OPTIMAL SIGNAL SETTINGS FOR THE VARIABLE CYCLE
TRAFFIC LIGHT

BY

JOHN P. LEHOCZKY

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1. Introduction

We consider an isolated intersection of two one-way one lane streets. In a recent paper by Lehoczky [7], a new model of a vehicle actuated traffic signal was introduced and discussed. This signal, called the variable cycle traffic signal, has the following control pattern. A cycle begins when the light turns green for the north flow. The light remains green for at least $g_1$ seconds and at most $g_2$ seconds. Each car in the initial queue and each arriving car extends the green time by $e_1$ seconds up to a maximum extension of $g_2$ seconds. The light turns red for the north, and $L_2$ seconds later it turns green for the east. The light stays green for the east for at least $r_1$ seconds and at most $r_2$ seconds. Each car in the initial queue and each arriving car extends the green time by $e_2$ seconds. At the conclusion of this phase, the light turns red for $L_1$ seconds, and the cycle is completed.

In [1], the steady state queueing behavior and the asymptotic stability conditions of the variable cycle traffic signal are discussed for the special case $g_2 = r_2 = \infty$. In this paper, two additional problems will be solved for this special case. First, the steady state expected total delay per cycle is derived for both east and north directions. Second, it is shown that $e_1 = e_2 = g_1 = r_1 = 1$ is the signal setting which simultaneously minimizes the total expected delay.

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per cycle for both the north and east directions. Thus this setting is optimal with respect to the expected delay per cycle criterion.

2. **Notation**

\[ Z_{i,n} = \text{the number of arrivals in the east direction during } [i,i+1] \]

of the \( n \)th cycle.

\[ L_n = \text{lost time to the east flow for the } n \text{th cycle due to yellow lights and the start up time of the first car.} \]

\[ V_n = \text{length of the east queue at the start of the } n \text{th cycle.} \]

\[ D_n = \text{total delay for the east flow during the } n \text{th cycle.} \]

\[ D_{1n} = \text{total delay for the east flow during } [0,t_n+L_n] \text{ of the } n \text{th cycle.} \]

\[ D_{2n} = \text{total delay for the east flow during } [t_n+L_n,t_n+L_n+r_n] \text{ of the } n \text{th cycle.} \]

\[ t_n = \text{north green time for the } n \text{th cycle.} \]

\[ r_n = \text{east green time for the } n \text{th cycle.} \]

\[ f(i,J) = P(Z_{0,n}^+ + \ldots + Z_{J-1,n} = i) \text{ when } Z_{0,n}^+ \ldots , Z_{J-1,n} \text{ are i.i.d and defined as above.} \]

For the special case \( r_2 = \infty \) to be discussed in this paper, the random variables \( \{V_{n,n=1,2,\ldots}\} \) must simply equal the number of arrivals during the lost time before the north cycle green time. Hence, if \( M_{n+1} \) is this time period, each of these random variables will have distribution given by \( P(V_n = V) = f(V,M_{n+1}) \text{ } V = 0,1,\ldots \text{ and } n = 1,2,\ldots . \)

If \( r_2 < \infty \), then \( V_n \) will consist of not only the arrivals during the lost time but also those vehicles left in the queue when the light changes to red.
3. **Assumptions**

a) Each car takes a constant time, $\tau$, to cross the intersection. We assume without loss of generality $\tau = 1$.

b) If the light is green for either direction and the queue is empty at time $k$ of the $n$th cycle, then if $i$ cars arrive during $[k,k+1)$, the queue size at time $k+1$ will be $i$. The first car arriving is not served immediately.

c) $L_n$, $n = 1, 2, \ldots$ are independent identically distributed random variables which are independent of all other random variables. $E[L_n] = \mu_2$, $\text{Var} [L_n] = \delta_2^2 < \infty$.

d) $M_n$, $n = 1, 2, \ldots$ are independent identically distributed random variables which represent the lost time to the north prior to the $n$th cycle. They are independent of all other random variables. $E(M_n) = \mu_1$, $\text{Var} (M_n) = \delta_1^2 < \infty$.

e) $\{Z_{i,j}\}$ are independent identically distributed random variables with $E(Z_{i,j}) = p_2$, $E(Z_{i,j}^2) = \tau_2 < \infty$. The north input process also consists of independent identically distributed random variables, and these two processes are independent.

f) $p_1 e_1 + p_2 e_2 < 1$.

g) $g_2 = r_2 = \infty$, $1 \leq r_1 < \infty$, $1 \leq g_1 < \infty$ with $e_1$ and $e_2$ positive integers.
4. **Expected Total Delay Per Cycle For The East Direction**

In this section, the delay to cars in the east flow for the $n^{th}$ cycle will be found. By arguments identical to those presented here, the delay to cars in the north flow can be found.

