SEQUENTIAL $\chi^2$ AND F TESTS AND THE RELATED CONFIDENCE INTERVALS

BY

D. SIEGMUND

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ABSTRACT

Sequential $\chi^2$ and F tests are proposed for comparing more than two treatments. Extending earlier results of the author, one obtains approximations to the significance level and power of the tests, and approximate confidence intervals for the non-centrality parameter $\theta$. For each fixed $\theta$ a confidence region is given for the angle $\omega$ which the treatment effect vector in canonical coordinates makes with a fixed direction, and hence an over-all confidence region is obtained for the pair $(\theta, \omega)$.

AMS 1970 Subject Classification. Primary 62L10

Key Words and Phrases. Sequential test, stopping rule, sequential confidence region.
SEQUENTIAL $\chi^2$ AND F TESTS AND THE RELATED CONFIDENCE INTERVALS

1. Introduction

For comparing two treatments in clinical trials, Armitage (1975) has suggested a class of sequential tests which he calls repeated significance tests. These tests have been studied numerically by McPherson and Armitage (1971) and theoretically by Siegmund (1977, 1978), who also described a method for obtaining confidence intervals related to these tests and suggested a modification of the tests to increase the precision of the estimates. See also Siegmund and Gregory (1979).

The goal of the present paper is to study analogous procedures for comparing more than two treatments. Simple examples involving three treatments might arise in comparing a drug at maximal dose, a drug at minimal dose, and placebo, or drug A, drug B, and drug A together with B.

For theoretical analysis it is convenient to assume that observations are made sequentially on vectors $X_n = (X_{1n}, \ldots, X_{rn})'$, $n = 1, 2, \ldots$ where $r$ denotes the number of treatments and $X_{ij}$ is the response of the $j$th patient assigned to treatment $i$. The results of this paper should be applicable to randomized schemes of allocating treatments which approximate this situation. The $X_{ij}$ are assumed to be independently and normally distributed with mean $\mu_{ij}$ and common variance $\sigma^2$. The following discussion is restricted to the special case that $\mu_{ij} = \mu_i$ for all $j$. The theory can easily be modified to
accommodate stratification on some concomitant variable, provided the allocation of treatments is approximately balanced within strata.

The one way analysis of variance model with \( \mu_{ij} = \mu_1 \) for all \( j \) is customarily reparameterized in terms of \( \mu = r^{-1} \sum_{i=1}^{r} \mu_i \) and \( \alpha_i = \mu - \mu_1 \), so \( \text{E}X_{ij} = \mu + \alpha_i \) with \( \text{E} \alpha_i = 0 \). The log likelihood ratio statistic for testing \( H_0 : \Sigma \alpha_i^2 = 0 \) against \( H_1 : \Sigma \alpha_i^2 > 0 \) based on \( n \) observations is

\[
(1) \quad Z_n = \frac{1}{2} \ln \frac{1 + n \Sigma (\bar{X}_{i} - \bar{X}_{-})^2 / \Sigma (X_{ij} - \bar{X}_{i})^2}{1 + \Sigma (\bar{X}_{i} - \bar{X}_{-})^2 / \Sigma (X_{ij} - \bar{X}_{i})^2}.
\]

A natural extension of the repeated significance test is as follows: given integers \( m_0 < m \) and constants \( 0 < c \leq a \), stop sampling at \( \min (T, m) \), where

\[
(2) \quad T = \inf \{ n : n \geq m_0, Z_n > a \},
\]

and reject \( H_0 \) if either \( T \leq m \) or \( T > m \) and \( Z_m > c \). Like the usual fixed sample size \( F \) test, this is a test of the parameter

\( \theta = \sigma^{-1} (\Sigma \alpha_i^2)^{1/2} \)

in the sense that the power function of the test,

\[
(3) \quad \text{pr}_\theta (T \leq m) + \text{pr}_\theta (T > m, Z_m > c),
\]

depends on the parameters \( \alpha_1, \alpha_2, \alpha_3, \mu, \) and \( \sigma \) only through the value of \( \theta \). Computation of the probabilities appearing in (3) will be discussed in Sections 2 and 4.

This test is designed to have a small expected sample size

\( E_0 \{ \min (T, m) \} \) for large \( \theta \), when the effects of the various treatments differ substantially from one another. However, like the usual
F test, it does not in itself provide information about the relative merits of the different treatments.

Let $C$ be an orthogonal $t \times t$ matrix with $c_{rj} = \frac{1}{\sqrt{2}} (j = 1, \ldots, r)$, and let $\Theta = \sigma^{-1} \alpha = (\theta_1, \ldots, \theta_r, 0)'$, where $\alpha = (\alpha_1, \ldots, \alpha_r)'$. Then $||\Theta|| = \Theta$. As an attempt to provide meaningful comparative information about the merits of the various treatments, in Sections 3 and 5 a joint confidence region is obtained for $\Theta$ and $\omega$ where $\omega$ denotes the angular coordinates of the vector $\Theta$ relative to a given direction in $r - 1$ space. Suppose, for example, that $r = 3$. Let $(c_{11}, c_{12}, c_{13}) = (1/\sqrt{2}, -1/\sqrt{2}, 0)$ and $(c_{21}, c_{22}, c_{23}) = (1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6})$, so $\Theta = (\theta_1, \theta_2, 0)'$, where $\theta_1 = (\alpha_1 - \alpha_2)/\sqrt{2} \sigma$ and $\theta_2 = (\alpha_1 + \alpha_2 - 2\alpha_3)/\sqrt{6} \sigma$. In this case $\omega$ may be taken to be the angle the vector $(\theta_1, \theta_2)$ makes with the $\theta_1$ axis in a $\theta_1, \theta_2$ plane.

