THE EXTINCTION PROBABILITY IN A CRITICAL BRANCHING PROCESS

BY

HOWARD J. WEINER

TECHNICAL REPORT NO. 18
MAY 30, 1978

PREPARED UNDER GRANT
DAAG29-77-G-0031
FOR THE U.S. ARMY RESEARCH OFFICE

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Partially supported under Office of Naval Research Contract N00014-76-C-0475 (NR-042-267) and issued as Technical Report No. 258.
The Extinction Probability in a
Critical Branching Process

by Howard J. Weiner*

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I. Introduction.

Let $Z(t)$ denote the number of cells alive at $t$ in a critical age-dependent Bellman-Harris branching process with cell lifetime distribution $G(t)$, $G(0^+) = 0$, non-lattice and assume

\begin{equation}
(1.1) \quad t^2 (1-G(t)) \to 0 \quad \text{as} \quad t \to \infty.
\end{equation}

\begin{equation}
(1.2) \quad 0 < \mu \equiv \int_0^\infty tdG(t) < \infty.
\end{equation}

Let the offspring generating function be denoted, for $0 \leq s \leq 1$,

\begin{equation}
(1.3) \quad h(s) \equiv \sum_{k=0}^{\infty} p_k s^k
\end{equation}

and

\begin{equation}
(1.4) \quad h'(1) = 1 = \sum_{k=1}^{\infty} kp_k \quad \text{(criticality)}
\end{equation}

and

\begin{equation}
(1.5) \quad 0 < \sigma^2 \equiv h''(1) = \sum_{k=2}^{\infty} k(k-1)p_k < \infty.
\end{equation}

See [1] Chapter 4 for details.

*Partial Support NONR at Stanford University
It is well-known that

\[
(1.6) \quad \lim_{t \to \infty} t P[Z(t) > 0] = \frac{2\mu}{\sigma^2} = b.
\]

Various proofs of (1.6) and the corresponding result for a critical Galton-Watson or discrete time process have appeared. See [1] Chapters 1, 4 for comments and references. For example, the proof of (1.6) in [2] uses the renewal theorem. The proof in [1], Chapter 4, gives the asymptotic form for a critical generating function, from which (1.6) is a special case, and relates the generating function to that of a critical Galton-Watson process, for which results are obtained. The proof given in this note is elementary and self-contained.

II. Comparisons and Iterations

**Definition.** For \(0 \leq s \leq 1, t \geq 0,\)

\[
(2.1) \quad F(s,t) = \sum_{k=0}^{\infty} P[Z(t) = k] s^k.
\]

It is well-known ([1], Chapter 4), and follows by the law of total probability that

\[
(2.2) \quad F(s,t) = s(1-G(t)) + \int_{0}^{t} h(F(s,t-u)) dG(u).
\]

Using the fact that

\[
(2.3) \quad P(t) = P[Z(t) > 0] = 1 - F(0,t),
\]

then from (2.2) it follows that, by a Taylor expansion on \(h(s)\) about \(s = 1,\)
\[(2.4) \quad 1 - P(t) = \int_{0}^{t} h(1 - P(t-u)) dG(u)\]

and

\[(2.5) \quad 1 - P(t) = \int_{0}^{t} \left[1 - P(t-u) + \frac{\sigma^2}{2} p^2(t-u) + o(P^2(t-u))\right] dG(u)\]

Rewriting (2.5),

\[(2.6) \quad P(t) = 1 - G(t) + \int_{0}^{t} P(t-u) dG(u) - \frac{\sigma^2}{2} \int_{0}^{t} p^2(t-u) dG(u) + f(t)\]

where \(f(t)\) denotes the remainder.