To find $E(D_n)$, we first condition on the random variables $t_n + L_n$, $V_n$, and $Q_{2n}$. As defined in [1], $Q_{2n} = \text{the east queue size at the start of the green time for the east flow during the } n^{th} \text{ cycle.}$

$$E(D_n | V_n, t_n + L_n) = \sum_{i=0}^{\infty} P(Q_{2n} = i + V_n | t_n + L_n, V_n)$$

$$\cdot [E(D_n | V_n, t_n + L_n, Q_{2n} = i + V_n) + E(D_{2n} | V_n, t_n + L_n, Q_{2n} = i + V_n)]$$

Each of the expressions on the right hand side of (1) will now be evaluated.

$$Q_{2n} = V_n + \sum_{i=0}^{t_n + L_n - 1} Z_{i, n-1}$$

(2)

$$P(Q_{2n} = i + V_n | t_n + L_n, V_n) = f(i, t_n + L_n | t_n + L_n)$$

Substitution of (3) into (1) yields

$$E(D_n | V_n, t_n + L_n) = \sum_{i=0}^{\infty} f(i, t_n + L_n | t_n + L_n) [E(D_n | Q_{2n} = i + V_n, V_n, t_n + L_n)$$

$$+ E(D_{2n} | Q_{2n} = i + V_n, V_n, t_n + L_n)] .$$

Now $E(D_{1n} | Q_{2n} = i + V_n, V_n, t_n + L_n)$ consists of two distinct parts. First, the original $V_n$ cars will be delayed the entire $t_n + L_n$ seconds. Second, there is the delay due to the $i$ cars arriving
during \([0, t_n + L_n]\) of the \(n\)th cycle. The former delay is clearly \(V_n(t_n + L_n)\). The latter may be computed as follows.

Each of the \(Z_{0,n-1}\) cars arriving during \([0,1]\) will on the average have a total delay of \(t_n + L_n - 1 + 1/2\), because each will wait during the interval \([1, t_n + L_n]\) and each is delayed an average of \(1/2\) during the interval \([0,1]\) assuming the cars arrive uniformly during this interval.

In general, each of the \(Z_{i,n-1}\) cars arriving during the interval \([i, i+1]\) will have a delay of \(t_n + L_n - i - 1 + 1/2\) for \(i = 0, 1, \ldots, t_n + L_n - 1\).

\begin{align*}
\tag{5} D_{ln} = V_n(t_n + L_n) + Z_{0,n-1}(t_n + L_n - 1/2) + Z_{1,n-1}(t_n + L_n - 3/2) \\
&\quad + Z_{2,n-1}(t_n + L_n - 5/2) + \ldots + Z_{t_n + L_n - 1,n-1}(1/2). \\
\end{align*}

But \(Z_{0,n-1}, \ldots, Z_{t_n + L_n - 1,n-1}\) are independent identically distributed with finite mean

\begin{align*}
\tag{6} E(Z_{i,n-1}|Z_{0,n-1} + \ldots + Z_{t_n + L_n - 1,n-1} = x) &= \frac{x}{t_n + L_n} \\
&\quad \text{for } i = 0, 1, \ldots, t_n + L_n - 1.
\end{align*}

\begin{align*}
\tag{6} E(D_{ln}|Q_{2n} = i + V_n, t_n + L_n, V_n) &= E(V_n(t_n + L_n) + Z_{0,n-1}(t_n + L_n - 1/2) \\
&\quad + Z_{1,n-1}(t_n + L_n - 3/2) + \ldots + Z_{t_n + L_n - 1,n-1}(1/2)|Q_{2n} = i + V_n, t_n + L_n) \\
\end{align*}
\[-V_n(t_n + L_n) + \frac{i}{t_n + L_n} \left\{ (t_n + L_n - 1) \cdot (t_n + L_n - 2) + \ldots + 1 + \frac{(t_n + L_n)}{2} \right\} \]

\[= V_n(t_n + L_n) + \frac{i}{t_n + L_n} \left\{ \frac{(t_n + L_n)(t_n + L_n - 1)}{2} + \frac{(t_n + L_n)}{2} \right\} \]

\[= V_n(t_n + L_n) + \frac{(t_n + L_n)}{2} = (V_n + \frac{i}{2})(t_n + L_n). \]

