A COMPLETE COUPLING PROOF OF BLACKWELL'S RENEWAL THEOREM

by

Hermann Thorisson
Chalmers University of Technology and Stanford University

TECHNICAL REPORT NO. 40
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A COMPLETE COUPLING PROOF OF BLACKWELL'S RENEWAL THEOREM

Hermann Thorisson

Department of Mathematics
Chalmers University of Technology
S-412 96 Göteborg
Sweden

Department of Statistics
Stanford University
Stanford, CA 94305
USA

Abstract

Blackwell's renewal theorem for non-lattice renewal processes with mean recurrence time \( m \) states that the expected number of renewals in a time-interval of length \( h \) tends to \( h/m \) as the interval goes to infinity:

\[
\mathbb{E}[N(t, t+h)] \to h/m \quad t \to \infty.
\]

This note presents a self-contained coupling proof of this result mending the drawbacks of earlier such proofs. Firstly, the proof is complete in the sense that it covers not only the case \( m < \infty \) but also \( m = \infty \). Secondly, the proof is fairly elementary in the sense that it does not rely on advanced results such as Hewitt-Savage 0-1-Law or the \( \varepsilon \)-recurrence of 0-mean random walks.
1. Introduction

The aim of this note is to present a fairly elementary and self-contained proof of the celebrated Blackwell’s Renewal Theorem.

1.1. The Theorem. Informally, the theorem may be explained as follows: A room is lighted by one light bulb. When it burns out a new one is installed immediately. Then, as time passes, the expected number of light bulb-installations in a time-interval of length $h$ will tend to $h/m$ where $m$ is the expected life length of a light bulb.

Mathematically, this becomes: Let $S = (S_n)^\infty_0$ be a renewal process, i.e.,

$$S_n = X_0 + \ldots + X_n$$

where $X_0, X_1, \ldots$ are independent non-negative random variables, $X_0$ with distribution $G$ and $X_n, n \geq 1$, with a common distribution $F$. Call $X_0$ (the time the first light bulb is installed) the delay, $X_1, X_2, \ldots$ (the life lengths of the successive light bulbs) the recurrence times and say that a renewal takes place at time $S_n$ (the time of the $(n+1)$th light bulb installation). If $X_0 \equiv 0$ then $S$ is zero-delayed. Put

$$m = \mathbb{E}[X_1] = \int_0^\infty (1 - F(x)) \, dx$$

and, for $t, h \geq 0$,

$$N(t, t + h] = \# \text{ renewals in the time-interval } (t, t + h];$$

thus $N(t, t + h] = N_{t+h} - N_t$ where

$$N_t = \inf\{n \geq 0; S_n > t\} = \# \text{ renewals up to and including time } t.$$}

Throughout the paper we shall assume that $F$ is non-lattice, i.e., for all $c > 0$ it holds that $\mathbb{P}(X_1 \in L_c) < 1$ where $L_c = \{0, \pm c, \pm 2c, \ldots\}$.

Blackwell’s Renewal Theorem. If $F$ is non-lattice then

$$\lim_{t \to \infty} \mathbb{E}[N(t, t + h)] = h/m$$

(1.1) $= 0$ if $m = \infty$

for all delay distributions $G$ and all $h > 0$.

This theorem was first proved by Blackwell (1948) although special cases had been treated by Täcklind (1945) and Doob (1948). Several proofs have been proposed since then, all non-probabilistic, the least complicated probably being the one based on Choquet’s theorem, cf. Feller
(1971). However, in the finite mean case, \( m < \infty \), Lindvall presented in 1977 a probabilistic proof based on the intuitively appealing coupling method and in 1978 Athreya, McDonald and Ney demonstrated a variation on the same theme. The basic idea of these papers is sketched below:

1.2. \( \varepsilon \)-coupling.  Let \( S' \) be a renewal process that is independent of \( S \), with the same recurrence time distribution \( F \) but a different delay distribution \( G' \). Let \( X'_n, N'_n \) have the obvious meaning. (In the light bulb example this amounts to lighting another room with light bulbs installed successively at times \( S'_n \).) The following proposition is well-known (and is an immediate consequence of Lemma 2.1 and formula (2.1) below).