Even more specifically suppose that there are treatments A and B, and a placebo which has no effect. It is presumably of considerable interest to distinguish among several sub-hypotheses under $H_1$. Three of these might be (i) the total treatment effect is due to A, (ii) the total treatment effect is due to B, or (iii) the total treatment effect is due equally to A and B. These subhypotheses might be modeled by $H_{11}: \alpha_1 = \sqrt{2}/3 \Theta \sigma$, $\alpha_2 = \sqrt{1}/6 \Theta \sigma$, $\alpha_3 = -\sqrt{1}/6 \Theta \sigma$; $H_{12}: \alpha_1 = -\sqrt{1}/6 \Theta \sigma$, $\alpha_2 = \sqrt{2}/3 \Theta \sigma$, $\alpha_3 = -\sqrt{1}/6 \Theta \sigma$; and $H_{13}: \alpha_1 = \sqrt{1}/6 \Theta \sigma$, $\alpha_2 = \sqrt{1}/6 \Theta \sigma$, $\alpha_3 = -\sqrt{2}/3 \Theta \sigma$; which are easily seen to be equivalent to $H_{11}: \omega = \pi/6$, $H_{12}: \omega = 5\pi/6$, and $H_{13}: \omega = \pi/2$.

A joint confidence region for $(\Theta, \omega)$ might suggest the superiority of treatment A, the superiority of Treatment B, or the advisability of a new experiment to decide between the two, etc.
The paper is arranged as follows: Sections 2 and 3 are devoted to the case when $\sigma$ is known. Approximations to the error probabilities and expected sample size are given in Section 2 and the confidence region is described in Section 3. Sections 4 and 5 give parallel results in the case of unknown $\sigma$. The required probability calculations are given in the appendices.

2. Testing When $\sigma$ Is Known

Because of its comparative simplicity, it is useful to consider first the special case that $\sigma$ is known. Without loss of generality assume that $\sigma = 1$.

The hypothesis testing problem described in the introduction is invariant under location changes and rotations. It is equivalent to testing $H_0 : \theta = 0$ against $H_1 : \theta > 0$, where, as above, $\Theta = C\alpha = (\theta_1, \ldots, \theta_{r-1}, 0)'$ and $\theta = ||\Theta||$. Let $Y_{ij} = \sum_k c_{ik} X_{kj}$,

$s_{1n} = \sum_{j=1}^n Y_{ij}$, and $s_n = (s_{1n}, \ldots, s_{r-1,n})'$. The log generalized likelihood ratio is $||s_n||^2/2n$. Let $0 < d < b$ and let $m_0 < m$ be positive integers. Let

$T = \inf\{n : n \geq m_0, ||s_n|| > b\sqrt{n}\}$ .

Consider the sequential test which stops sampling at $\min(T, m)$ and rejects $H_0 : \theta = 0$ if either $T \leq m$ or $T > m$ and $||s_m|| > d\sqrt{m}$. By rotational invariance the distribution of $T$ and the power function of this test depend on the parameters only through $\theta$. The power function is
\[ (5) \quad \text{pr}_{\theta}(T \leq m) + \text{pr}_{\theta}(T > m, \|s_m\| > d\sqrt{m}) . \]

This test is analogous to that described in Section 1 if one puts \( a = b^2/2, c = d^2/2 \), and identifies the two log likelihood ratios, \( Z_n \) and \( \|s_n\|^2/2n \). Asymptotic approximations to the probabilities entering into (5) may be given as \( m \to \infty \) and \( b \to \infty \) in such a way that

\[ (6) \quad 0 < \theta_1 = b/\sqrt{m} . \]

Under the null hypothesis \( \theta = 0 \), the methods of Woodrooffe (1976, 1978) or Lai and Siegmund (1977) yield

\[ (7) \quad \text{pr}_{0}(T \leq m) \sim 2\left[ \Gamma\left(\frac{1}{2}(r-1)\right) \right]^{-1} a^{\frac{r-1}{2}} \int_{0}^{\infty} \frac{\nu(x)}{x} dx , \]

where

\[ \nu(x) = 2 \exp\left(-2 \sum_{1}^{\infty} \phi(- \frac{1}{2} \frac{x}{\sqrt{n}}) / x^2 \right) . \]

An integral equivalent to that appearing in (7) has been tabulated by Woodrooffe (1979). Alternatively, one may evaluate this integral approximately by using the small \( x \) approximation \( \nu(x) \approx \exp(-.583x) \) (Siegmund, 1979), tables of the exponential integral, and the value

\[ \int_{2}^{\infty} \frac{\nu(x)}{x} dx \approx .224 \ . \]

The approximation (7) seems to be very accurate when \( m_0 = 1 \), the only value considered in this paper in the case of known \( \sigma \). When \( d \) is fairly small compared to \( b \), a simple and moderately accurate
upper bound to the second term on the right hand side of (5) is just

$$\text{pr}_\theta(d \sqrt{m} < \|s_m\| \leq b \sqrt{m})$$

which may be obtained exactly from the $\chi^2$ distribution with $r - 1$ degrees of freedom.

By way of contrast the methods of Siegmund (1977, 1978) for computing (5) in the case $\theta \neq 0$ and $r = 2$ require a considerably more subtle argument when $r > 2$. The sum in (5) may be rewritten as

$$\text{pr}_\theta(\|s_m\| > d \sqrt{m}) + \text{pr}_\theta(T < m, \|s_m\| \leq d \sqrt{m})$$

The first probability in (8) may be obtained exactly from the non-central $\chi^2$ distribution. Whenever $(b - d) \sqrt{m}$ is large and $\theta$ is not too small, one may neglect the second term in (7). However, if $(b - d) \sqrt{m}$ is small, for example, if $d = b$, or if one wants to examine

$$\text{pr}_\theta(T \leq m) = \text{pr}_\theta(\|s_m\| > b \sqrt{m}) + \text{pr}_\theta(T < m, \|s_m\| \leq b \sqrt{m})$$

as one aspect of the experimental design or to obtain the confidence intervals suggested below, then it becomes necessary to approximate the second term in (8). Now

$$\text{pr}_\theta(T < m, \|s_m\| \leq d \sqrt{m}) = \int_{\|s_m\| \leq d \sqrt{m}} \text{pr}_\theta(T < m \mid \|s_m\|, \omega_m) \text{dpr}_\theta$$

where $\omega_m$ denotes the angular components of the polar coordinates of the vector $s_m$, and by sufficiency the conditional probability does not depend on the value of $\theta$. The appearance of the angle $\omega_m$ in order to invoke sufficiency in the multidimensional case makes the subsequent calculation sufficiently complicated that it has been relegated to Appendix A.
When \( r = 3, \ d = b, \) and the discrete random walk is replaced by a continuous parameter Brownian motion, the result for all \( \theta \neq \theta_1 = b/\sqrt{m} \) is

\[
\text{pr}_\theta(T < m, \|s_m\| \leq b\sqrt{m}) \sim (\theta_1/\theta)^{1/2} \phi(\sqrt{m}(\theta_1 - \theta))/\sqrt{m} \theta .
\]

This is the same as the one-dimensional \( (r = 2) \) case except for the appearance of the factor \((\theta_1/\theta)^{1/2}\).

