JOINT DISTRIBUTIONS FOR TOTAL PROGENY IN A CRITICAL BRANCHING PROCESS

BY

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Joint Distributions for Total Progeny in a
Critical Branching Process

by Howard J. Weiner *

I. Introduction. Let

(1.1) \( Z(t) \) denote the number of cells alive at time \( t \) in a critical
age-dependent branching process ([1], Ch. 4) as follows. At time \( t = 0 \), a
new cell starts the process and has random lifetime with continuous distribution
function

(1.2) \[ 0 \leq G(t) < 1, \ G(0+) = 0. \]

Assume

(1.3) \[ t^2 (1-G(t)) \to 0 \quad \text{as} \quad t \to \infty \]

and denote

(1.4) \[ 0 < \mu = \int_0^\infty tdG(t). \]

At the end of its life the cell is replaced by \( k \) new cells with
probability \( p_k \). Define

(1.5) \[ h(s) = \sum p_k s^k. \]

Assume, for some \( \varepsilon > 0 \),

(1.6) \[ h(1+\varepsilon) < \infty. \]

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This allows for differentiation of \( h(s), 0 \leq s \leq 1 \), under the summation sign, and also implies that

\[
(1.7) \quad \sum_{k=1}^{\infty} k^n p_k < \infty \text{ for all } n \geq 1.
\]

The basic assumption of criticality is that

\[
(1.8) \quad m = \sum_{k=1}^{\infty} k p_k = 1.
\]

Each new cell proceeds as the parent cell, independent of the past and of other cells.

Let

\[
(1.9) \quad N(t) \text{ denote the number of total progeny born by } t \text{ in a critical age-dependent branching process satisfying (1.1) - (1.8)}.
\]

It is the purpose of this note to obtain a limit theorem for the joint distribution of \( N(\alpha t)/t^2 \) and \( N(t)/t^2 \) given \( Z(t) > 0 \), where \( 0 < \alpha < 1 \), and to indicate an extension. The method involves comparison with a corresponding Galton-Watson process and fractional linear generating function for number of offspring so that iterates may be explicitly computed.

II. Iterations and Approximations.

Definitions

\[
(2.1) \quad F(s_1, s_2, t_0, t_1) = E \left[ \frac{N(t_0)}{s_1} \frac{N(t_0 + t_1)}{s_2}; Z(t_0 + t_1) = 0 \right]
\]

\[
(2.2) \quad H(s_1, s_2, t_0, t_1) = E \left[ \frac{N(t_0)}{s_1} \frac{N(t_0 + t_1)}{s_2} \right].
\]
By the law of total probability,

\[(2.3) \quad F(s_1, s_2, t_0, t_1) = s_1 s_2 \left[ \int_0^{t_0} h(F(s_1, s_2, t_0 - u, t_1)) dG(u) + \int_{t_0}^{t_0 + t_1} h(F(1, s_2, 0, t_0 + t_1 - u)) dG(u) \right], \]

\[F(s_1, s_2, 0, 0) = 0\]

and

\[(2.4) \quad H(s_1, s_2, t_0, t_1) = s_1 s_2 \left[ \int_0^{t_0} h(H(s_1, s_2, t_0 - u, t_1)) dG(u) + \int_{t_0}^{t_0 + t_1} h(H(1, s_2, 0, t_0 + t_1 - u)) dG(u) + 1 - G(t_0 + t_1) \right].\]

**Definitions**

\[(2.5) \quad F(s, t) \equiv E(s^N(t); Z(t) = 0).\]

\[(2.6) \quad H(s, t) = E(s^N(t)).\]

Then

\[(2.7) \quad F(s, t) = s \int_0^t h(F(s, t-u)) dG(u)\]

\[F(s, 0) = 0\]

and

\[(2.8) \quad H(s, t) = s \left[ 1 - G(t) + \int_0^t h(H(s, t-u)) dG(u) \right].\]
Define the iterative schemes

\[(2.9) \quad F_{n+1}(s_1, s_2, t_0, t_1) = s_1 s_2 \int_{0}^{t_0} h(F_n(s_1, s_2, t_0 - u, t_1)) dG(u) + s_1 s_2 \int_{t_0}^{t_0 + t_1} h(F(1, s_2, 0, t_0 + t_1 - u)) dG(u)\]

with

\[(2.10) \quad F_0(s_1, s_2, t_0, t_1) = F(s_1 s_2, t_1) = F(1, s_1 s_2, 0, t_1),\]