Proposition 1.0.  Put, for \( 0 \leq x \leq \infty \),

\[
\gamma(x) = \int_0^x (1 - F(y))dy \quad \text{ (= } m \text{ when } x = \infty). \tag{1.2}
\]

Suppose \( m < \infty \) and define a distribution \( G_\infty \) by

\[
G_\infty(x) = \gamma(x)/m, \quad 0 \leq x < \infty. \tag{1.3}
\]

If \( G = G_\infty \) then, for all \( t, h \geq 0 \),

\[
\mathbb{E}[N(t, t + h)] = h/m. \tag{1.4}
\]

Thus, if we choose \( G' = G_\infty \) then \( \mathbb{E}[N'(t, t + h)] \) equals the limit \( h/m \) in (1.1) for all \( t \) and (1.1) should follow if we can show that \( S \) behaves asymptotically in the same way as \( S' \) in some sense. This is achieved by coupling: Suppose there were random indices \( \nu \) and \( \nu' \) such that \( S_\nu = S'_\nu \). Then we could modify \( S' \) by replacing \( X'_{\nu+k} \) by \( X_{\nu+k} \) for \( k \geq 1 \), i.e., by letting the renewals of \( S' \) from time \( S'_\nu \) onward coincide with the renewals of \( S \), (when light bulbs are installed at exactly the same time in both rooms tear down the wall between them and use one light bulb to light both rooms); this would not alter the fact that \( S' \) is a renewal process with delay distribution \( G' = G_\infty \). Thus asymptotically \( N(t, t + h] \) and \( N'(t, t + h] \) would coincide and (1.1) would follow by taking expectations.

Now there is one complication: \( S \) and \( S' \) will never meet, i.e., \( \mathbb{P}(\nu < \infty) = 0 \). This can be resolved by only requiring that, for all \( \varepsilon > 0 \), \( S \) and \( S' \) come \( \varepsilon \)-close. An application of Hewitt-Savage 0-1-Law (Lindvall) or a reference to the \( \varepsilon \)-recurrence of 0-mean random walks (Athreya, McDonald and Ney) shows that this occurs with probability one. Replace \( X'_{\nu'(\varepsilon)+k} \) by \( X_{\nu(\varepsilon)+k} \), \( n \geq 1 \), where

\[
\nu(\varepsilon) = \inf\{n \geq 0; |S_n - S'_n| \leq \varepsilon \text{ for some } n'\}, \quad \text{(interpret } \inf \theta = \infty) \tag{1.5}
\]

\[
\nu'(\varepsilon) = \inf\{n' \geq 0; |S_n - S'_n| \leq \varepsilon \text{ for some } n\},
\]
to obtain that from time \( S_{\nu(e)} \) onward the renewals of \( S \) are \( \varepsilon \)-close to the renewals of \( S' \) and everything works out fine by sending \( \varepsilon \downarrow 0 \) in the end.

1.3. Our \( \varepsilon \)-coupling. In the present paper we follow the coupling scheme sketched above and add to it what is needed to mend the following drawbacks:

Firstly, the proof outlined above does not cover the infinite mean case, \( m = \infty \). The key to our solution of this problem is the following proposition (proved in Section 2.1 below).

**Proposition 1.1.** For finite constants \( a > 0 \) define distributions \( G_a \) by

\[
G_a(x) = \begin{cases} 
\gamma(x)/\gamma(a), & 0 \leq x \leq a \\
1, & x > a.
\end{cases}
\]

If \( G = G_a \) then for all \( t \geq 0 \) and \( h \geq 0 \)

\[
\mathbb{E}[N(t, t + h)] \leq h/\gamma(a).
\]

Secondly, in spite of the elementary coupling idea the proof is not elementary due to the reliance on Hewitt-Savage 0-1 Law or the \( \varepsilon \)-recurrence of 0-mean random walks. Our proof is based on the following rather elementary result (proved in Section 2.2 below).