Similar considerations apply to approximating the expected sample size,

\[
E_\theta \min(T, m) = E_\theta T - \int_{(T > m)} E_\theta (T - m \|s_m\|, \omega_m) \text{dpr}_\theta .
\]

For Brownian motion, as \( b \to \infty \)

\[
E_\theta T = (b^2 - r + 1)/\theta^2 + o(1) ,
\]

e.g., Siegmund (1977).

For \( \theta \) in a neighborhood of \( \theta_1 \), say \( \theta = \theta_1 + \xi/\sqrt{m} \), it is possible to approximate the integral on the right hand side of (11). Details are contained in Appendix B. For Brownian motion with \( r = 3 \) the result is that

\[
\int_{(T > m)} E_\theta (T - m \|s_m\|, \omega_m) \text{dpr}_\theta = 2\sqrt{m}(\theta - \theta_1/2)^{-1} r^{\pi/2} \phi(-\xi/\cos \omega) \cos \omega \text{d}\omega/\sqrt{2\pi}
\]

\[
-2\theta_1^{-2} r^{\pi/2} [\{\xi^2(1/\cos \omega - 1)^2 + 2\xi \phi(\xi/\cos \omega) - 2\xi \phi(-\xi/\cos \omega) \} \text{d}\omega/\sqrt{2\pi}
\]

\[+ o(1)\]
as $b \to \infty$. Here $\phi$ and $\Phi$ are the standard normal density and distribution function respectively, and the integrals may be easily evaluated numerically.

In the case $r = 3$, the author has conducted fairly extensive simulations, which indicate that the approximations suggested above are reasonably accurate. Table 1 gives a very abbreviated report of these simulations. For the theoretical calculations, the approximations (7), (10), (12), and (13) were used. The error probability simulations were based on 2500 repetitions, and importance sampling was used for variance reduction (Siegmund, 1975). The ± figures are one estimated standard error. The expected sample size simulations involved 900 repetitions.

The expected sample size approximation given by (12) and (13) under the assumption that $\theta = \theta_1 + \xi/\sqrt{m}$ breaks down for small $\theta$, although in this case $E_{\theta} \min(T,m)$ is about equal to $m$ and its precise value does not seem particularly important. However, if one writes

$$E_{\theta} \min(T,m) = m - \int_{(T<m)} (m-T)d\Pr_{\theta},$$

one may find an approximation to the integral in (14) for fixed $\theta < \theta_1 = b/\sqrt{m}$ as $b \to \infty$ and hence give a reasonable approximation to the expected sample size in the region where the previous approximation is not accurate. To avoid overburdening the paper, the details of this analysis have been omitted.
TABLE 1
ERROR PROBABILITIES AND EXPECTED SAMPLE SIZE
FOR KNOWN \( \sigma \). \( b = c = 3.4\), \( m_0 = 1\), \( m = 100\)

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \Pr_{\theta}(T \leq m) )</th>
<th>( E_{\theta} \min(T,m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theoretical</td>
<td>Monte Carlo</td>
</tr>
<tr>
<td>0</td>
<td>.045</td>
<td>.043 ± .001</td>
</tr>
<tr>
<td>.2</td>
<td>.211</td>
<td>.225 ± .004</td>
</tr>
<tr>
<td>.3</td>
<td>.536</td>
<td>.552 ± .014</td>
</tr>
<tr>
<td>.4</td>
<td>.846</td>
<td>.841 ± .008</td>
</tr>
<tr>
<td>.5</td>
<td>.976</td>
<td>.975 ± .002</td>
</tr>
<tr>
<td>.6</td>
<td>.983</td>
<td>.982 ± .0001</td>
</tr>
<tr>
<td>.7</td>
<td>( 1 \cdot 10^{-4} )</td>
<td>( 1 \cdot 5 \cdot 10^{-4} \pm 3 \cdot 10^{-5} )</td>
</tr>
</tbody>
</table>
3. Confidence Regions When $\sigma$ Is Known

For the one-dimensional case ($r = 2$), Siegmund (1978) has described a method for obtaining confidence intervals related to repeated significance tests. For general $r \geq 3$ those methods lead immediately to confidence intervals for $\theta = ||\theta||$ based on $T' = \min(T, m)$ and $||s_{T'}||$. Suppose that $[\theta_*(T', ||s_{T'}||), \theta^*(T', ||s_{T'}||)]$ is this interval with confidence coefficient $\gamma_1$, i.e.,

\begin{equation}
    \Pr_\theta\{\theta_*(T', ||s_{T'}||) \leq \theta \leq \theta^*(T', ||s_{T'}||)\} \geq 1 - \gamma_1.
\end{equation}

Assume also that $r = 3$, although the results seem capable of extension to higher dimensions. Let $\omega$ denote the angle that the vector $(\theta_1, \theta_2)$ makes with the positive $\theta_1$ axis in the $\theta_1\theta_2$ plane, and let $\omega_n$ be the corresponding angle for $s_n$.