\[
(7) \quad \mathbb{E}(D_{1n} | Q_{2n} = i + V_n, t_n + L_n, V_n) = (V_n + \frac{i}{2})(t_n + L_n) \quad \text{and substitution into (4) yields} \]

\[
(8) \quad \mathbb{E}(D_n | V_n, t_n + L_n) = V_n(t_n + L_n) + \frac{t_n + L_n}{2} \sum_{i=0}^{\infty} f(i, t_n + L_n | t_n + L_n) \]

\[+ \sum_{i=0}^{\infty} f(i, t_n + L_n | t_n + L_n) \mathbb{E}(D_{2n} | Q_{2n} = i + V_n, V_n, t_n + L_n) \]

\[= \sum_{i=0}^{\infty} f(i, t_n + L_n | t_n + L_n) \mathbb{E}(D_{2n} | Q_{2n} = i + V_n, V_n, t_n + L_n) \]

\[= V_n(t_n + L_n) + (t_n + L_n)^2 \frac{D_2}{2} + \sum_{i=0}^{\infty} f(i, t_n + L_n | t_n + L_n) \]

\[\cdot \mathbb{E}(D_{2n} | Q_{2n} = i + V_n, V_n, t_n + L_n) \]

We now turn to \( \mathbb{E}(D_{2n} | Q_{2n} = i + V_n, V_n, t_n + L_n) = \mathbb{E}(D_{2n} | Q_{2n} = i + V_n). \)

Suppose we graph the number of cars in the east queue against time for the time interval \([t_n + L_n, t_n + L_n + r_n).\)
First, it should be clear that there may be times when the queue is empty, but the light is still green for the east flow. An example of this would be if $Q_{2n} = 1$, $r_1 = 10$, $e_2 = 10$. Then the light will remain green for at least 20 seconds, since the one car extends $r_1 = 10$ by 10 seconds; however, if no cars arrive during this time period, the queue will be empty for 19 of those 20 seconds.

Second, $E(D_{2n} | Q_{2n} = 1 + V_n)$ is the expected area under the curve illustrated in Figure 1. This holds because if $c$ cars are in the queue at time $k$, then these cars are delayed a total of $c$ during $[k,k+1)$. To the area under the curve, we must add the delay to the cars arriving during $[k,k+1)$. The sum of these two represents the total delay during $[t_n + L_n, t_n + L_n + r_n)$. Thus we must find the expected area under the curve.

We define

- $f_1$ = the time at which the east queue first reaches size 1
- $f_1 = 0$ if $Q_{2n} = 1$.
- $f_0$ = the time at which the east queue first becomes empty.

We assume for the moment $Q_{2n} > 0$. Hence the total area under
the curve may be divided into two parts: the area in the interval 
\([t_n + L_n, f_0] \) and the area in the interval \([f_0, t_n + L_n + r_n] \). We now find each of these components.

![Figure 2](image)

The first area may be found by using the theory of storage. A dam has initial height \( Q_{2n} \). After each second, one unit is released and a random number of units added, \( Z_0, n, Z_1, n, \ldots \).

Let \( A(X) = \text{expected area under the curve with initial height } X, X > 0 \).

\[
A(X) = X + \sum_{i=0}^{\infty} P(i \text{ arrivals in } [0, 1))[\frac{1}{2} + A(X+i-1)]
\]

or

\[
A(X) = X + \frac{P_0}{2} + \sum_{i=0}^{\infty} f(i, 1)A(X+i-1).
\]

Equation (9) follows from consideration of the number of cars arriving during \([0, 1]\). If 1 cars arrive, then \( X + \frac{1}{2} \) is the total delay during \([0, 1]\) and the curve moves to \((1, X+i-1)\). From the i.i.d. input assumption, \( A(X+i-1) \) represents the expected total delay for the time interval \([1, f_0]\).
A second equation may be found by decomposing the area under the curve in Figure 2 into the following sub-areas defined by:

(a) \((0,0), (0,1), (f_1,1), (f_1,0)\)
(b) \((0, q_{2n}), (0,1), (f_1,1)\), the curve on the top.
(c) \((f_1,0), (f_1,1), (f_0,0)\), the curve on the top.