**Proposition 1.2.** Let \( S, S' \) be independent. For \( \varepsilon > 0 \) let \( \nu(\varepsilon) \) be defined by (1.5) and let

\[
\tau(\varepsilon) = \sup\{S_n; |S_n - S'_{n'}| \leq \varepsilon \text{ for some } n'\}.
\]

Then either

\[
\mathbb{P}(\tau(\varepsilon) < \infty) = 1 \quad \text{for all } \varepsilon, G, G',
\]

or

\[
\mathbb{P}(\nu(\varepsilon) < \infty) = 1, \quad \text{for all } \varepsilon, G, G'.
\]

Proposition 1.2 allows us to split into two cases:

The **non-coupling case** (see Section 3.1 below). This is the case when (1.8) holds. Here straightforward calculations show that \( \mathbb{E}[N(t, t + h)] \to 0 \) and that \( m = \infty \); this implies (1.1).

The **coupling case** (see Section 3.2 for \( m < \infty \) and Section 3.3 for \( m = \infty \)). This is the case when (1.9) holds. If \( m < \infty \) then the above coupling argument applies. In order to proceed in the same way when \( m = \infty \) we would need a delay distribution \( G' \) rendering \( \mathbb{E}[N'(t, t + h)] = h/m = 0 \) which of course is nonsense. However, by Proposition 1.1 there is a delay distribution \( G' = G_a \) such
that \( \mathbb{E}[N'(t, t+h)] \) is bounded above by a number that is arbitrarily close to 0. The same coupling argument as before now shows that asymptotically \( \mathbb{E}[N(t, t+h)] \) will be close to \( \mathbb{E}[N'(t, t+h)] \) and (1.1) follows.

Below we make the above ideas precise. Section 2 contains the proofs of Propositions 1.1 and 1.2 together with one additional preparation for Section 3 where Blackwell’s Renewal Theorem is proved. Section 4 finishes off with some remarks.

2. Preparations

Let \( S \) and \( S' \) be independent and let \( \mathbb{P}_0, \mathbb{E}_0 \) indicate that both \( S \) and \( S' \) are zero-delayed, i.e., \( S_0 = 0 = S'_0 \).

2.1. Proof of Proposition 1.1. Let \( U(t) = \mathbb{E}_0[N_t] \) be the renewal function. It is well-known that

\[
\mathbb{E}[N(t, t+h)] = (G \ast U)(t, t+h) = (G \ast U)(t+h) - (G \ast U)(t)
\]

where \( \ast \) denotes convolution. We shall need the following:

Lemma 2.1. For all \( t, h \geq 0 \) it holds that \( (\gamma \ast U)(t, t+h) = h \).

Proof. For \( t \geq 0 \)

\[
t = \int_0^t (1 - F(x))dx + \int_0^t F(x)dx = \gamma(t) + \int_0^t F(t-x)dx.
\]

Thus \( \mu(t) = t \) solves the renewal equation \( \mu = \gamma + \mu \ast F \). But it is well-known that \( \mu = \gamma \ast U \) is the only solution that is bounded on finite intervals and thus \( (\gamma \ast U)(t) = t \), and we are through.

In order to prove Proposition 1.1 put

\[
\varphi_a(x) = \frac{\gamma(x)}{\gamma(a)} - G_a(x) = \begin{cases} 
0, & 0 \leq x \leq a \\
\frac{1}{\gamma(a)} \int_a^x (1 - F(y))dy, & x > a
\end{cases}
\]

Then, with \( G = G_a \),

\[
\mathbb{E}[N(t, t+h)] = (G_a \ast U)(t, t+h)
\]

\[
= \frac{1}{\gamma(a)} (\gamma \ast U)(t, t+h) - (\varphi_a \ast U)(t, t+h)
\]

\[
\leq h/\gamma(a)
\]

where the final step follows from Lemma 2.1 and the fact that \( (\varphi_a \ast U)(t, t+h) \) is non-negative.
2.2. Proof of Proposition 1.2. We shall need the following result:

Lemma 2.2. For all \( x \in (-\infty, \infty) \) it holds that

\[ (2.2) \quad \forall \varepsilon > 0 \exists m, m': P_0(|x + S_m - S_m'| \leq \varepsilon) > 0. \]