and

\[(2.11) \quad H_{n+1}(s_1, s_2, t_0, t_1) = s_1 s_2 \int_{0}^{t_0} h(H_n(s_1, s_2, t_0 - u, t_1)) dG(u) + s_1 s_2 \int_{t_0}^{t_0 + t_1} h(H(1, s_2, 0, t_0 + t_1 - u)) dG(u) + s_1 s_2 (1 - G(t_0 + t_1)),\]

with

\[(2.12) \quad H_0(s_1, s_2, t_0, t_1) = s_1 H(s_2, t_1) = H(s_1, s_2, 0, t_1)\]

\[(2.13) \quad D_{n+1}(s, t) = s \int_{0}^{t} h(D_n(s, t - u)) dG(u)\]

with

\[(2.14) \quad D_0(s, t) = 0\]

\[(2.15) \quad C_{n+1}(s, t) = s \left[ 1 - G(t) + \int_{0}^{t} h(C_n(s, t - u)) dG(u) \right]\]

with

\[(2.16) \quad C_0(s, t) = s\]
(2.17) \[ K_{n+1}(s_1, s_2) = s_1 s_2 h(K_n(s_1, s_2)) G(t_0) + 1 - G(t_0) \]

with

(2.18) \[ K_0(s_1, s_2) = s_1 F(s_2, t_1) + 1 - G(t_0) \]

(2.19) \[ J_{n+1}(s_1, s_2) = s_1 s_2 h(J_n(s_1, s_2)) \]

with

(2.20) \[ J_0(s_1, s_2) = s_1 H(s_2, t_1) = F(s_1, s_2, 0, t_1) \]

(2.21) \[ L_{n+1}(s) = s h(L_n(s)) \]

with

(2.22) \[ L_0(s) = 0 \]

(2.23) \[ R_{n+1}(s) = s h(R_n(s)) \]

with

(2.24) \[ R_0(s) = s. \]

Denote

(2.25) \[ G^{(n)}(t) \]

to be the n-th convolution of G evaluated at t.

**Lemma 1.**

(2.26) \[ 0 \leq F(s, t) - D_n(s, t) \leq G^{(n)}(t) \]

(2.27) \[ 0 \leq L_n(s) - D_n(s, t) \leq 1 - G^{(n)}(t) \]

(2.28) \[ 0 \leq C_n(s, t) - H(s, t) \leq G^{(n)}(t) \]
(2.29) \[ 0 \leq C_n(s,t) - R_n(s) \leq 1 - G^{(n)}(t) \]

(2.30) \[ 0 \leq H_n(s_1,s_2,t_0,t_1) - H(s_1,s_2,t_0,t_1) \leq G^{(n)}(t_0) \]

(2.31) \[ 0 \leq H_n(s_1,s_2,t_0,t_1) - J_n(s_1,s_2) \leq 1 - G^{(n)}(t_0) \]

(2.32) \[ 0 \leq F(s_1,s_2,t_0,t_1) - F_n(s_1,s_2,t_0,t_1) \leq G^{(n)}(t_0) \]

(2.33) \[ 0 \leq K_n(s_1,s_2) - F_n(s_1,s_2,t_0,t_1) \leq 1 - G^{(n+1)}(t_0) \]

**Proof.** Only (2.32) and (2.33) will be explicitly proved. The other results are similar or simpler.

For (2.32), let \( n = 0 \). Then, assuming \( t_1 > t_0 \)

(2.34) \[ F(s_1,s_2,t_0,t_1) = E(s_1 s_2 ; Z(t_0 + t_1) = 0) \]

\[ \geq E(s_1 s_2 \sum_{i=1}^{N_i(t)} ; Z(t_1) = 0) \]

\[ \geq E((s_1 s_2) ; Z(t_1) = 0) = F_0(s_1,s_2,t_0,t_1) \]

since path considerations yield that

(2.35) \[ N(t_0 + t_1) \geq N(t_1) + \sum_{i=1}^{N_i(t_0)} N_i(t_0) \]

where \( \{N_i(t_0)\} \) are I.I.D. as \( N(t_0) \) and independent of the \((Z(t_1), N(t_1))\) process.
Similarly, if \( t_0 > t_1 \),

\[
(2.35) \quad F(s_1, s_2, t_0, t_1) \geq E(s_1^{N(t_0)} s_2^{N(t_0)} s_3^{\sum_{i=1}^{N_1(t_0)}} ; Z(t_0) = 0)
\]

\[
= E((s_1 s_2)^{N(t_0)} ; Z(t_0) = 0) \geq E \left[ (s_1 s_2)^{N(t_0)} ; Z(t_1) = 0 \right]
\]

\[
= E \left[ (s_1 s_2)^{N(t_1)} ; Z(t_1) = 0 \right].
\]