Evaluation of each of these areas yields:

area (a) \(= f_1\)
area (b) \(= A(q_{2n} - 1)\)
area (c) \(= A(1)\)

\[(10) \quad A(X) = A(X-1) + A(1) + E(f_1 | X) \quad X \geq 1, \quad A(0) = 0\]

By an application of Wald's equation, we find

\[E(f_1 | X) = E(\text{first emptiness time} | \text{initial height} = X-1) = \frac{X-1}{1-p_2}\]

\[(11) \quad A(X) = A(X-1) + A(1) + \frac{X-1}{1-p_2}, \quad X \geq 1, \quad A(0) = 0\]

Equation \((11)\) may be iterated as follows:

\[
A(X) = A(X-1) + A(1) + \frac{X-1}{1-p_2} = A(X-2) + 2A(1) + \frac{(X-1)+(X-2)}{1-p_2} \\
= A(X-3) + 3A(1) + \frac{(X-1)+(X-2)+(X-3)}{1-p_2} = \ldots \\
= XA(1) + \frac{(X-1)+(X-2)+\cdots+1}{1-p_2}
\]

\[
\ldots
\]

\[(12) \quad A(X) = XA(1) + \frac{X(X-1)}{2(1-p_2)}, \quad X \geq 0 \text{ an integer}\]
Substitution of (12) into (9) yields:

\[ XA(1) + \frac{X(X-1)}{2(1-p_2)} = X + \frac{P_2}{2} + \sum_{i=0}^{\infty} f(i, 1) \left[ (X+1-l)A(1) + \frac{(X+1-l)(X+1-2)}{2(1-p_2)} \right] \]

or

\[ A(1)[X-(X-1+p_2)] = X + \frac{P_2}{2} + \frac{(X-1)(X-2)}{2(1-p_2)} + \frac{P_2(X-1+X-2)}{2(1-p_2)} + \frac{\tau_2}{2(1-p_2)} + \frac{X(X-1)}{2(1-p_2)} \]

\[ \therefore \]

(13) \quad \quad A(1) = \frac{P_2(1-p_2) + \tau_2 + 2-3p_2}{2(1-p_2)^2} = \frac{1}{2(1-p_2)^2} + \frac{1-P_2}{2(1-p_2)^2} + \frac{P_2^2 + \tau_2}{2(1-p_2)^2}. \]

Substitution of (13) into (12) yields:

\[ A(X) = X \left[ \frac{1}{2(1-p_2)} + \frac{1-P_2}{2(1-p_2)^2} \right] + \frac{X(X-1)}{2(1-p_2)} = \frac{x^2}{2(1-p_2)} + \frac{X(1-P_2^2 + \tau_2)}{2(1-p_2)^2} \]

\[ \therefore \]

(14) \quad \quad A(X) = \frac{x^2}{2(1-p_2)} + \frac{X(1-P_2^2 + \tau_2)}{2(1-p_2)^2} \quad X \text{ an integer } \geq 0. \]

We note \( 1-p_2^2 + \tau_2 = 1-P_2^2 + \tau_2 \geq 1-p_2 > 0. \)

Thus the expected delay during the interval \([0, f_0]\) is

\[ \frac{Q_{2n}}{2(1-p_2)} + \frac{Q_{2n}(1-P_2^2 + \tau_2)}{2(1-p_2)^2}. \]

We now turn to the evaluation of the delay during the time interval \([f_0, t_n + L_n + r_n]\). At time \(f_0\), the green time has been extended to \(r_1 + f_0(e_2 - 1)\), since each car crossing the intersection \((f_0\) in all since one crosses each second) has extended the light by \(e_2\) seconds
beyond the initial length $r_1$. $f_0$ seconds have elapsed, hence $r_1 + f_0 (e_2 - 1)$ seconds remain, and the queue is empty.

Suppose in general that $s$ seconds remain, and the queue is empty.

Let $W_s =$ total expected delay when the queue is empty and $s$ seconds remain.

**Lemma 1.** Suppose the queue is empty at time $t$ and $s$ seconds of green time remain ($s > 0$). Then with probability 1, there exists a time point $t' > t$ such that the queue is empty and $s-1$ seconds of green time remain.

**Proof.** Consider $\{(u_i, v_i), i=1, 2, \ldots\}$ where $u_i$ is the sequence of time points beyond $t$ at which the queue is empty and $v_i$ is the green time remaining at time $u_i$. Since $r_n < \infty$ w.p.1, the sequence has only finitely many members w.p.1. Consider $u_{i+1}$. $u_{i+1} - u_i - 1$ represents the number of cars served during $[u_i, u_{i+1})$, hence it also represents the number of arrivals during $[u_i, u_{i+1})$. This follows, because the queue is not empty during $[u_i, u_{i+1}).$ Thus at time $u_{i+1}$

$$v_{i+1} = v_i + (u_{i+1} - u_i - 1) e_2 - (u_{i+1} - u_i),$$

since $u_{i+1} - u_i$ seconds have elapsed. But then $v_{i+1} = v_i - 1 + (u_{i+1} - u_i - 1)(e_2 - 1)$ and $(u_{i+1} - u_i - 1)(e_2 - 1) \geq 0$. Hence either $v_{i+1} = v_i - 1$ or $v_{i+1} \geq v_i$.