**Lemma 1** ([1]) \(P(t) \downarrow 0\).

**Proof.** A simpler and elementary proof will be given here. Clearly

\[(2.7) \quad P(t) \downarrow C \geq 0.\]

Assume \(C > 0\). Split the integral on the right side of (2.5) into \(\int_{0}^{t} = \int_{0}^{t/2} + \int_{t/2}^{t}\). For large \(t\), the integral \(\int_{t/2}^{t}\) is \(o(t^{-2})\). Then it is clear that

\[(2.8) \quad 1 - C = 1 - C + \frac{\sigma^2}{2} C + o(1),\]

which is a contradiction of \(C > 0\). This proves the lemma.

\[(2.9) \quad \text{Let } G^{(n)}(t) = \text{n}^{\text{th}} \text{ iterate of } G \text{ evaluated at } t.\]

Define the iterative sequences

\[(2.10) \quad U_{n+1}(t) = \int_{0}^{t} h(U_n(t-u)) dG(u)\]

\(U_0(t) = 1.\)
\[ I_{n+1}(t) = \int_{0}^{t} h(I_{n}(t-u))dG(u) \]

\[ I_{o}(t) = \begin{cases} 
1, & t \leq T \\
1 - \frac{b}{h}, & t > T 
\end{cases} \]

for \( T \geq b \).

**Lemma 2** Under assumptions (1.1) - (1.5), for \( n \geq 0 \), and all \( t > T \),

\[ 0 \leq U_{n}(t) - U(0,t) \leq G^{(n)}(t) \]

\[ 0 \leq U_{n}(t) - I_{n}(t) \leq G^{(n)}(t) \]

\[ |I_{n}(t) - I_{o}(t)| \leq k(t), \]

where, for \( t \to \infty \),

\[ tk(t) \to 0. \]

**Proof.** Eq. (212) holds for \( n = 0 \).

Assume (2.12) for \( n \). Then, omitting arguments,

\[ 0 \leq U_{n+1} - F = \int_{0}^{t} h(U_{n}) - h(F) dG \leq \int_{0}^{t} (U_{n} - F) dG \]

\[ \leq \int_{0}^{t} G^{(n)}(t-u) dG = G^{(n+1)}(t) \]

where the left inequality follows from the induction hypothesis and the monotonicity of \( h \), and the right inequalities from the mean value theorem, the fact that \( h' \left( \frac{1}{2} \right) = 1 \), and the induction hypothesis.

To show (2.13), observe that for \( t > T \)

\[ 0 \leq U_{o} - I_{o} \leq \frac{b}{t} < 1 = G^{(o)}(t). \]
Assume (2.13) for \( n \) by induction. Then, arguing as in (2.16),

\[
0 \leq U_{n+1} - I_{n+1} = \int_0^t (h(U_n) - h(I_n))dG \leq \int_0^t (U_n - I_n)dG \\
\leq \int_0^t G^{(n)}(t-u)dG(u) = G^{(n+1)}(t).
\]

To show (2.14) write \( I_1(t) \) as

\[
I_1(t) = \int_0^{t/2} + \int_{t/2}^t h(I_0(t-u))dG(u).
\]

Note that the second integral \( \int_{t/2}^t \) in (2.19) is \( o(t^{-2}) \) by (1.1).

The first integral \( \int_0^{t/2} \) may be written by a Taylor expansion of \( h(s) \) about \( s = 1 \) as \( t \gg T \)

\[
I_1(t) = o(t^{-2}) + \int_0^{t/2} h(1 - \frac{b}{t-u})dG(u)
\]

\[
= o(t^{-2}) + \int_0^{t/2} \left[ h(1) - \frac{b}{t-u} h'(1) + \frac{b^2}{2(t-u)^2} h''(1)
\right. \\
\left. + o(t^{-2}) \right] dG(u)
\]