By induction, as

\[
(2.36) \quad 0 \leq F - F_0 \leq 1 = G(0)(t_0)
\]

and

\[
(2.37) \quad 0 \leq F - F_1 = s_1 s_2 \int_0^{t_0} (h(F) - h(F_0))dG \leq \int_0^{t_0} (F - F_0)dG \leq G(t_0),
\]

if it is assumed that

\[
(2.38) \quad 0 \leq F - F_n \leq G^{(n)}(t_0),
\]

then

\[
(2.39) \quad 0 \leq F - F_{n+1} = s_1 s_2 \int_0^{t_0} (h(F) - h(F_n))dG \leq \int_0^{t_0} (F - F_n)dG \leq G^{(n+1)}(t_0)
\]

proving (2.32).

To show (2.33), for \( n = 0 \),

\[
(2.40) \quad K_0 - F_0 = s_1 E \left[ s_2^{N(t_1)} ; Z(t_1) = 0 \right] - E \left[ (s_1 s_2)^{N(t_1)} ; Z(t_1) = 0 \right] + 1 - G(t_0)
\]
and hence

\begin{equation}
(2.41) \quad 0 \leq K_0 - F_0 \leq 1 - G(t_0).
\end{equation}

Also, for \( n = 1 \),

\begin{equation}
(2.42) \quad 0 \leq K_1 - F_1 = s_1 s_2 \int_0^{t_0} h(K_0) - h(F_0) dG(u) + 1 - G(t_0)
- s_1 s_2 \int_{t_0}^{t_0 + t_1} h(F) dG(u)
\end{equation}

and

\begin{equation}
(2.43) \quad K_1 - F_1 \leq \int_0^{t_0} (1 - G(t_0 - u)) dG(u) + 1 - G(t_0) = 1 - G(2)(t_0).
\end{equation}

By induction, assume \((2.33)\) for \( n \). Then

\begin{equation}
(2.44) \quad 0 \leq K_{n+1} - F_{n+1} \leq \int_0^{t_0} (h(K_n) - h(F_n)) dG + 1 - G(t_0)
\end{equation}

\begin{align*}
K_{n+1} - F_{n+1} &\leq \int_0^{t_0} (K_n - F_n) dG + 1 - G(t_0) \\
&\leq \int_0^{t_0} (1 - G(n)(t_0 - u)) dG(u) + 1 - G(t_0) = 1 - G(n+1)(t_0),
\end{align*}

completing \((2.33)\).

**Lemma 2.** Let \( h_1(s), h_2(s) \) satisfy \((1.5)-(1.8)\) and assume

\begin{equation}
(2.45) \quad \sigma_1^2 \equiv h_1''(1) < h_2''(1) \equiv \sigma_2^2.
\end{equation}
Then there exists an \( 0 < s_0 < 1 \), and an integer \( M > 0 \) such that for 
\( s_1 > s_0, s_2 > s_0 \) and all \( n > m > M \),

\[
E_1(s_1^m s_2^n) \leq E_2(s_1^m s_2^n)
\]  

and

\[
E_1(s_1^m s_2^n; Z_n = 0) \leq E_2(s_1^m s_2^n; Z_n = 0)
\]

where \( N_m, N_n, Z_n \) are from G-W processes governed by \( h_1(s) \) and \( h_2(s) \), respectively.

**Proof.** As \( n > m \to \infty \), for \( h_i(s), i = 1,2 \)

\[
E_i \left[ \begin{array}{cc} N_m & N_n \\ s_1 & s_2 \end{array} \right] \to E_i \left[ \begin{array}{c} N_m \\ s_1 s_2 \end{array} \right]
\]

and

\[
E_i(s_1^m s_2^n; Z_n = 0) \to E_i \left[ \begin{array}{c} N_m \\ s_1 s_2 \end{array} \right; Z = 0] = E_i(s_1 s_2)^N
\]

where \( N, Z \) are bona-fide r.v.s. and

\[
P[Z = 0] = 1
\]

for the critical case.

To prove the lemma, it therefore suffices to show that there exists an \( 1 > s_0 > 0 \) such that for \( s > s_0 \),

\[
E_1(s^n) < E_2(s^n).
\]
This proof, due to N. Knueppel, will now be given.