But $r_n < \infty$ w.p.1. implies that there exists $n$ such that $v_n = 0$ w.p.1. Hence there exists $t' = u_k$ for some $k$ at which $v_k = s - 1$. Q.E.D.

**Lemma 1** allows the delay during $[f_0, r_n - t + L_n]$ to be divided into $r_1 + f_0 (e_2 - 1)$ parts. Let $(u_i, i=0, 1, \ldots, r_1 + f_0 (e_2 - 1))$ be the sequence of time points at which the queue is first empty with $i$ seconds of green time remaining. Thus the delay during $[f_0, r_n + L_n + r_n]$ equals
the sum of the delays during \([u_{i\_1}, u_{i\_1-1}]\). By the i.i.d. input assumption each of these delays must have the same expectation. Hence \(W_s = sW_1\) for \(s \geq 0\) an integer, and for our problem the delay during 

\([f_{0\_1}, n_{n\_1} + r_{n\_1}] = (r_{1\_1} + f_{0\_1}(e_{2\_1}-1))W_1\). Thus we need only find \(W_1\).

Suppose the queue is empty, one second of green time remains, and \(i\) cars arrive during the next second; then \(A(i)\) is the expected total delay until the queue is next empty. If the time to the next emptiness is \(J\) seconds, then \(W_J(e_{2\_1}-1)\) is the expected total delay after emptiness.

\[W_1 = \sum_{i=1}^{\infty} f(i,1) \left[ \frac{1}{2} + A(i) + \sum_{j=1}^{\infty} P(J \text{ seconds until the next emptiness}) \right] \]

\(W_J(e_{2\_1}-1)\)

\[P(J \text{ seconds until the next emptiness was found by Lloyd in [2]}}\]

to be \(\frac{1}{J} f(J-i, J)\).

\[W_1 = \frac{p_2}{2} + \sum_{i=1}^{\infty} f(i,1) \left( \frac{i^2}{2(1-p_2)} + \frac{1(1-p_2)^2+p^2}{2(1-p_2)^2} + W_1(e_{2\_1}-1) \sum_{i=1}^{\infty} f(i,1) \right)

\[\sum_{j=1}^{\infty} f(J-i, J) \)

\[= \frac{p_2}{2} + \frac{\tau_2(1-p_2)+p_2(1-p_2)^2+p^2}{2(1-p_2)^2} + W_1(e_{2\_1}-1) \sum_{i=1}^{\infty} f(i,1) \sum_{j=0}^{\infty} f(J, J+i) \]

By equation (20) in [1], \(\sum_{J=0}^{\infty} f(J, J+i) = \frac{1}{1-p_2}\)
\[
\sum_{i=1}^{\infty} \frac{f(i,1)}{1-p_2} \sum_{j=0}^{\infty} f(j,j+1) = \frac{p_2}{1-p_2}
\]

\[
W_1 = \frac{p_2^2}{2} + \frac{\tau_2 (1-p_2) + p_2 (1-p_2^2+\tau_2)}{2(1-p_2)^2} + W_1 (e_2-1) \frac{p_2}{1-p_2}
\]
or

\[
W_1 = \frac{2p_2^2 - 3p_2^2 + \tau_2}{2(1-p_2)(1-p_2e_2)} \quad \text{after simplification.}
\]

\[
2p_2^2 - 3p_2^2 + \tau_2 = 2p_2 - 2p_2^2 + \epsilon_2^2 = 2p_2(1-p_2) + \epsilon_2^2 \geq 0 \quad \therefore \quad W_1 \geq 0
\]

\[
W_s = s \left( \frac{2p_2^2 - 3p_2^2 + \tau_2}{2(1-p_2)(1-p_2e_2)} \right), \quad s \geq 0
\]