Proof. Let \( D \) be the set of all \( x \in (-\infty, \infty) \) such that (2.2) holds. It is readily checked that \( a, b \in D \) implies \( a + b \in D \) and that \( a \in D \) implies \( -a \in D \); thus \( D \) is an additive subgroup of \((-\infty, \infty)\). It is also readily checked that if \( a_n \in D \) and \( a_n \to a \) then \( a \in D \); thus \( D \) is closed and in particular \( c = \inf D \cap (0, \infty) \in D \). If \( c > 0 \) then it is easy to deduce that \( D = L_c \), but due to the assumption that \( F \) is non-lattice we can find \( a \in D \) and \( b \in D \) such that \( a/b \) is non-rational contradicting \( D = L_c \). Thus \( c = 0 \) from which it easily follows that \( D \) is dense in \((-\infty, \infty)\). Since \( D \) is closed this yields \( D = (-\infty, \infty) \) and the proof is complete.

We shall prove Proposition 1.2 by showing that if (1.9) does not hold then (1.8) is true. The following is a loose outline of the argument: Let \( S_0 = S'_0 = 0 \). Due to the above lemma, it is possible to come close to achieving an arbitrary difference \( S_m - S'_m \approx y \) (event A below). Since (1.9) does not hold, we can find a \( y \) such that it is possible to continue from \( S_m - S'_m \approx y \) without \( S_n - S'_n \) ever coming close to 0 (event B below). Thus it is possible that \( S_n - S'_n \) eventually stays away from 0 (event C below). Now let \( S_0, S'_0 \) be arbitrary. Then, whenever \( S_n - S'_n \) is close to 0, there is a positive probability \( (p = P_0(C)) \) that eventually this does not happen again. Thus \( S_n - S'_n \) comes close to 0 only finitely often. We plug in the \( \varepsilon \)-details below.

Let us assume that (1.9) does not hold. Then there is a \( \delta > 0 \) and a \( y \in (-\infty, \infty) \) such that

\[ (2.3) \quad P_0(|y + S_n - S'_n| > 3\delta, \forall n \geq 0, n' \geq 0) > 0. \]

Due to Lemma 2.2 there are \( m, m' \) such that

\[ P_0(A) > 0 \quad \text{where} \quad A = \{|S_m - S'_m - y| \leq \delta\}. \]

Since \( (S_{m+n} - S_m)_{n=0}^{\infty} \) and \( (S'_{m+n} - S'_m)_{n=0}^{\infty} \) are independent zero-delayed renewal processes with recurrence time distribution \( F \), (2.3) yields

\[ P_0(B) > 0 \quad \text{where} \quad B = \{|y + (S_n - S_m) - (S'_n - S'_m)| > 3\delta, \forall n \geq m, n' \geq m'\}. \]

Observe that \( A \) and \( B \) are independent and

\[ A \cap B \subseteq C \quad \text{where} \quad C = \{|S_n - S'_n| > 2\delta, \forall n \geq m, n' \geq m'\}. \]
Thus

\[ p = P_0(C) \geq P_0(A \cap B) = P_0(A)P_0(B) > 0. \]

Now let \( S_0, S'_0 \) have arbitrary distributions \( G, G' \), respectively. Put \( \nu_1 = \nu(\delta), \nu'_1 = \nu'(\delta) \) and for \( k \geq 1 \)

\[ \nu_{k+1} = \inf \{ n > \nu_k; |S_n - S'_{n'}| \leq \delta \text{ for some } n' > \nu'_k \}, \]
\[ \nu'_{k+1} = \inf \{ n' > \nu'_k; |S_n - S'_{n'}| \leq \delta \text{ for some } n > \nu_k \}. \]