Using \( h(1) = h'(1) = 1 \), and the expansion, \( 0 < u < t/2 \),

\[
\frac{1}{t-u} = \frac{1}{t} \left( 1 + \frac{u}{t} + o\left(\frac{u}{t}\right) \right)
\]

in the right side of (2.20) yields

\[
I_1(t) = o(t^{-2}) + G(t) - \frac{b}{t} \int_0^{t/2} \left( 1 + \frac{u}{t} + o\left(\frac{u}{t}\right) \right) dG(u)
\]

\[
+ \frac{b^2 \sigma^2}{2t^2} \int_0^{t/2} \left[ 1 + o\left(\frac{u}{t}\right) \right] dG(u)
\]

as again by (1.1), an integration by parts yields

\[
\int_t^\infty udG(u) = o(t^{-1}).
\]
It follows that

\[(2.24) \quad I_1(t) = I_0(t) + f(t)\]

where

\[(2.25) \quad 0 \leq |f(t)| \leq K < \infty\]

and as \(t \to \infty\),

\[(2.26) \quad t^2 f(t) \to 0.\]

Then one may write

\[(2.27) \quad I_2(t) = \int_0^t h(I_0(t-u)+f(t-u))dG(u) \geq \int_0^t \{h(I_0(t-u))+h'(I_0(t-u))f(t-u)\}dG(u).\]

Note that for \(t \gg T\),

\[(2.28) \quad I_1(T) = \int_0^T h(I_0(T-u))dG(u)\]

and

\[(2.29) \quad h(1-\varepsilon)G(T) \leq I_1(T) \leq h(1 - \frac{a}{T})G(T),\]

for some \(a > 0, \varepsilon > 0\),

\[(2.30) \quad I_2(t) \leq \int_0^t \{h(I_0(t-u)) + h'(\max\{h(1 - \frac{a}{T})G(T),1 - \frac{\gamma}{t}\})f(t-u)\}dG(u)\]

for some \(a > 0, \gamma > 0\).
For all \( t >> T \), (2.27) - (2.30) yield that

\[
|I_2(t) - I_1(t)| \leq (1 - \frac{\alpha}{t}) \int_0^t f(t-u) dG(u).
\]

Hence, repeating the argument of (2.27) - (2.31),

\[
|I_3(t) - I_2(t)| \leq (1 - \frac{\alpha}{t})^2 \int_0^t f(t-u) dG^{(2)}(u),
\]

where

\[
0 < \alpha \text{ is a constant.}
\]

An induction yields that

\[
|I_n(t) - I_0(t)| \leq \sum_{\ell=0}^{\infty} (1 - \frac{\alpha}{t})^{\ell} f \ast G^{(\ell)}(t) = k(t),
\]

where "\( \ast \)" denotes the usual convolution integral.

The right side of (2.34) is now broken up into a number of parts, and upper bounds for each part is obtained.

\[
k(t) = \sum_{\ell=0}^{t^2} \sum_{\ell=0}^{\infty} (1 - \frac{\alpha}{t})^{\ell} f \ast G^{(\ell)}(t).
\]

Since

\[
|f| \ast G^{(\ell)}(t) \leq K
\]

the second term of (2.35) is dominated by

\[
\sum_{\ell=0}^{t^2} (1 - \frac{\alpha}{t})^{\ell} \leq \frac{Kte^{-\alpha t}}{\alpha}.
\]

\[
K \sum_{\ell=0}^{t^2} (1 - \frac{\alpha}{t})^{\ell} \leq \frac{Kte^{-\alpha t}}{\alpha}.
\]
The first term on the right of (2.35) is now written as

\[
(2.38) \quad \frac{t^2}{\sum_{\ell=0} \left(1 - \frac{\alpha^\ell}{t}\right) f^{(\ell)}(u)}(t) = \sum_{\ell=0} \left(1 - \frac{\alpha^\ell}{t}\right) \left[ \int_0^{t-t/\ln t} f(t-u)dG^{(\ell)}(u) + \int_{t-t/\ln t}^t f(t-u)dG^{(\ell)}(u) \right].
\]

The first term on the right side of (2.38) is bounded above by

\[
(2.39) \quad \frac{t}{\alpha} o\left(\frac{(\ln t)^2}{t^2}\right) = o(t^{-1}),
\]

by the asymptotic behavior of \( f \) and summing the geometric series.

