
\beta'_0(t) = 0 .
\]

Thus for the given sine wave,
\[
\hat{\alpha}_k(t) = e^{i\omega t} \hat{H}(\omega) J(\omega) \left[ A_k C_k(\omega) + i B_k D_k(\omega) \right]
\]
\[
\hat{\beta}_k(t) = e^{i\omega t} \hat{H}(\omega) J(\omega) \left[ -B_k C_k(\omega) + i A_k D_k(\omega) \right]
\]
whence, corresponding to Nerlove's equation (3.12),
\[
S(t) = \sum_{k=1}^{6} \hat{\alpha}_k(t) .
\]

[This follows from the symmetry of our working, where we must put \( J = 0 \) in the right hand side of (3.12).] Thus
\[
S(t) = e^{i\omega t} \hat{H}(\omega) J(\omega) \sum_{k=1}^{6} \left[ A_k C_k(\omega) + i B_k D_k(\omega) \right] .
\]

From this we obtain the transfer function for the seasonal component. Similarly, when \( X(t) = e^{i\omega t} \), the trend-cycle and irregular component becomes
\[
X(t) - S(t) = e^{i\omega t} \left\{ 1 - \hat{H}(\omega) J(\omega) \sum_{k=1}^{6} \left[ A_k C_k(\omega) + i B_k D_k(\omega) \right] \right\}
\]
and the transfer function follows.

Since these transfer functions are in general complex, we can best study their properties by computing their amplitudes and phase angles.
as $\omega$ varies. Thus

$$Am[S] = |\tilde{H}(\omega)|^{-P} |J(\omega)| \left\{ \left[ \sum A_k C_k(\omega) \right]^2 + \left[ \sum B_k D_k(\omega) \right]^2 \right\}^{1/2}$$

$$\text{Ph}[S] = -P \text{Ph}[\tilde{H}(\omega)] + \text{Ph}[J(\omega)] + \tan^{-1} \frac{\sum B_k D_k(\omega)}{\sum A_k C_k(\omega)}.$$

The amplitude and phase of $X(t) - S(t)$ follow either directly from the transfer function, or more easily from the Argand diagram showing the triangle with sides $\text{Tr}[S]$, $1$ and $1 - \text{Tr}[S]$.

In Figures 11 - 16, the amplitude and phase angle of these transfer functions can be seen, for no, one and two quasi-difference operations. With no quasi-differencing, the transfer functions are both real. For the seasonally adjusted series, the transfer function is always positive. For the seasonal component, however, the function often goes negative, hence the jagged appearance of the corresponding phase angle and amplitude graphs.
5. Comments on Hannan-Nerlove Method

In our analysis of the Hannan-Nerlove filters, we see the effect of using certain pre-whitening and recolouring processes in conjunction with these filters. The pre-whitening processes are the various quasi-differencing operations suggested by Nerlove; and the corresponding recolouring processes ensure that a time series of observations at one of the seasonal frequencies would be filtered right out of the seasonally adjusted series.

The difference between the various pre-whitening procedures is negligible at all but the lowest frequencies. At these low frequencies, a greater proportion of the trend-cycle appears with the seasonal component on using the higher orders of quasi-differences. From this point of view alone, we might suggest that quasi-differences not be used. However, this effect is small; and other effects dominate it, such as the basic need to pre-whiten, and the overall conclusions of this study.

6. Conclusions

Although we have already made some comments on the individual procedures, the principal conclusion of our analysis of the two seasonal adjustment procedures by the method of transfer functions is as follows.