It is well-known and easy to prove--see Proposition 1 in Section 5 for a more novel result along similar lines--that

\begin{equation}
    \Pr_{\theta, \omega}(\omega_{T'} \in d\omega \mid T', ||s_{T'}||) = \exp\{\theta ||s_{T'}|| \cos(\omega - \omega)\} d\omega / 2\pi I_0(\theta ||s_{T'}||),
\end{equation}

where $I_0$ is the usual modified Bessel function. Hence, for fixed $\gamma_2$, one may determine $u(x)$ such that

\begin{equation}
    \Pr_{\theta, \omega}(|\omega_{T'} - \omega| \leq u(\theta ||s_{T'}||) \mid T', ||s_{T'}||) = 1 - \gamma_2
\end{equation}

and then

10
\begin{equation}
\begin{aligned}
pr_{\theta, \omega}(\theta^*(T', \|s_T',\|) \leq \theta \leq \theta^*(T', \|s_T',\|), |\omega_{T'} - \omega| \leq u(\|s_T',\|)) \\
= \int_{\theta^*(T', \|s_T',\|) \leq \theta \leq \theta^*(T', \|s_T',\|)} pr_{\theta, \omega}(|\omega_{T'} - \omega| \leq u(\|s_T',\|)) |T', \|s_T',\|) dpr_{\theta, \omega} \\
\geq (1 - \gamma_1)(1 - \gamma_2),
\end{aligned}
\end{equation}

so \( \theta^*(T', \|s_T',\|) \leq \theta \leq \theta^*(T', \|s_T',\|), \omega_{T'} - u(\|s_T',\|) \leq \omega \leq \omega_{T'}, \)

\( + u(\|s_T',\|) \) defines a \((1 - \gamma_1)(1 - \gamma_2)\) 100\% confidence region for \((\theta, \omega)\).

A simple approximation to the function \( u \) may be obtained by recalling that for large \( \nu \), the von Mises distribution on the circle with density \( \exp(\nu \cos(\omega))/2\pi I_0(\nu) \) is approximately normally distributed with mean 0 and variance \( \nu^{-1} \). For numerical purposes it seems better to use a normal distribution with variance \( (\nu - \frac{1}{2})^{-1} \) instead of \( \nu^{-1} \) (Mardia, 1972). Hence, \( u(x) \approx z_{\gamma_2} / \sqrt{x - 1/2} \), where \( z_{\gamma_2} \)

is defined by \( 1 - \Phi(z_{\gamma_2}) = \gamma_2/2 \); and the above confidence region becomes approximately

\begin{equation}
\theta^*(T', \|s_T',\|) \leq \theta \leq \theta^*(T', \|s_T',\|), |\omega_{T'} - \omega| \leq z_{\gamma_2} / \sqrt{\|s_T'\| - 1/2}.
\end{equation}

For fixed \( T', s_T' \), this region in the \( \omega \theta \) plane is the shaded portion of Figure 1.
Suppose that $\theta$ is sufficiently large that $\Pr_{\theta}(T > m)$ is negligible, so in effect $T' = T$. A simple argument based on the strong law of large numbers shows that as $b \to \infty$

$$||s_T|| \sim b\sqrt{T} \sim b\sqrt{(b/\theta)^2} = b^2/\theta,$$

so the region (18) in the $\omega$ coordinate is unconditionally, approximately

$$|\omega_T - \omega| \leq z_{\gamma_2} / \sqrt{b^2 - 1/2}.$$

This observation may be useful at the design stage if one would like to know whether his confidence region will be helpful in distinguishing among subhypotheses of $H_1$ as described in the Introduction.
For example, suppose that $b = 3.4$ as in Table 1 and $\gamma = .05$, so the width of the interval about $\omega_T$ given by (19) is 

$$2(1.96) / \sqrt{(3.4)^2 - .5} = 1.18.$$ 

This interval may, albeit with small probability, simultaneously cover the values of $\omega$ specified by $H_{11}: \omega = \pi/6$ and $H_{13}: \omega = \pi/2$ or $H_{12}: \omega = 5\pi/6$ and $H_{13}$. If this is regarded as a serious defect, one may wish to increase $b$ in order to decrease the probability of such an ambiguity.

One can give a crude approximation to the actual coverage probability of (19) by writing

$$p_{\theta, \omega}(|\omega_T - \omega| > z / \sqrt{b^2 - 1/2})$$

$$= E_{\theta, \omega}[p_{\theta, \omega}(\sqrt{\theta} \|s_T\| - 1/2 | \omega_T - \omega| > z / (\theta \|s_T\|-1/2)/(b^2 - 1/2) \|s_T\|)]$$

$$\approx 2E_{\theta}[1 - \Phi(z / (\theta \|s_T\|-1/2)/(b^2 - 1/2))]$$

by the normal approximation to the von Mises distribution. If one neglects excess over the stopping boundary, several Taylor series expansions show that this last expression is

$$2(1 - \Phi(z) + b^{-1} z \phi(z)(1 + z^2/4) + o(b^{-2})) .$$

For $b = 3.4$ as in Table 1 and $z = 1.96$, this approximation adjusts the nominal .05 by a correction term of .038, for a total non-coverage probability of .088. A more detailed analysis taking excess over the boundary into account reduces this to about 0.08 for $\theta$ about equal to .6. Simulations show that these figures are approximately correct.

An example is given in Section 5 for the unknown variance case.
4. Testing When \( \sigma \) Is Unknown

This section contains results similar to those of Section 2, but it is no longer assumed that \( \sigma \) is known. The test is defined by the stopping rule (2), and its power function is the expression (3). The general approach is patterned after that of Siegmund (1977, 1978) in the one-dimensional case. For calculations under \( H_0 \), the theory invoked in Section 2 continues to hold, although there are some additional technical difficulties. Under \( H_1 \) a Brownian motion approximation reduces the required computations to a similar result with \( \sigma \) known.

Suppose initially that \( \theta \neq 0 \). If as \( m \to \infty \), \( \theta \) is proportional to \( m^{-\frac{1}{2}} \), then \( Z_n \) given by (1) behaves asymptotically like the log likelihood ratio \( \|s_n\|^2/2n\sigma^2 \) of Section 2, with known \( \sigma \). Hence to approximate the probabilities in (3) and the expected sample size \( E_\theta \min(T,m) \), it seems reasonable to rewrite (2) as

\[
T = \inf \{ n : n \geq m_0, \sqrt{2nZ_n^2} > b\sqrt{n} \} ,
\]

where \( b = \sqrt{2a} \), and hope that \( \sqrt{2nZ_n^2} \) behaves enough like the norm of an \( r-1 \)-dimensional Brownian motion process to provide useful numerical results.