The second term on the right side of (2.38) is split into the three terms, ignoring distinctions between \( c \) and \( c \),

\[
(2.40) \quad \sum_{\ell=0}^{\sqrt{t/\ln t}} + \sum_{\ell=\sqrt{t/\ln t}} f(t-u)dG^{(\ell)}(u).
\]

Since \( |f| \leq 1 \), Chebyshev's inequality yields

\[
(2.41) \quad \int_{t-t/\ln t}^t |f(t-u)|dG^{(\ell)}(u) \leq \frac{a\ell}{(t-t/\ln t - \ell\mu)^2},
\]

where

\[
(2.42) \quad 0 < a^2 = \int_0^{\infty} (t-\mu)^2 dG(u).
\]

An application of (2.41) to the first term of (2.40) yields

\[
(2.43) \quad \left| \sum_{\ell=0}^{\sqrt{t/\ln t}} \int_{t-t/\ln t}^t f(t-u)dG^{(\ell)}(u) \right| \leq \frac{B}{t(\ln t)^2}
\]

for some \( B > 0 \).
The Central Limit theorem may be applied to the second and third terms of (2.40) since the summation index $\ell$ is large.

For $\ell \geq \sqrt{t/\ln t},$

\[(2.44) \quad G^{(\ell)}(t) - G^{(\ell)}(t - t/\ln t) \sim \frac{\ell}{\sqrt{\pi}} \left( \frac{t - \ell u}{a \sqrt{\ell}} - \frac{t - t/\ln t - \ell u}{a \sqrt{\ell}} \right).\]

Applying (2.44) to the second term of (2.40) and using the mean value theorem and

\[(2.45) \quad \varphi(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}\]

yields

\[(2.46) \quad \left| \sum_{\ell = \sqrt{t/\ln t}}^{t} \int_{t-t/\ln t}^{t} f(t-u)dG^{(\ell)}(u) \right| \leq \sum_{\ell = \sqrt{t/\ln t}}^{t} \varphi(\frac{\ell - \ell u}{a \sqrt{\ell}} \frac{t}{a(\ln t) \sqrt{\ell}}) \leq C e^{-t/2 \ln t} \frac{e^{3/2}}{(a \ln t)^{3/2}}, \]

where $C$ is a positive constant.

The Central Limit theorem is applied to the third term of (2.40), and since the arguments are large, use will be made of the standard approximation

\[(2.47) \quad 1 - \xi(x) \sim \frac{\beta}{x} \varphi(x)\]

for some $\beta > 0$, as $x \to \infty$. 
Applying (2.44) and (2.47) to the third term of (2.40) yields

\[
(2.48) \quad \left| \sum_{\ell=t}^{t^2} \int_{t-t/ln t}^{t} f(t-u)dG(u) \right| \\
\leq \beta \sum_{\ell=t}^{t^2} \phi(t/\ell) \left[ \frac{a\sqrt{\ell}}{t-t/ln t - \ell/t} - \frac{a\sqrt{\ell}}{t-\ell/t} \right] \\
\leq \frac{\beta t^2 \phi(-\mu \sqrt{t/ln t})}{a} \left( \frac{1}{t^4} \right)^{1/2} \leq ye^{-\delta t \ln t} \frac{\ln t}{ln t}.
\]

Now (2.37), (2.39), (2.43), (2.46), (2.48) applied to (2.34) yield that for all sufficiently large \( t \), equation (2.14) holds.

Theorem 1. Under assumptions (1.1) to (1.5)

\[
(1.6) \quad \lim_{t \to \infty} tP[Z(t) > 0] = b.
\]

Proof. Combine (2.12) - (2.15) to yield

\[
(2.49) \quad |P[Z(t) > 0] - \frac{b}{t}| \leq G(n)(t) + o(t^{-1}).
\]

Let \( n \to \infty \). The weak law of large numbers yields the result.