A Taylor expansion of p. 22 of [1] shows that for $s > s_1 > 0$,

$$h_1(s) < h_2(s).$$

Since

$$E_i(s^n)^N \downarrow E_i(s^N), \quad i = 1, 2,$$

for $s > s_0$,

$$E_i(s^N) > s_1.$$

For $n = 1$ and $s > s_0 > s_1$,

$$E_1(s^1) = sh_1(s) < sh_2(s).$$

Assume that for $s > s_0$,

$$s_1 < E_1(s^n) < E_2(s^n).$$

Then for $s > s_0$,

$$E_1(s^{n+1}) = sh_1(E_1(s^n)) < sh_2(E_1(s^n))$$

$$< sh_2(E_2(s^n)) = E_2(s^{n+1}),$$

completing the proof of lemma 2.
Define the iterations

\[(2.58) \quad T(s_1, s_2, m, n) = E(s_1^m s_2^n; Z_n = 0)\]

with

\[(2.59) \quad T(s_1, s_2, 0, n-m) = s_1 E(s_2^{n-m}; Z_{n-m} = 0) = s_1 I_{n-m}(s_2).\]

\[(2.60) \quad U(s_1, s_2, m, n) = E\left[\begin{bmatrix} N_m & N_n \\ s_1 & s_2 \end{bmatrix}\right]\]

with

\[(2.61) \quad U(s_1, s_2, 0, n-m) = s_1 E\left[\begin{bmatrix} N_{n-m} \\ s_2 \end{bmatrix}\right] = s_1 R_{n-m}(s_2),\]

where \(Z_n, N_m, N_n\) are from a critical G-W process with \(h''(1) = \sigma^2\).

Lemma 3.

\[(2.62) \quad F(s_1, s_2, t_0, t_1) - H(s_1, s_2, t_0, t_1) \leq T(s_1, s_2, n, r+n) - U(s_1, s_2, n, r+n) + 2G(r)(t_1) + 2G(n)(t_0) + 1 - (G(t_0))^{n+1},\]

and

\[(2.63) \quad F(s_1, s_2, t_0, t_1) - H(s_1, s_2, t_0, t_1) \geq T(s_1, s_2, n, r+n) - U(s_1, s_2, n, r+n) - 2(1 - G(r)(t_1)) - 2(1 - G(n)(t_0)).\]
Proof. From (2.30) - (2.33),

\[
(2.64) \quad K_n - J_n - 2(1 - G_n(t_0)) \leq F(s_1, s_2, t_0, t_1) - H(s_1, s_2, t_0, t_1)
\]

\[
\leq 2G_n(t_0) + K_n - J_n.
\]

From (2.26) to (2.29), for any \( r \geq 1 \),

\[
(2.65) \quad s_1(L_r(s_2) - (1 - G_r(t))) + 1 - G(t_0) \leq K_0(s_1, s_2)
\]

\[
\leq s_1(L_r(s_2) + G_r(t)) + 1 - G(t_0),
\]

and

\[
(2.66) \quad s_1(R_r(s_2) - G_r(t)) \leq J_0(s_1, s_2) \leq s_1(R_r(s_2) + 1 - G_r(t)).
\]

For a critical generating function \( h \), note that for \( a > 0, b > 0, a + b \leq 1 \), the mean value theorem yields that

\[
(2.67) \quad h(a+b) \leq h(a) + b
\]

\[
\quad h(a) - b \leq h(a-b).
\]

Note that

\[
(2.68) \quad T(s_1, s_2, m+1, n+1) = s_1 s_2 h(T(s_1, s_2, m, n))
\]

\[
(2.69) \quad U(s_1, s_2, m+1, n+1) = s_1 s_2 h(U(s_1, s_2, m, n)).
\]
Then (2.58) - (2.61), (2.64) - (2.69) together with (2.17) - (2.20)
upon successive application of (2.67) yield, for \( r \geq 1 \),

\[
(2.70) \quad K_1 = s_1 s_2 h(K_0) G(t_0) + 1 - G(t_0) \\
\leq s_1 s_2 G(t_0) h(s_1 L_{1, r}(s_2)) + 1 - (G(t_0))^2 + G(r)(t),
\]
or

\[
K_1 \leq T(s_1, s_2, 1, r+1) + G(r)(t) + 1 - (G(t_0))^2.
\]

\[
(2.71) \quad K_1 \geq s_1 s_2 G(t_0) h(s_1 L_{1, r}(s_2)) - (1 - G(r)(t))) + 1 - G(t_0),
\]

from which it follows that

\[
(2.72) \quad K_1 \geq T(s_1, s_2, 1, r+1) - (1 - G(r)(t))) s_1 s_2
\]

\[
(2.73) \quad K_2 = s_1 s_2 h(K_1) G(t_0) + 1 - G(t_0) \\
\leq s_1 s_2 G(t_0) h(T(s_1, s_2, 1, r+1)) + G(r)(t) + 1 - (G(t_0))^3.
\]
or

\[
(2.74) \quad K_2 \leq T(s_1, s_2, 2, r+2) + G(r)(t) + 1 - (G(t_0))^3.
\]