If \( \tilde{\mu} \) denotes the mean vector and \( \tilde{\sigma}^2 \) the variance of the approximating Brownian motion, the asymptotic theory of the preceding paragraph requires only that \( \|\tilde{\mu}\| \sim \theta \) and \( \tilde{\sigma} \sim 1 \) as \( \theta \to 0 \). The following argument suggests values for \( \tilde{\mu} \) and \( \tilde{\sigma} \) which are consistent with these requirements, and which appear to give more accurate
results for $\theta$ not close to 0. For fixed $\theta \neq 0$, a linear Taylor series approximation yields

\[
\sqrt{2n} \frac{Z_n}{n} \approx n \sqrt{r \log(1+\theta^2/r)} + \{(1+\theta^2/r) \sqrt{r \log(1+\theta^2/r)}\}^{-1} \\
\cdot \left[ \sigma^{-2} \sum_{i=1}^{r} (\alpha_i e_i - n\mu_i) - (2r)^{-1} \theta^2 \sum_{i=1}^{n} \sum_{j=1}^{r} \{(X_{ij} - \mu_i)^2/\sigma^2 - 1\} \right] .
\]

(21)

The random walk on the right hand side of (21) has increments with expectation $\sqrt{r \log(1+\theta^2/r)}$ and variance $(1+\theta^2/r)^{-2} \{r \log(1+\theta^2/r)\}^{-1} \theta^2 (1+\theta^2/2r)$. Matching moments to the analogous expansion of the norm of an $r-1$ dimensional Brownian motion suggests that these are appropriate values for $\|\tilde{\mu}\|$ and $\tilde{\sigma}^2$. Table 2 below indicates that this approximation is reasonably accurate.

Consider now the case $\theta = 0$, and write

\[
pr_0(T \leq m) = pr_0(Z_{m_0} > a) + pr_0(m_0 < T \leq m) .
\]

(22)

The first term on the right hand side of (22) may be obtained directly from tables of the $F$ distribution. The second may be approximated asymptotically by the methods of Siegmund (1977) or Woodroofe (1978) with the following result: if $a \to \infty$, $m_0 \to \infty$, and $m \to \infty$ in such a way that for fixed values $0 < \theta_1 < \theta_0 < \infty$, $\frac{1}{2} m_0 \log(1+\theta_0^2/r) = a = \frac{1}{2} rm \log(1+\theta_1^2/r)$, then for $r \geq 2$,
\[ p_{0}(m_0 < T \leq m) \sim 2r \left\{ \frac{r-1}{2} \right\}_2 \left[ \Gamma \left( \frac{r}{2} - 1 \right) \right]^{-1} a^\frac{r-1}{2} e^{-a} \]

\[
\cdot \int_{\theta}^{\theta_0} v_r(x) \sqrt{1 + x^2/r} x^{r-2} \left\{ \log(1 + x^2/r) \right\}^{\frac{r-1}{2}} dx .
\]

The function \( v_r \) is analogous to the function \( v \) in (7), but is much more difficult to calculate. Woodroffe (1979) has given a general method for computing \( v_r \), which involves the numerical integration of a function of the characteristic function of the linear Taylor series approximation to \( Z_n \) (cf. (21)). One can simplify this numerical burden considerably by using the small \( x \) approximation

\[ v_r(x) \approx \exp\left\{-0.583x/(1 + x^2/r)\right\} \]

(Siegmund, 1979). For \( r = 3 \), the cost is about a 10-15% increase in the right hand side of (23).

It should be noted that under the asymptotic assumptions yielding (23), the first term on the right hand side of (22) is negligibly small, and hence there is no good mathematical reason for including it. However, the author's extensive numerical experimentation with (23) when \( r = 3 \), the corresponding results for the one sample \( t \) statistic, and the case when \( \sigma \) is known but \( m_0 > 1 \) indicate that including this term yields much more accurate numerical results.

Table 2 compares the approximations suggested here with the results of a Monte Carlo experiment. In this example \( c \) is considerably smaller than \( a \). As suggested in Section 2, for \( \theta = 0 \) the second term in (3) is approximated by the upper bound \( p_{0}(c < Z_m < a) \).
The Monte Carlo estimates of $\Pr_\theta(T \leq m)$ and of the significance level were based on 2500 repetitions, and importance sampling was used for variance reduction (Siegmund, 1975). The estimates of the power for $\theta > 0$ and of the expected sample size were averages of 900 repetitions.

**TABLE 2**

SEQUENTIAL F TEST  
$r = 3, a = 6.48, c = 3.38 (b = 3.6, d = 2.6), m_0 = 7, m = 49$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\Pr_\theta(T \leq m)$</th>
<th>Power</th>
<th>$E_\theta \min(T, m)$</th>
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<td>Th.</td>
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<tr>
<td>0</td>
<td>.021</td>
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<td>.056</td>
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<tr>
<td>.3</td>
<td>.16</td>
<td>.18 ± .00</td>
<td>.39</td>
</tr>
<tr>
<td>.4</td>
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<td>.64</td>
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<tr>
<td>.5</td>
<td>.60</td>
<td>.60 ± .01</td>
<td>.85</td>
</tr>
<tr>
<td>.6</td>
<td>.81</td>
<td>.82 ± .01</td>
<td>.96</td>
</tr>
<tr>
<td>.7</td>
<td>.94</td>
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<td>.99</td>
</tr>
<tr>
<td>.8</td>
<td>.99</td>
<td>.99 ± .00</td>
<td>1.00</td>
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5. Confidence Regions When $\sigma$ Is Unknown

The methods and results of this section are similar to Section 3, but the distribution theory is more complicated. Suppose that $r = 3$. As before, let $\omega$ denote the angle made by the vector $(\theta_1, \theta_2)$ with the positive $\theta_1$ axis in a $\theta_1 \theta_2$ plane, and let $\omega_n$ be the corresponding angle for $s_n$. Let $W_n = n \sum_{i,j} (X_{ij} - \bar{X}_i)^2 / \sum_{i,j} (X_{ij} - \bar{X}_i)^2$. The role of the von Mises distribution in the case of known $\sigma$ is assumed by the following result—cf. (16).