For every variant of both the procedures, in the neighbourhood of the "seasonal" frequencies the transfer function of the non-seasonal terms shows substantial "troughs". At the actual frequencies \( \omega_k = \frac{\kappa \pi}{6} \), \( k = 1, 2, \ldots, 6 \), the transfer function in every instance is zero. From the relation \( f_\omega(\omega) = |L(\omega)|^2 f_\omega(\omega) \), therefore, the spectral density of the seasonally adjusted series must likewise show troughs that reach zero at the seasonal frequencies. (Of course, the estimated spectrum will merely show a "dip", the depth of which will depend on the width of window used in estimation.)
In contrast to this, we should expect to find the following: If the time series shows any underlying trend, we expect the seasonally adjusted series to have considerable power at and near the origin. Likewise if there are any underlying periodicities at frequencies not near the seasonal frequencies, we expect them to show corresponding peaks in the spectral density functions of the series both before and after seasonal adjustment. Apart from these, the seasonally adjusted series is just the "irregular" component, which should surely have a fairly smooth spectrum. In other words, we expect the spectral density function of the seasonally adjusted series to be quite smooth except perhaps for peaks at the origin or other non-seasonal frequencies. Thus our hopes will not be realised if either of the two seasonal adjustment filters discussed in this paper are used.

An extreme example which shows how wrong the present methods are, is as follows. We consider a series of observations in which there are no seasonal components, and whose spectrum, therefore, we take to be smooth, showing no peaks at the seasonal frequencies. On analysis by the methods described here, however, even this series will yield a seasonal component and a "seasonally adjusted" series whose spectrum will have large troughs at the seasonal frequencies. Is this what we expect from a seasonal adjustment procedure? Surely this time series ought not to be affected by any seasonal adjustment procedure.

Therefore we must ask if there is much justification for the correction methods analysed here, when they give a seasonally adjusted series whose spectrum has such pronounced troughs at the seasonal frequencies. In other words, we suggest that in terms of the spectral density function,
the methods of seasonal adjustment described here over-compensate considerably for the seasonal effects.

Lest this paper appear entirely negative in its import, we suggest that in future the seasonal adjustment ought to be done in the light of the spectrum of the time series or its estimate rather than in a blind, mechanical fashion expected to apply to all possible series. In series where the power in the "seasonal" peaks is high compared with the irregular component at these frequencies, we shall require a filter whose transfer has a considerable dip at the seasonal frequencies. The dip, too, may vary from one frequency to another, but will never reach zero. On the other hand, when the power in the seasonal component is low, our filter must be more gentle in its effect at the seasonal frequencies.

For each given time series, therefore, our first task is to separate and estimate the seasonal and irregular components at the seasonal frequencies. Only then can we design the appropriate filter. Estimation of the seasonal components is discussed fully by the author in his Ph.D. thesis [Nexet (1964)], and it is hoped at a later date to discuss the use of these estimates for designing a linear filter to remove these components.

In the meantime these conclusions are offered to those working in the area of practical time series analysis for further study and comment, and in the hope of eventual effect upon actual practice.

Finally, it is useful to compare our theoretical conclusions with available practical results. Nerlove (1964) gives the spectral analyses of various unemployment series, both before and after seasonal adjustment. Because the methods of seasonal adjustment are both non-time-variant and one (the BLS) is non-linear, and because of the "smoothing"
effect of spectral estimation, the full over-compensation effect described in this paper is not seen. Even so, the spectra of the seasonally adjusted series still show effects of one kind or another at the seasonal frequencies.

In the ELS-adjusted series (Figs. 1.1A, 2.1A, 3.1A, 4.1A, and 5.1A), the spectra at the seasonal frequencies are lower than the local average, thus supporting the idea that this analysis over-estimates the seasonal term. In general, we see that the ELS method removes more of the power than is contained in the "seasonal" peaks alone -- and it does this from all the frequencies, not only the seasonal ones.

The spectra for the Hannan-adjusted series (Figs. 1.2A, 2.2A, 3.2A, 4.2A, 5.2A), on the other hand, still show slight peaks at the seasonal frequencies, and so at first sight suggest that the seasonal adjustment is insufficient. However, they give the impression that the spectral density function of the adjusted series is just a smoother version of that for the unadjusted series. In fact, in the final figure there seems to be more power in the series after the supposed removal of seasonality.

On their showing here, then, both the methods of seasonal adjustment discussed here leave much to be desired in actual use.
REFERENCES