Thus the expected total delay after the queue first empties is

\[
[r_1 + p_0 (e_2-1)] \left[ \frac{2p_2^2 - 3p_2^2 + \tau_2}{2(1-p_2)(1-p_2e_2)} \right]
\]

\[
\mathbb{E}(p_{2n}|Q_{2n}=1+y_n) = \frac{(i+y_n)^2}{2(1-p_2)} + \frac{(i+y_n)(1-p_2^2+\tau_2)}{2(1-p_2)^2}
\]

\[
+ \left[ r_1 + (e_2-1) \mathbb{E}(p_0|Q_{2n}=1+y_n) \right] \left[ \frac{2p_2^2 - 3p_2^2 + \tau_2}{2(1-p_2)(1-p_2e_2)} \right]
\]

\[
= \frac{(i+y_n)^2}{2(1-p_2)} + \frac{(i+y_n)(1-p_2^2+\tau_2)}{2(1-p_2)^2} + \left[ r_1 + (e_2-1) \frac{i+y_n}{1-p_2} \right] \left[ \frac{2p_2^2 - 3p_2^2 + \tau_2}{2(1-p_2)(1-p_2e_2)} \right]
\]
\[
(20) \quad \sum_{i=0}^{\infty} f(i, t_n + L_n | t_n + L_n) E(D_{2n} | q_{2n-1} = i + V_n) = \frac{(t_n + L_n)^2 \sigma_2^2 + (t_n + L_n)^2 \mu_2^2}{2(1-p_2)}
\]
\[
+ \frac{2\gamma_n p_2 (t_n + L_n) + \gamma_n^2}{2(1-p_2)} + \frac{(p_2 (t_n + L_n) + V_n)(1-p^2 - p_2^2 + \tau_2)}{2(1-p_2)^2}
\]
\[
+ \left[ r_1^{e_2-1} \left( \frac{p_2 (t_n + L_n) + V_n}{1-p_2} \right) \right] \left[ \frac{2p_2 - 3p_2^2 + \tau_2}{2(1-p_2)(1-p_2e_2)} \right]
\]

Combining (20) and (8) we find

\[
(21) \quad E(D_n | t_n + L_n, V_n) = V_n (t_n + L_n) + (t_n + L_n)^2 \frac{p_2}{2} + \frac{(t_n + L_n)^2 \sigma_2^2 + (t_n + L_n)^2 \mu_2^2}{2(1-p_2)}
\]
\[
+ \frac{2\gamma_n p_2 (t_n + L_n) + \gamma_n^2}{2(1-p_2)} + \frac{(p_2 (t_n + L_n) + V_n)(1-p^2 - p_2^2 + \tau_2)}{2(1-p_2)^2}
\]
\[
+ \left[ r_1^{e_2-1} \left( \frac{p_2 (t_n + L_n) + V_n}{1-p_2} \right) \right] \left[ \frac{2p_2 - 3p_2^2 + \tau_2}{2(1-p_2)(1-p_2e_2)} \right]
\]

Now \( E[E(D_n | t_n + L_n, V_n) | t_n + L_n] = E[D_n | t_n + L_n] \) and

\[ \begin{align*}
E(V_n) &= p_2 \mu_2, \quad \text{Var}(V_n) = \text{Var}(Z_{0, n} + \cdots + Z_{L_n-1, n}) \\
&= \text{Var}(E(Z_{0, n} + \cdots + Z_{L_n-1, n} | L_n)) \\
+ E(\text{Var}(Z_{0, n} + \cdots + Z_{L_n-1} | L_n)) &= \text{Var}(p_2 L_n) + E(L_n \sigma_2^2) = p_2^2 \sigma_2^2 + \mu_2 \sigma_2^2
\end{align*} \]

under the assumption that \( r_2 = \infty \), hence \( P(V_n = v) = f(V, L_2), \ V \geq 0 \).
\[
+ \left[ r_1 + \left( e_2 - 1 \right) \frac{3p_2(t_n + L_n) + p_2 \mu_2}{1 - p_2} \right] \cdot \left[ \frac{2p_2 - 3p_2^2 + \tau_2}{2(1 - p_2)(1 - p_2^2e_2)} \right].
\]