Conditionally on \( \{ \nu_k < \infty \} \) \( (S_{\nu_k+n} - S_{\nu_k})_{n=0}^{\infty} \) and \( (S'_{\nu'_k+n} - S'_{\nu'_k})_{n=0}^{\infty} \) are independent zero-delayed renewal processes with recurrence time distribution \( F \) and thus with

\[ C_k = \{ \nu_k < \infty \} \cap \{ (S_{\nu_k+n} - S_{\nu_k}) - (S'_{\nu'_k+n} - S'_{\nu'_k}) \leq 2\delta, \forall n \geq m, n' \geq m' \} \]

we have

\[ P(C_k| \nu_k < \infty) = P_0(C) = p > 0. \]

Further,

\[ \{ \nu_{k+m} < \infty \} \subseteq \{ \nu_k < \infty \} \cap C_k^c \]

and thus

\[ (2.4) \quad P(\nu_{k+m} < \infty) \leq P(\nu_k < \infty)(1 - p). \]

A repeated application of (2.4) with \( k = (n-1)m + 1, k = (n-2)m + 1, \ldots, k = 1 \) yields

\[ P(\nu_{nm+1} < \infty) \leq (1 - p)^n \rightarrow 0 \quad \text{as } n \rightarrow \infty \]

and since \( \{ \nu_{nm+1} < \infty \} \downarrow \{ \tau(\delta) < \infty \} \) as \( n \rightarrow \infty \) we have established (1.8) for \( \epsilon = \delta \). Now observe that

\[ \{ |S_n - S'_{n'}| \leq 2\delta \text{ for some } n' \} = \{ |\delta + S_n - S'_{n'}| \leq \delta \text{ for some } n' \} \cup \{ |S_n - S'_{n'} - \delta| \leq \delta \text{ for some } n' \} \]

and thus \( \tau(2\delta) = \max\{ \tau_\delta, \tau'_\delta \} \) where \( \tau_\delta \) and \( \tau'_\delta \) are the \( \tau(\delta)'s \) we get by replacing \( X_0 \) and \( X'_0 \) by \( \delta + X_0 \) and \( \delta + X'_0 \), respectively. Since (1.8) holds for \( \epsilon = \delta \) this shows that it also holds for \( \epsilon = 2\delta \). Repeat this argument to obtain that (1.8) holds for \( \epsilon = 2^k \delta \). But \( \tau(\epsilon) \) is increasing in \( \epsilon \) and thus (1.8) holds for all \( \epsilon > 0 \) completing the proof.

2.3. A useful lemma. It is well-known that \( IE[N(t,t+h)] \leq IE_0[N_h] = U(h) \). Here we need a slightly stronger result.
Lemma 2.3. Suppose $A$ is an event that is independent of $(X_{N_t+n})_{t=1}^{\infty}$. Then for all $G$ and all $h \geq 0$

(2.5) \[ \mathbb{E}[N(t,t+h)|I_A] \leq U(h) \mathbb{P}(A) \]

where $I_A = 1$ or $0$ according as $A$ occurs or not.

Proof. Put $S^{(t)}_n = X_{N_t+1} + \ldots + X_{N_t+n}$, $n \geq 0$, and

\[ N_h^{(t)} = N[S_n, S_n + h] = \inf\{n \geq 0; S_n^{(t)} > h\}. \]

Since $(t,t+h] \subseteq (t, S_N + h]$ and since $S$ has no renewal in $(t, S_N)$ we have $N(t,t+h] \leq N_h^{(t)}$.

This yields the inequality in

\[ \mathbb{E}[N(t,t+h)|I_A] \leq \mathbb{E}[N_h^{(t)} I_A] = \mathbb{E}[N_h^{(t)}] \mathbb{E}[I_A] = U(h) \mathbb{P}(A) \]

while the first equality follows from the fact that $N_h^{(t)}$ is independent of the event $A$ and the second from the fact that $S^{(t)}$ is a zero-delayed renewal process with recurrence time distribution $F$. 