From (2.72),

\[
(2.75) \quad K_2 \geq s_1 s_2 h(K_1) \geq s_1 s_2 h(T(s_1, s_2, 1, r+1)) - (s_1 s_2)^2 (1 - G(r)(t))
\]
or

\[(2.76) \quad K_2 \geq T(s_1, s_2, 2, r+2) - (s_1s_2)^2(1 - G(r)(t)). \]

By induction, assume that

\[(2.77) \quad K_n \leq T(s_1, s_2, n, r+n) + G^{(r)}(t) + 1 - (G(t_0))^{n+1} \]

and

\[(2.78) \quad K_n \geq T(s_1, s_2, n, r+n) - (s_1s_2)^n(1 - G(r)(t)). \]

Then

\[(2.79) \quad K_{n+1} = s_1s_2G(t_0)h(K_n) + 1 - G(t_0) \]

\[\leq s_1s_2h(T(s_1, s_2, n, r+n)) + G^{(r)}(t) + 1 - (G(t_0))^{n+2} \]

or

\[(2.80) \quad K_{n+1} \leq T(s_1, s_2, n+1, r+n+1) + G^{(r)}(t) + 1 - (G(t_0))^{n+2}, \]

completing the induction started by (2.77).

In the other direction, using (2.78),

\[(2.81) \quad K_{n+1} \geq s_1s_2h(K_n) \geq s_1s_2h(T(s_1, s_2, n, r+n)) - (s_1s_2)^{n+1}(1 - G(r)(t)) \]

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or

\[(2.82) \quad K_{n+1} \geq T(s_1, s_2, n+1, r+n) - (s_1 s_2)^{n+1}(1 - G^{(r)}(t)),\]

completing the induction started by (2.78).

A similar argument to that of (2.70) - (2.82) yields

\[(2.83) \quad U(s_1, s_2, n, r+n) - G^{(r)}(t) \leq J_n \leq U(s_1, s_2, n, r+n) + 1 - G^{(r)}(t).\]

Hence (2.64), (2.77), (2.78), (2.83) yield

\[(2.84) \quad F(s_1, s_2, t_0, t_1) - H(s_1, s_2, t_0, t_1) \geq T(s_1, s_2, n, r+n) - U(s_1, s_2, n, r+n) - 2(1 - G^{(r)}(t)) \geq 2(1 - G^{(n)}(t_0))\]

and, omitting the same arguments as in (2.84),

\[(2.85) \quad F - H \leq T - U + 2G^{(r)}(t) + 1 - (G(t_0))^{n+1} + 2G^{(n)}(t_0).\]

Now set \(t = t_1\). This completes lemma 3.

Let

\[(2.86) \quad h_0(s, \sigma^2) = \frac{\sigma^2 + (2 - \sigma^2)s}{\sigma^2(1-s) + 2}.\]

Let

\[(2.87) \quad U_0(s_1, s_2, m, n)\]

and

\[(2.88) \quad T_0(s_1, s_2, m, n)\]
denote the respective quantities \( U, T \) obtained for \( h_0 \) of (2.86).

**Lemma 4.** For the critical generating function (2.86), it follows that

\[
(2.88) \quad \lim_{n \to \infty} \left( \frac{n \sigma^2}{2} \right) \langle U_0(e^{-\theta_1/n^2}, e^{-\theta_2/n^2}, n_0, n(1-n)) - T_0(e^{-\theta_1/n^2}, e^{-\theta_2/n^2}, n_0, n(1-n)) \rangle \\
= \lim_{n \to \infty} \mathbb{E} \left[ \exp \left( -\frac{1}{n} (\theta_1 N_0, [n]) + \theta_2 N_0, n \right) \right] \left[ Z_{on} > 0 \right] \\
= \frac{4 \sqrt{\sigma^2 \theta_2 (\theta_1 + \theta_2)}}{\left( \sqrt{\theta_2} + \sqrt{\theta_1 + \theta_2} \right)^2 \sinh \left( \alpha \sqrt{2 \sigma^2 (\theta_1 + \theta_2)} \right)} \\
+ \frac{\left( \sqrt{\theta_1 + \theta_2} - \sqrt{\theta_2} \right)^2 \sinh \left( (1-\alpha) \sqrt{2 \sigma^2 \theta_2} + \alpha \sqrt{2 \sigma^2 (\theta_1 + \theta_2)} \right)}{\left( \sqrt{\theta_2} + \sqrt{\theta_1 + \theta_2} \right)^2 \sinh \left( \alpha \sqrt{2 \sigma^2 \theta_2} \right)} \\
+ 2 \theta_1 \sinh \left( (1-\alpha) \sqrt{2 \sigma^2 \theta_2} \right)
\]