Now \( E(t_n + L_n) = E(t_n) + \mu_2, \ E(t_n + L_n)^2 = E(t_n^2) + 2\mu_2 E(t_n) + \delta_2^2 + \mu_2^2 \)

\[
\therefore \ E(D_n) = E(E(D_n | t_n + L_n)) \quad \text{and hence}
\]

\[
(23) \quad E(D_n) = \frac{1}{2} \mu_2 \mathbb{E}(\mu_2 + E(t_n)) + \frac{p_2}{2} \left( E(t_n^2) + 2\mu_2 E(t_n) + \delta_2^2 + \mu_2^2 \right)
+ \frac{(\mu_2 + E(t_n))^2}{2(1 - p_2)} + \frac{p_2 E(t_n^2) + 2\mu_2 E(t_n) + \delta_2^2 + \mu_2^2}{2(1 - p_2)}
+ \frac{2p_2 \mu_2 (\mu_2 + E(t_n)) + p_2 \delta_2^2 + \mu_2^2 e_2^2 + p_2 \mu_2^2}{2(1 - p_2)}
+ \frac{(p_2 E(t_n) + \mu_2^2 + p_2 \mu_2^2)(1 - p_2^2 - p_2^2 + \tau_2)}{2(1 - p_2)^2}
+ \left[ r_1 + \left( e_2 - 1 \right) \frac{3p_2 \mu_2 + E(t_n)}{1 - p_2} \right] \cdot \left[ \frac{2p_2 - 3p_2^2 + \tau_2}{2(1 - p_2)(1 - p_2^2e_2)} \right].
\]

It has been shown, [1], that for \( p_1 e_1 + p_2 e_2 < 1 \) there is a steady state distribution and \( E(t_n) \rightarrow E(t), \ E(t_n^2) \rightarrow E(t^2) \). Thus \( E(D_n) \rightarrow E(D) \), the steady state expected total delay for the east flow. We had the following formulas from [1]:

\[
E(t) = \frac{e_1 + e_1 \left( \frac{a}{1 - b} \right)}{1 - p_1 e_1}, \quad E(t^2) = \frac{e_1^2}{(1 - p_1 e_1)^2} + \frac{e_1^2}{(1 - b^2)} \left( k_1 + k_2 \frac{a}{1 - b} \right) + \frac{(e_1 + e_1 \left( \frac{a}{1 - b} \right))^2 e_2^2}{(1 - p_1 e_1)^3}
\]

where
\[ a = p_1^{\mu_1} + \frac{p_1 r_1}{1-p_2 e_2} + \frac{p_1 p_2 e_2^{\mu_2}}{l-p_2 e_2} + \frac{p_1 p_2 e_2 g_1}{(1-p_1 e_1)(1-p_2 e_2)} \]

\[ b = p_1 e_1 + p_2 e_2 \]

\[ k_1 = \frac{p_1^2 p_2^2 e_2^2 e_1^2}{(1-p_2 e_2)^2 (1-p_1 e_1)^3} + \frac{p_1^2 p_2^2}{(1-p_1 e_1)^2 (1-p_2 e_2)^2} + \frac{g_1 p_2 e_2^2}{(1-p_1 e_1)(1-p_2 e_2)^2} \]

\[ + \frac{\mu_2^2 p_1 e_2^2}{(1-p_2 e_2)^2} + \left( p_2^{\mu_2} + \frac{p_2 g_1}{1-p_1 e_1} \right) \left( \frac{e_1 e_2}{l-p_2 e_2} + \frac{p_1^2 e_2^2}{(1-p_2 e_2)^2} \right) \]

\[ + \frac{p_2^2 p_1 e_2^2}{(1-p_2 e_2)^2} + \mu_1^2 + \frac{p_1^2}{l-p_2 e_2} + \frac{p_1^2 e_2^2}{(1-p_2 e_2)^3} \]

\[ k_2 = \frac{e_1 e_2 p_2}{(1-p_1 e_1)(1-p_2 e_2)} \left( 1 + \frac{p_1^2 p_2 e_2 e_1^2}{(1-p_1 e_1)^2 (1-p_2 e_2)} \right) + \frac{p_2^2 e_1 e_2}{(1-p_1 e_1)(1-p_2 e_2)^3} \]

\[ (24) \ E(D) = \left( \frac{p_2^2}{2(1-p_2)} \right) E(t^2) + \frac{E(t)}{2(1-p_2)^2} \left( c_2^2(1-p_2) + \mu_2^2 p_2^2(1-p_2) \right) \]

\[ + p_2(1-p_2-p_2^2 r_2) + \left( e_2 - 1 \right) p_2 \left( \frac{2 p_2 - 3 p_2^2 + r_2}{(1-p_2)^2} \right) \left( \frac{\mu_2 e_2}{l-p_2} + \frac{p_2^2 e_2}{1-p_2} \right) \]

\[ + \frac{r_2^2}{1-p_2} + \frac{p_2^2}{(1-p_2)^2} - \frac{p_2^2}{(1-p_2)^2} + \frac{p_2^2}{(1-p_2)^2} + r_1 \left( \frac{2 p_2 - 3 p_2^2 + r_2}{2(1-p_2)(1-p_2 e_2)} \right) \]