III. Extension. The result of Theorem 1 can be strengthened by the method.

Theorem 2. Under the assumptions (1.2) - (1.5) and in addition, for \( t \to \infty \),

\[
(3.1) \quad t^3(1-G(t)) \to 0
\]
and

(3.2) \[ h^{(3)}(1) < \infty, \]

then for \( t \) large,

(3.3) \[ P[Z(t) > 0] \sim \frac{b}{t} + \frac{c \ln t}{t^2} \]

for some unspecified constant \( c \).

Remark. Conditions (3.1) and (3.2) are not as stringent as in [2], where more terms in the asymptotic expansion of \( P(t) \) are given.

Outline of Proof.

Define

(3.4) \[ J_{n+1}(t) = \int_0^t h(J_n(t-u))dG(u) \]

\[ J_0(t) = \begin{cases} 1, & t \leq T \\ 1 - \frac{b}{t} - \frac{c \ln t}{t^2}, & t > T \end{cases} \]

for \( T \gg b \).

As in the proof of (2.13) one obtains

(3.5) \[ 0 \leq U_n(t) - J_n(t) \leq e^{(n)}(t). \]

An expansion of \( h \) about 1 to four terms in the Taylor expansion yields that

(3.6) \[ J_1(t) = J_0(t) + o(t^{-3}). \]
From this and a similar tedious sequence of estimations as in Theorem 1, one obtains

\[(3.7) \quad |J_n(t) - J_0(t)| \leq c^{(n)}(t) + o(t^{-2})\]

and again one lets \( n \to \infty \) and applies the weak law of large numbers to complete the argument.
REFERENCES


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<tr>
<td>4. TITLE (and Subtitle(s))</td>
<td>5. TYPE OF REPORT &amp; PERIOD COVERED</td>
</tr>
<tr>
<td>The Extinction Probability in a Critical Branching Process</td>
<td>TECHNICAL REPORT</td>
</tr>
<tr>
<td>7. AUTHOR(s)</td>
<td>6. PERFORMING ORG. REPORT NUMBER</td>
</tr>
<tr>
<td>Howard J. Weiner</td>
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<tr>
<td>9. PERFORMING ORGANIZATION NAME AND ADDRESS</td>
<td>10. PROGRAM ELEMENT, PROJECT, TASK AREA &amp; WORK UNIT NUMBERS</td>
</tr>
<tr>
<td>Department of Statistics</td>
<td>P-14435-M</td>
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<td>Stanford University</td>
<td></td>
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<tr>
<td>Stanford, CA 94305</td>
<td></td>
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<tr>
<td>11. CONTROLLING OFFICE NAME AND ADDRESS</td>
<td>12. REPORT DATE</td>
</tr>
<tr>
<td>U.S. Army Research Office</td>
<td>May 30, 1978</td>
</tr>
<tr>
<td>Post Office Box 12211</td>
<td></td>
</tr>
<tr>
<td>Research Triangle Park, NC 27709</td>
<td>13. NUMBER OF PAGES</td>
</tr>
<tr>
<td>14. MONITORING AGENCY NAME &amp; ADDRESS (if different from Controlling Office)</td>
<td>15. SECURITY CLASS. (of this report)</td>
</tr>
<tr>
<td></td>
<td>UNCLASSIFIED</td>
</tr>
<tr>
<td>16. DISTRIBUTION STATEMENT (of this Report)</td>
<td>18a. DECLASSIFICATION/DOWNGRADING SCHEDULE</td>
</tr>
<tr>
<td>Approved for Public Release; Distribution Unlimited.</td>
<td></td>
</tr>
<tr>
<td>17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)</td>
<td></td>
</tr>
<tr>
<td>18. SUPPLEMENTARY NOTES</td>
<td>Extinction probability, critical branching process, integral equations iterations.</td>
</tr>
<tr>
<td>The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents. This report partially supported under Office of Naval Research Contract N00014-76-C-0475 (NR-042-267) and issued as Technical Report No. 258.</td>
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<td>20. ABSTRACT (Continue on reverse side if necessary and identify by block number)</td>
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