**Proposition 1.** Let $\tau$ be any stopping time defined in terms of $W_2, W_3, \ldots$. Then

$$
\Pr_{\Theta, \omega} (\omega_{\tau} \in d\omega | \tau, W_{\tau}) = \frac{\int_0^{\infty} \exp\{\theta x \sqrt{\tau W_{\tau}/(1+W_{\tau})} \cos(w-\omega)\} \frac{x^{3\tau-2} \exp(-x^2/2)dx}{2\pi \int_0^{\infty} \int_0^{\infty} \{\theta x \sqrt{\tau W_{\tau}/(1+W_{\tau})\}} x^{3\tau-2} \exp(-x^2/2)dx} \, dw.
$$

A proof of Proposition 1 is sketched in Appendix C.

Straightforward application of Laplace's asymptotic formula to these integrals shows that for large $\tau$, the conditional distribution of $\omega_{\tau}$ given $\tau$ and $W_{\tau}$ is approximately normal with expectation $\omega$ and variance $1/\tau \theta \sqrt{3W_{\tau}/(1+W_{\tau})}$.

Let $T' = \min(T, m)$. It follows that for given $\theta$, an approximate $(1-\gamma_2)100\%$ confidence interval for $\omega$ is given by

$$
(25) \quad \omega_{T', z_{\gamma_2}/\sqrt{\tau T'} \{3W_{T'}/(1+W_{T'})\}}^{1/6} \leq \omega \leq \omega_{T', z_{\gamma_2}/\sqrt{\tau T'} \{3W_{T'}/(1+W_{T'})\}}^{1/6},
$$

where $1 - \Phi(z_{\gamma_2}) = \frac{1}{2} \gamma_2$. As in Section 2, this may be combined with
a confidence interval for $\theta$ to give an approximate confidence region for $\theta$ and $\omega$.

The reasoning leading to (19) in Section 3 shows that for large $\theta$ the conditional interval given by (25) may be further approximated by the unconditional interval

$$|\omega_T - \omega| \leq \frac{z_{\chi_2}}{\sqrt{2a \theta^2/3 \log(1+\theta^2/3) \sqrt{1+\theta^2/3}}}.$$

Since this result is useful primarily for design purposes, and since $\{\theta^2/3 \log(1+\theta^2/3) \sqrt{1+\theta^2/3}\}^{1/2} \approx 1$ for a wide range of $\theta$ values, this interval may be replaced by the approximation

(26) $$|\omega_T - \omega| \leq \frac{z_{\chi_2}}{\sqrt{2a}}.$$

Table 3 reports the outcome of a Monte Carlo experiment to check the accuracy of these approximations. The value of $z_{\chi_2}$ is 1.96 for a nominal non-coverage probability of .05. The interval $I_w$ denotes (25); whereas $I_a$ denotes (26) with $2a$ replaced by $2a - \frac{1}{2}$ to make (26) consistent with (19) whenever $T \leq m$, and $I_a$ is (25) if $T > m$. The stopping rule is the same as that studied in Table 2. The number of repetitions of the Monte Carlo experiment is 900. From the results it appears that (25) has approximately the nominal non-coverage probability. That for (26) is larger by roughly the amount predicted by the analysis of Section 3.
\begin{table}
\centering
\caption{Confidence Interval Simulations}
\begin{tabular}{lll}
\hline
$\theta$ & Estimated $pr_{\theta,\omega}(\omega \notin I_w)$ & Estimated $pr_{\theta,\omega}(\omega \notin I_a)$ \\
\hline
.3 & .053 & .084 \\
.4 & .060 & .088 \\
.5 & .059 & .089 \\
.6 & .047 & .075 \\
.7 & .054 & .076 \\
.8 & .045 & .061 \\
\hline
\end{tabular}
\end{table}

APPENDIX A

Heuristic Derivation of (9)

Let \( \{X(t): 0 \leq t < \infty\} \) be two-dimensional Brownian motion with
\( E_\theta X(t) = (\theta t, 0)' \) and covariance matrix \( t \cdot \text{Identity} \). Let \( 0 < b < \infty \) and for some \( m_0 > 0 \)
\[ T = \inf \{ t: t \geq m_0, \|X(t)\| > b\sqrt{t} \} \] .

Let \( m \to \infty \) and \( b \to \infty \) in such a way that \( bm^{-\frac{1}{2}} = \theta_1 \).

Theorem A. For each \( 0 < \theta < \infty, \theta \neq \theta_1 \)
\[ \text{pr}_\theta (T < m, \|X(m)\| < b\sqrt{m}) \sim (\theta_1/\theta)^{\frac{1}{2}} \frac{\phi(\sqrt{m}(\theta_1-\theta))}{\sqrt{m} \theta} . \]

The following heuristic argument stresses those aspects of the problem that arise for the first time in the two-dimensional case. For the one-dimensional case and some of the technique for converting the heuristics below into a rigorous proof, see Siegumund (1978).

Suppose initially that \( \theta > \theta_1 \). Let \( \rho_t = \|X(t) - (\theta t, 0)'\| \) and let \( \gamma_t \) denote the angle formed by the vector \( X(t) - (\theta t, 0)' \) and the negative \( x \) direction. Let \( \omega_t \) denote the angle formed by \( X(t) \) and the positive \( x \) direction. Then--see Figure 2--
\[ \text{pr}_\theta (T < m, \|X(m)\| < b\sqrt{m}) \]

\begin{equation}
(A1) \quad \int \int \text{pr}_\theta (T < m, \|X(m)\| = z_m (\rho, \gamma), \omega_m = \omega_m (\rho, \gamma)) \text{pr}_\theta (\rho_m \in d\rho, \gamma_m \in d\gamma) . \tag{A1} \end{equation}

\((\theta-\theta_1)m < \rho < \infty, \quad \gamma_m (\rho) < \gamma < \gamma_m (\rho)\)
By sufficiency the conditional probability does not depend on \( \theta \).