where \( N_{0m} \) is the total progeny and number alive, respectively, in a critical G-W process at generation \( m \) with offspring generating function \( h_0(s, \sigma^2) \).

**Proof.** The proof follows the method of Lindvall ([2] pp. 318-319).

For \( 0 < m < n \), with \( N_{0m}, N_{0n}, Z_{0n} \) from a critical G-W process with offspring generating function \( h_0(s, \sigma^2) \), one may write

\[
(2.89) \quad \mathbb{E}(s_1^{N_{0m}}, s_2^{N_{0n}}, s_3^{Z_{0n}}) = \mathbb{E} \left( \begin{array}{c} Z_{0m} \\
\sum_{i=1}^{N_{0m}} N_{0,n-m,i} \\
\sum_{i=1}^{Z_{0m}} Z_{0,n-m,i} \\
Z_{0m} \end{array} \right) \mathbb{E} \left( \begin{array}{c} s_1^{N_{0m}} \\
s_2^{\sum_{i=1}^{N_{0m}} N_{0,n-m,i}} \\
s_3^{\sum_{i=1}^{Z_{0m}} Z_{0,n-m,i}} \\
s_3 \end{array} \right)
\]

where \( \{N_{0,n-m,i}\} \) are I.I.D. as \( N_{0,n-m} \), the \( \{Z_{0,n-m,i}\} \) are I.I.D. as \( Z_{0,n-m} \), and both sets of r.v.s. are independent of the \( (Z_{0m}, N_{0m}) \) part of the process, and \( N_{0,n-m,i} \) and \( Z_{0,n-m,j} \) are independent for \( i \neq j \), with \( N_{0,n-m,i} \) and \( Z_{0,n-m,i} \) from the same critical G-W process. Hence
(2.90) \[ E(s_1^{N_{0m}} s_2^{N_{0n}} s_3^{Z_{0n}}) = h_m(s_1 s_2, h_{n-m}(s_2, s_3)) \]

where

(2.91) \[ h_r(s_1, s_2) = E(s_1^{N_{0r}} s_2^{Z_{0r}}). \]

To express \( h_r(s_1, s_2) \) in terms of \( h_0(s) = h_0(s, \sigma^2) \) and its iterates, note that

(2.92) \[ h_1(s_1, s_2) = s_1 h_0(s_1 s_2) \]

and

(2.93) \[ h_{r+1}(s_1, s_2) = E(s_1^{N_{0, r+1}} s_2^{Z_{0, r+1}}) = E \left[ E(s_1^{N_{0r,i}} s_2^{Z_{0r,i}}) \right] \]

where \( N_{0r,i} \) and \( Z_{0r,i} \) are from the same process, and \( N_{0r,j} \) and \( Z_{0r,i} \) are independent for \( i \neq j \), and the \( \{N_{0r,i}\} \) are I.I.D. as \( N_{0r} \), and \( \{Z_{0r,i}\} \) are I.I.D. as \( Z_{0r} \).

Hence

(2.94) \[ h_{r+1}(s_1, s_2) = s_1 h_0(h_r(s_1, s_2)). \]

A tedious but straightforward induction using (2.90) yields that

(2.95) \[ h_n(s_1, s_2) = \frac{P_{1,n}(s_1) + s_2 P_{2,n+1}(s_1)}{P_{3,n-1}(s_1) + s_2 P_{4,n}(s_1)} \]

where \( P_{1n} \), denote the \( n \)-th degree polynomials to be determined. Relation (2.95) yields
(2.96)  
(a) \( p_{1,n+1}(s) = sp_{3,n-1}(s) - (2p-1)sp_{1,n}(s) \)

(b) \( p_{2,n+2}(s) = sp_{3,n}(s) - (2p-1)sp_{2,n+1}(s) \)

(c) \( p_{3,n}(s) = p_{3,n-1}(s) - pp_{1,n}(s) \)

(d) \( p_{4,n+1}(s) = p_{4,n}(s) - pp_{2,n+1}(s) \).