3. Proof of Blackwell's Renewal Theorem

3.1. The non-coupling case. Let $G$ be arbitrary and put $G' = G$ to obtain

\[ \mathbb{P}(N(t,t+h] \geq 1)^2 = \mathbb{P}(N(t,t+h] \geq 1, N'(t,t+h] \geq 1) \]

\[ \leq \mathbb{P}(\tau(h) > t) \to 0 \quad \text{as} \quad t \to \infty. \]

This and Lemma 2.3 with $A = \{N(t,t+h] \geq 1\}$ yields

(3.1) \[ \mathbb{E}[N(t,t+h)] = \mathbb{E}[N(t,t+h)|I_{\{N(t,t+h] \geq 1\}}] \]

\[ \leq U(h) \mathbb{P}(N(t,t+h] \geq 1) \to 0, \quad t \to \infty. \]

If $m < \infty$ put $G = G_{\infty}$ to obtain the following contradiction:

\[ 0 < \frac{h}{m} = \mathbb{E}[N(t,t+h)] \to 0 \quad \text{as} \quad t \to \infty. \]

Thus $m = \infty$ and (3.1) proves (1.1).

3.2. The coupling case, $m < \infty$. Clearly the event $\{\nu(e) = n, \nu'(e) = n'\}$ is determined by $X_0, \ldots, X_n$ and $X'_0, \ldots, X'_n$, and thus we may replace $X'_{\nu'(e)+k}$ by $X_{\nu(e)+k}$ for $k \geq 1$ without
affecting the fact that $S'$ is a renewal process with recurrence distribution $F$ and delay distribution $G'$. After this modification we have (for $\varepsilon < h/2$)

$$N'(t + \varepsilon, t + h - \varepsilon)I_{\{S_{\nu(\varepsilon)} \leq t\}} \leq N(t, t + h)I_{\{S_{\nu(\varepsilon)} \leq t\}} \leq N'(t - \varepsilon, t + h + \varepsilon)I_{\{S_{\nu(\varepsilon)} \leq t\}}$$

Subtract $(N'(t, t + h) - N'(t + \varepsilon, t + h - \varepsilon))I_{\{S_{\nu(\varepsilon)} > t\}}$ on the left, drop $I_{\{S_{\nu(\varepsilon)} \leq t\}}$ on the right and add $N(t, t + h)I_{\{S_{\nu(\varepsilon)} > t\}}$ in the middle and on the right to obtain

$$N'(t + \varepsilon, t + h - \varepsilon) - N'(t, t + h)I_{\{S_{\nu(\varepsilon)} > t\}}$$

$$\leq N(t, t + h)$$

$$\leq N'(t - \varepsilon, t + h + \varepsilon) + N(t, t + h)I_{\{S_{\nu(\varepsilon)} > t\}}.$$  

Taking expectations and applying Lemma 2.3 with $A = \{S_{\nu(\varepsilon)} > t\}$ yields

$$\mathbb{E}[N'(t + \varepsilon, t + h - \varepsilon)] - U(h)\mathbb{P}(S_{\nu(\varepsilon)} > t)$$

$$\leq \mathbb{E}[N(t, t + h)]$$

$$\leq \mathbb{E}[N'(t - \varepsilon, t + h + \varepsilon)] + U(h)\mathbb{P}(S_{\nu(\varepsilon)} > t).$$

Put $G' = G_\infty$ and subtract $\mathbb{E}[N'(t, t + h)] = h/m$ on all three sides to obtain

$$|\mathbb{E}[N(t, t + h)] - h/m| \leq \frac{2\varepsilon}{m} + U(h)\mathbb{P}(S_{\nu(\varepsilon)} > t)$$

$$\to \frac{2\varepsilon}{m} \text{ as } t \to \infty$$

$$\to 0 \text{ as } \varepsilon \downarrow 0$$

completing the proof in the finite mean case.

3.3. **The coupling case, $m = \infty$.** Put $G' = G_a$ and apply $\mathbb{E}[N'(t - \varepsilon, t + h + \varepsilon)] \leq (h + 2\varepsilon)/\gamma(a)$ to the second inequality in (3.2) to obtain

$$0 \leq \mathbb{E}[N(t, t + h)] \leq \frac{h + 2\varepsilon}{\gamma(a)} + U(h)\mathbb{P}(S_{\nu(\varepsilon)} > t)$$

$$\to \frac{h + 2\varepsilon}{\gamma(a)} \text{ as } t \to \infty$$

$$\to \frac{h + 2\varepsilon}{m} = 0 \text{ as } a \to \infty$$

and the proof is complete.
4. Remarks

Remark 4.1 (On stationarity when $m = \infty$). In the proof of Proposition 1.1 observe that since $\varphi_a(x) = 0$ for $x \leq a$ we have $(\varphi_a * U)(t,t + h) = 0$ for $t + h \leq a$ and thus, for these values of $t$ and $h$, the inequality in (1.7) is actually an equality. In fact, it is easily seen that with $G = G_a$ the point process $N$ is stationary in the time-interval $[0,a]$.