As in the one-dimensional case, only those values of \( \rho \) which are \((\theta - \theta_1)m + O(1)\) contribute asymptotically. It is convenient to set \( \eta = \rho - (\theta - \theta_1)m \) and think of \( z_m \) and \( w_m \) as functions of \((\eta, \gamma)\).

Consider the conditional probability in (A1) by looking backward over a sample path from its endpoint at \( \|X(m)\| = z, \omega_m = w \). This conditional probability would be unchanged if the coordinate system of the conditional process were rotated to have one axis in the \( w \) direction and the other perpendicular to this direction. Since for bounded \( \eta \), \( \delta = \theta_1 m - z_m(\eta, \gamma) \) is also bounded as \( \theta_1 m \to \infty \), oscillations of the conditional Brownian motion in the direction
perpendicular to the \( w \) direction do not contribute asymptotically to the conditional probability. Also, since \( \|X(m)\| \sim m\theta_1 \), the component of the conditional process in the \( w \) direction will behave like a one-dimensional Brownian motion with drift \(-\theta_1\). Hence, as \( m \to \infty \), for fixed \( \eta = \rho - (\theta - \theta_1)m \) and all \( |\gamma| < \gamma^*_m(\eta) \), the conditional probability in (A1) is asymptotically equal to the probability that a one-dimensional Brownian motion with drift \(-\frac{1}{2}\theta_1\) even attains the level \( \delta_m = \theta_1 m - z_m(\eta, \gamma) \), to wit

\[
(A2) \quad \exp(-\theta_1 \delta_m).
\]

Observe that by Pythagoras

\[
(A3) \quad z^2 = (\theta_1 m - \delta)^2 = \rho^2 \sin^2 \gamma + (\theta m - \rho \cos \gamma)^2.
\]

Setting \( \delta = 0 \) and \( \gamma = \gamma^* \) yields

\[
(A4) \quad \cos \gamma^* = \left\{ \theta m - (\theta_1^2 m/\theta - \rho^2/\theta m) \right\}/2\rho.
\]

Putting \( \rho = (\theta - \theta_1)m + \eta \) with \( \eta \) fixed shows, after some calculation, that as \( m \to \infty \)

\[
\gamma^*_m(\eta) \sim \left\{ 2\theta_1 \eta/(\theta - \theta_1)\theta m \right\}^{1/2}.
\]

Solving (A3) for \( \delta \) and expanding the result in a Taylor series gives for fixed \( \eta \) and all \( |\gamma| < \gamma^*_m(\eta) \)

\[
\delta = \eta - \theta(\theta - \theta_1)\eta y^2/2\theta_1 + O(\frac{1}{m}).
\]

Substituting this into (A2) and putting the result into (A1) yields
\[ \Pr \{ T < m, \|X(m)\| < b\sqrt{m} \} \]

\[ \sim \int_{(\theta - \theta_1)m}^{\infty} \int_{-\gamma^*}^{\gamma^*} \exp\{-\theta_1 \delta_m (\rho, \gamma)\} \frac{dy}{2\pi} \rho \exp(-\rho^2/2m) d\rho/m \]

\[ \sim 2(\theta - \theta_1) \phi\{(\theta - \theta_1)^{1/2}\} \int_{0}^{\infty} \int_{0}^{\infty} \frac{\{2\theta_1 \eta/(\theta - \theta_1)\theta m\}^{1/2}}{\exp[m \theta (\theta - \theta_1) \gamma^2/2]} \frac{dy}{\sqrt{2\pi}} \exp(-\theta_1 \eta) d\eta, \]

which after the change of variables \( u = [(\theta - \theta_1)m]^{1/2} \gamma \) and a change in the order of integration becomes the desired result.

The case \( 0 < \theta < \theta_1 \) involves the same probabilistic ideas with a more elaborate geometry. One must consider the integrals over \( 0 < \rho < (\theta_1 - \theta)m \) and \( (\theta_1 - \theta)m < \rho < \infty \) separately. For the first of these the range of integration for \( \gamma \) is \((-\pi/2, \pi/2)\) and the resulting asymptotic expression is \( \{(\theta_1 - \theta)/\theta\}^{1/2} \phi\{\sqrt{m}(\theta_1 - \theta)\}/\sqrt{m} \theta \).

For the second, the relevant range of values for \( \gamma \) is \( |\gamma| > [2\theta_1 \eta/m \theta (\theta_1 - \theta)]^{1/2} \), where \( \eta = \rho - (\theta_1 - \theta)m \), and the result is \( [(\theta_1/\theta)^{1/2} - ((\theta_1 - \theta)/\theta)^{1/2}] \phi\{\sqrt{m}(\theta_1 - \theta)\}/\sqrt{m} \theta \). These results add up to the same algebraic expression as in the case \( \theta > \theta_1 \), although there is no obvious reason why this should be so. The details of this argument have been omitted.

It is worth noting that the simplicity of the formulae derived here depends essentially on the fact that asymptotically \( \gamma^*_m(\eta) \to 0 \). For the expected sample size calculation in Appendix B, this is not the case; and the resulting formulae are considerably more complicated.
APPENDIX B

Heuristic Derivation of (13)

Let \( \{X(t), 0 \leq t < \infty\} \) and \( T \) be as in Appendix A. An asymptotic analysis of the expected sample size may be based on (10) and (11).

The most meaningful result and the most accurate for numerical purposes is obtained by letting \( \theta \) vary with \( m \) so that as \( m \to \infty \) and \( b \to \infty \), there exists a constant \( \xi \) such that

\[
(Bl) \quad \theta = \theta_1 + \xi / \sqrt{m}.
\]

As usual, \( \theta_1 = b / \sqrt{m} \) is a fixed, positive number.

The following computation is based on a picture similar to that in Appendix A, but there are important differences. The most important of these is that with the scaling (B.1), the angle of integration is not vanishingly small.