From the theory of difference equations one may solve pairs (2.96) (a) and (c) and pair (2.96) (b) and (d) and from initial conditions obtained from explicit formulas for \( h_1(s_1,s_2) \) and \( h_2(s_1,s_2) \) one substitutes a solution

(2.97) \( p_{in} = A_i r^n, \ 1 \leq i \leq 4 \)

to obtain

(2.98) \( p_{in} = A_{0i} r_1^n + A_{1i} r_2^n, \ 1 \leq i \leq 4 \)

where \( \{A_{0i}\}, \{A_{1i}\}, r_1, r_2 \) are explicitly determined.

Writing \( n\alpha \) instead of \([n\alpha]\), which will not affect a limit, it follows that

(2.99) \[
\lim_{n \to \infty} \mathbb{P} \left[ \exp \left\{ -\frac{1}{n} \left( \theta_1 N_0, n\alpha + \theta_2 N_0n \right) \right\} \mid Z_{0n} > 0 \right] \\
= \lim_{n \to \infty} h_{n\alpha}(h_{n(1-\alpha)}(1, e^{-\theta_2/n^2}), e^{-(\theta_1 + \theta_2)/n^2}) - h_{n\alpha}(h_{n(1-\alpha)}(0, e^{-\theta_2/n^2}), e^{-(\theta_1 + \theta_2)/n^2}) \\
= 1 - h_n(1,0)
\]
A tedious but straightforward computation using (2.95), (2.98) in (2.99) yields the result of lemma 4.

Let

\[(2.100) \quad \frac{2p}{q} = \sigma^2\]

where \(0 < p < 1\) and \(q = 1 - p\).

For \(0 < \epsilon \ll p\), denote

\[(2.101) \quad p_1 = p + \epsilon, \quad q_1 = 1 - p_1\]

and

\[(2.102) \quad p_2 = p - \delta(\epsilon), \quad q_2 = 1 - p_2\]

where

\[(2.103) \quad p_1 q_1 = p_2 q_2.\]

Denote

\[(2.104) \quad \sigma_i^2 = \frac{2p_i}{q_i}, \quad i = 1, 2.\]

**Corollary.** For \(1 \leq i, j \leq 2, i \neq j\), and \(0 < \epsilon \leq \epsilon_0 \ll p\), and \(0 < \alpha < 1\), and if (2.100) - (2.103) hold, then

\[(2.105) \quad \lim_{n \to \infty} n |U_0(e^{-\theta_1/n^2}, e^{-\theta_2/n^2}, n\alpha, n(1-\alpha), \sigma_i^2)_{n, \epsilon \leq \epsilon_0} - T_0(e^{-\theta_1/n^2}, e^{-\theta_2/n^2}, n\alpha, n(1-\alpha), \sigma_j^2)| \leq C\]

where \(C < \infty\) is a positive constant.
Proof. This is a straightforward if tedious computation of $U_0, T_0$ using
the method of difference equations of the previous lemma, noting that from
(2.103), the constant term in the expansion of $U_0 - T_0$ cancels out, leaving
terms of order $\frac{1}{n}$ and lower in $n$.

Theorem. Under the assumptions (1.1) to (1.8)

$$
\lim_{t \to \infty} \mathbb{E} \left[ \exp \left\{ \frac{1}{\mu^2} \left( \theta_1 N(\alpha t) + \theta_2 N(t) \right) \right\} Z(t) > 0 \right]
$$

$$
= \left( \sqrt{\frac{2\sigma_2^2 \theta_2}{\theta_1^2 + \theta_2^2}} \right) \left[ \frac{1}{\sqrt{\theta_1^2 + \theta_2^2}} \sinh \left[ \frac{\alpha}{\mu} \frac{2\sigma_2^2 \theta_1}{\theta_1^2 + \theta_2^2} \right] \right.
$$

$$
+ \left( \frac{\sqrt{\theta_1^2 + \theta_2^2}}{\mu} \right)^2 \sinh \left[ \frac{(1-\alpha)\sqrt{2\sigma_2^2 \theta_1}}{\mu} \right]
$$

$$
+ 2 \alpha \sinh \left[ \frac{(1-\alpha)\sqrt{2\sigma_2^2 \theta_1}}{\mu} \right] \right]^{-1}.
$$

Proof. From lemma 3, let, for $0 < \epsilon < \epsilon_0 < p$, where $\sigma^2 = \frac{2p}{q}$,