Remark 4.2. It follows from the argument in Section 3.1 that if $m < \infty$ then $P(\nu(\epsilon) < \infty) = 1$ for all $\epsilon, G, G'$.

Remark 4.3 (Discrete aperiodic renewal process). It is well-known that if the $S_n$'s take values in $L_c$ for some $c > 0$ and $P(X_1 \in L_c) < 1$ for all $c' > c$ then (1.1) holds with $h$ restricted to $L_c$. Clearly the argument in the present paper may be modified to give an elementary proof of this result, simply drop the epsilons. However, the author prefers to treat the result in this case as a consequence of the basic limit theorem of Markov chains (see Thorisson (1985) for a proof of that result along the above lines) using the fact that $V_n = S_{N[0,nc]} - nc$ forms an irreducible aperiodic Markov chain and thus $E[N\{nc\}] = P(V_n = 0)U(0) \rightarrow \pi_0 U(0)$ where $\pi_0 = P(X_1 > 0)/m$, note that $U(0) = 1/IP(X_1 > 0)$.

Remark 4.4 (Two-sided random walk with positive drift). If we drop the condition that the $X_n$'s are non-negative and only assume that $0 < m = E[X_1] \leq \infty$ then (1.1) still holds. This was proved by Blackwell (1953) by applying the result in the one-sided case to the imbedded ladder height renewal process $\tilde{S}_0 = S_{N_0}, \tilde{S}_1 = S_{N_{\tilde{S}_0}}, \ldots$ (observe that now $N_t = \inf\{n \geq 0; S_n > t\}$ no longer equals $N[0,t] = \#\{n \geq 0; 0 \leq S_n \leq t\}$). In Athreya, McDonald and Ney (1978) the two-sided case is attacked directly under the additional condition that $m < \infty$. We have presented our argument in the one-sided case in order to keep the presentation at a more elementary level but the approach may be applied directly in the two-sided case with the following straightforward modification: Redefine

$$\gamma(z) = \int_0^\infty P(S_{N_0} > y)dy \quad (= E_0[N_0]m \text{ when } z = \infty).$$

Instead of Lemma 2.1, establish that $\gamma * U(t) = E[N_0]t$ where $U(t) = E_0[N[0,t]]$. Then Proposition 1.1 still holds and also Proposition 1.0 with $G_\infty = \gamma/E_0[N_0]m$. Finally, instead of (2.5) establish that $E[N(t,t+h)I_A] \leq U[-h,h]IP(A)$. 

10
References


**REPORT DOCUMENTATION PAGE**

<table>
<thead>
<tr>
<th>1. REPORT NUMBER</th>
<th>40</th>
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**SUPPLEMENTARY NOTES**

**KEY WORDS (Continue on reverse side if necessary and identify by block number)**

**ABSTRACT (Continue on reverse side if necessary and identify by block number)**

(See reverse side for abstract.)
Blackwell's renewal theorem for non-lattice renewal processes with mean recurrence time $m$ states that the expected number of renewals in a time-interval of length $h$ tends to $h/m$ as the interval goes to infinity:

$$E[N(t, t+h)] \to h/m \quad t \to \infty.$$ 

This note presents a self-contained coupling proof of this result mending the drawbacks of earlier such proofs. Firstly, the proof is complete in the sense that it covers not only the case $m < \infty$ but also $m = \infty$. Secondly, the proof is fairly elementary in the sense that it does not rely on advanced results such as Hewitt-Savage 0-1-Law or the $\varepsilon$-recurrence of 0-mean random walks.