Theorem B. Under the assumption (B.1), as \( m \to \infty \)

\[
(B2) \quad \int_{(T > m)} E_{\theta}[T - m | X(m)] \, d\theta = K_1 \sqrt{m} + K_2 + o(1),
\]

where \( K_1 \) and \( K_2 \) depend on \( \theta \) and \( \theta_1 \), and are given explicitly on the right hand side of (13).

The analogous one-dimensional result is proved in Siegmund and Gregory (1979). As in Appendix A, the argument given here is heuristic and is limited to the simpler case \( \xi > 0 \). The notation of Appendix A is used again here, except that now \( \eta \) is defined by \( \rho = \sqrt{m}(\xi + \eta) \). The analytic starting point is the expression
and the appropriate geometric relations are illustrated in Figure 3.

For fixed $\eta$ and large $m$, oscillations of the Brownian motion in the $y$ direction make a negligible contribution to the conditional expectation in (B3), which hence can be computed by the one-dimensional result of Siegmund and Gregory (1979) to be

\begin{equation}
\tag{B4}
\frac{u}{(\theta - \frac{1}{2} \theta_1^m)} - \frac{u^2}{m \theta_1^m} + o \left( \frac{u^2}{m} \right)
\end{equation}
By Pythagoras \( v = \theta_1 m - (\theta_1^2 m^2 - \rho^2 \sin^2 \gamma)^{1/2} = (\xi + \eta)^2 \sin^2 \gamma / 2 \theta_1 + O(m^{-1}) \)
and hence

\[ u = \rho \cos \gamma - (\theta - \theta_1) m - v = \sqrt{m} ((\xi + \eta) \cos \gamma - \xi) - (\xi + \eta)^2 \sin^2 \gamma / 2 \theta_1 + O(m^{-1}). \]

After substituting this expression into (B4), which in turn is substituted into (B3), the problem has been reduced to one of calculating a double integral. By using (A4), the range of integration

\( 0 < \eta < \infty, \quad 0 < \gamma < \gamma^*(\eta) \)

can be re-expressed as

\( 0 < \gamma < \pi / 2, \quad \eta_*(\gamma) < \eta < \infty, \)

where \( \eta_* \) satisfies

\[
\eta_* = \xi ((\cos \gamma)^{-1} - 1) + \eta_* \xi / \sqrt{m} \cos \gamma + \eta_*^2 / 2 \theta \sqrt{m} \cos \gamma ,
\]

so

\[
\eta_* = \xi ((\cos \gamma)^{-1} - 1) + \xi^2 ((\cos \gamma)^{-1} - 1) / \sqrt{m} \cos \omega
\]

\[
+ \xi^2 ((\cos \gamma)^{-1} - 1)^2 / 2 \theta \sqrt{m} \cos \gamma + o(1/\sqrt{m}) .
\]

The integral over \( \eta \) can be calculated explicitly to give the final result.
APPENDIX C

Proof of Proposition 1

Without loss of generality it may be assumed that \( \omega = 0 \). Let \( \mu_\theta^{(n)} \) denote the joint distribution of \((W_2, \omega_2), \ldots, (W_n, \omega_n)\). A tedious calculus exercise and considerations of sufficiency show that

\[
(C1) \quad \frac{d\mu_\theta^{(n)}}{d\mu_0^{(n)}}
= c_{3n-1} \exp(-n\theta^2/2) \int_0^\infty \exp\left\{\frac{\theta \sqrt{n}W_n}{(1+W_n)} \cos \omega \right\} x^{3n-2} e^{-x^2/2} dx,
\]

where \( C_\theta = (\int_0^\infty t^{\nu-1} e^{-t^2/2} dt)^{-1} \). To prove the proposition, it suffices to show for all \( n = 2, 3, \ldots \) and bounded, measurable functions \( g, h \) that

\[
\int (\tau=n) h(W_n) g(\omega_n) d\rho_\theta
= \int_{(\tau=n)} h(W_n) \frac{\int_0^\infty \int_0^{\pi/2} g(w) \exp\left\{\frac{\theta \sqrt{n}W_n}{(1+W_n)} \cos \omega \right\} dw \cdot x^{3n-2} e^{-x^2/2} dx}{2\pi \int_0^\infty \int_0^{\pi/2} \left\{\frac{\theta \sqrt{n}W_n}{(1+W_n)} \right\} x^{3n-2} e^{-x^2/2} dx} d\rho_\theta.
\]

when \( \theta = 0, \omega_n \) is distributed uniformly and independently of \( W_2, \ldots, W_n \). Hence by \( (C1) \)

\[
(C2)
= \int (\tau=n) h(W_n) \int_{-\pi}^{\pi} g(\omega) c_{3n-1} e^{-n\theta^2/2} \int_0^\infty \exp\left\{\frac{\theta \sqrt{n}W_n}{(1+W_n)} \cos \omega \right\} d\omega \cdot (2\pi)^{-1} dw x^{3n-2} e^{-x^2/2} dx d\lambda_0^{(n)},
\]

28
where \( \lambda_{(n)}^{(n)} \) denotes the joint distribution of \( W_2, \ldots, W_n \). Putting \( g = 1 \) in (C2) shows that

\[
\frac{d\lambda_{(n)}}{d\lambda_0} = C_{3n-1} e^{-n\theta^2/2} \int_0^\infty I_0 \left( \theta x \sqrt{nW_1/(1+W_1)} \right) x^{3n-2} e^{-x^2/2} dx,
\]

where \( I_0(\beta) = (2\pi)^{-1} \int_{-\pi}^\pi \exp(\beta \cos w) dw \) is the usual modified Bessel function. Substituting this last relation back into (C2) completes the proof.
REFERENCES


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ABSTRACT

Sequential $\chi^2$ and $F$ tests are proposed for comparing more than two treatments. Extending earlier results of the author, one obtains approximations to the significance level and power of the tests, and approximate confidence intervals for the non-centrality parameter $\theta$. For each fixed $\theta$ a confidence region is given for the angle $\omega$ which the treatment effect vector in canonical coordinates makes with a fixed direction, and hence an over-all confidence region is obtained for the pair $(\theta, \omega)$. 