(2.107) \begin{align*}
(a) \quad r_1 &= \left[ \frac{t_1(1+\epsilon)}{\mu} \right] \\
(b) \quad r_2 &= \left[ \frac{t_1(1-\epsilon)}{\mu} \right] \\
(c) \quad n_1 &= \left[ \frac{t_0(1+\epsilon)}{\mu} \right] \\
(d) \quad n_2 &= \left[ \frac{t_0(1-\epsilon)}{\mu} \right].
\end{align*}

Then by lemma 3 and the lemma 3 of Ch. 4 of [1], p. 158--160.
(2.108) \[ F(s_1, s_2, t_0, t_1) - H(s_1, s_2, t_0, t_1) \]
\[ \leq T(s_1, s_2, n_1, r_{1} + n_1) - U(s_1, s_2, n_1, r_{1} + n_1) + o(t_0^{-1}) + o(t_1^{-1}) \]

and

(2.109) \[ F(s_1, s_2, t_0, t_1) - H(s_1, s_2, t_0, t_1) \]
\[ \geq T(s_1, s_2, n_2, r_{2} + n_2) - U(s_1, s_2, n_2, r_{2} + n_2) + o(t_0^{-1}) + o(t_1^{-1}). \]

Now, using lemma 2 in (2.108), (2.109) yields, with assumptions
(2.100) - (2.104), for \( r_i, n_i \) sufficiently large, \( i = 1, 2 \), and \( \epsilon < \epsilon_0 \ll p \),

(2.110) \[ F(s_1, s_2, t_0, t_1) - H(s_1, s_2, t_0, t_1) \]
\[ \leq T_0(s_1, s_2, n_1, r_{1} + n_1, \sigma_1^2) - u_0(s_1, s_2, n_1, r_{1} + n_1, \sigma_2^2) + o(t_0^{-1} + t_1^{-1}) \]

and

(2.111) \[ F(s_1, s_2, t_0, t_1) - H(s_1, s_2, t_0, t_1) \]
\[ \geq T_0(s_1, s_2, n_2, r_{2} + n_2, \sigma_2^2) - u_0(s_1, s_2, n_2, r_{2} + n_2, \sigma_1^2) + o(t_0^{-1} + t_1^{-1}) \]

where

(2.112) \[ \sigma_1^2 > \sigma_2^2 > \sigma_2^2 \]

and

(2.113) \[ \sigma_i^2 = 2p_i / q_i, \quad i = 1, 2 \]

with

(2.114) \[ p_i = p + \epsilon_i, \quad \text{as in (2.101) - (2.103)}. \]
Now, set, for $0 < \alpha < 1$,

\[(2.115) \quad (a) \quad t = n\mu
\]

\[(b) \quad t_0 = n\alpha\mu
\]

\[(c) \quad t_1 = n(1-\alpha)\mu \quad -\frac{\theta_1}{n^2}
\]

\[(d) \quad s_1 = e^{-\frac{\theta_1}{n^2}}, \quad s_2 = e^{-\frac{\theta_2}{n^2}}.
\]

Multiply (2.110) and (2.111) by $n$.

Then let $\varepsilon \to 0$, then $n \to \infty$, noting that by the corollary, these limits may be interchanged.

Since, for fixed $\sigma^2 > 0$,

\[(2.116) \quad \mathbb{E} \left[ \exp \left\{ -\frac{1}{t} \left( \theta_1 N(\alpha t) + \theta_2 N(t) \right) \right\} \right]
\]

\[= \mathbb{H}(e^{-\frac{\theta_1}{t^2}}, e^{-\frac{\theta_2}{t^2}}, e^{\alpha t(1-\alpha)t} - e^{\alpha t(1-\alpha)t})
\]

and by [1], Ch. 4,

\[(2.117) \quad \lim_{{t \to \infty}} t \mathbb{P}[Z(t) > 0] = \frac{2\mu}{\sigma^2},
\]

then lemma 4 and the corollary together with (2.116), (2.117) and the substitution of $\theta_i/\mu^2$ for $\theta_i$, $i = 1, 2$, then yields the result of the theorem.
References


Joint Distributions for Total Progeny in a Critical Branching Process

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Joint distribution, asymptotic, total progeny, critical branching process, iteration

Let \( N(t) \) denote the total progeny born by time \( t \) in a critical age-dependent branching process. A limit law for the joint distribution of \( N(xt)/t^2 \) and \( N(t)/t^2 \) conditioned on the event that the process is not extinct at \( t \) is obtained, where \( 0 < x < 1 \).