\[ + p_2^2 e_2 \left( \frac{2 p_2 - 3 p_2^2 + r_2}{(1-p_2)^2 (1-p_2 e_2)} \right) \text{ with } E(t^2) \text{ and } E(t) \text{ defined above.} \]
Thus we have found the total expected delay for the east cycle. It may easily be seen that the total expected delay for the north cycle may be obtained by identical arguments. The result will be the same as (24) except the relevant parameters must be changed (for example \( p_2 \) becomes \( p_1 \), \( p_1 \) becomes \( p_2 \), \( g_1 \) becomes \( r_1 \), etc.).

5. **Optimum Signal Settings.**

We wish to determine the values of the parameters \( g_1, r_1, e_1 \) and \( e_2 \) which minimize \( E(D) \) subject to \( p_1 e_1 + p_2 e_2 < 1 \). We notice the following:

a is an increasing function of \( r_1, g_1, e_1 \), and \( e_2 \).
b is an increasing function of \( e_1 \) and \( e_2 \).
k_1 is an increasing function of \( r_1, g_1, e_1 \), and \( e_2 \).
k_2 is an increasing function of \( r_1, g_1, e_1 \), and \( e_2 \).
E(D) is an increasing function of \( a, b, k_1, k_2, d_1, r_1, e_1 \), and \( e_2 \).

In order to minimize \( E(D) \), one must choose the smallest possible values of \( a, b, k_1, k_2, g_1, r_1, e_1 \) and \( e_2 \). Since \( 1 \leq g_1, r_1 < \infty \), we immediately find \( g_1 = r_1 = 1 \) to be optimum. \( e_1 \) and \( e_2 \) are restricted to be positive integers, hence \( e_1 = e_2 = 1 \) is also optimum.

The signal setting which minimize the total steady state per cycle delay for the east flow is \( g_1 = r_1 = e_1 = e_2 = 1 \). It is clear that this setting will also minimize the total steady state per cycle delay for the north direction by inspection after writing (24) modified for the north flow.
For this setting

\[
a = p_1^2 + \frac{p_1(1-p_1)+p_1p_2}{(1-p_1)(1-p_2)} + \frac{p_1p_2}{1-p_2}
\]

\[
b = p_1 + p_2
\]

\[
k_1 = \frac{\frac{p_1^2}{p_2} + \frac{p_1^2}{p_2(1-p_2)^3}}{(1-p_2)^2} + \frac{\frac{2}{p_1p_2} + \frac{2}{p_1(1-p_2)^2} + \frac{\mu_2^2}{(1-p_2)^2}}{(1-p_2)^2} + \frac{a^2}{2}
\]

\[
+ \frac{\frac{p_2}{p_2^2} + \frac{p_2}{p_2(1-p_2)^2} + \frac{2}{p_1p_2} + \frac{2}{p_1(1-p_2)^2} + \frac{\mu_2^2 + 2p_2^2}{(1-p_2)^2}}{(1-p_2)^2} + \frac{p_1p_2}{(1-p_2)^3}
\]

\[
k_2 = \frac{\frac{2}{p_1^2} + \frac{2}{p_1(1-p_2)^2}}{(1-p_1)(1-p_2)^2} \left( 1 + \frac{\frac{2}{p_1p_2} + \frac{2}{p_1(1-p_2)^2}}{(1-p_2)^2} \right) + \frac{c_1^2}{(1-p_1)(1-p_2)^3} + 2ab
\]

and

\[
E(\Delta) = E(t^2) \frac{p_2^2}{(1-p_2)} + E(t) \left( \frac{\frac{2}{p_1p_2} + \frac{2}{(1-p_2)^2} + \frac{p_2^2}{(1-p_2)^2}}{2(1-p_2)^2} \right)
\]

\[
+ \frac{\frac{\mu_2^2}{2} + \frac{p_2^2}{1-p_2} + \frac{p_2^2}{1-p_2} + \frac{2\mu_2^2 + 2p_2^2}{(1-p_2)^2} - \frac{p_2^2}{(1-p_2)^2}}{(1-p_2)^2} - \frac{p_2^2}{(1-p_2)^2}
\]

\[
+ \frac{p_2^2}{(1-p_2)^2} + \frac{1}{2} \frac{2p_2^2 + \mu_2^2}{p_2^2 + \mu_2^2}
\]

\[
E(t) = \frac{1 + \frac{a}{1-b}}{1-p_1}
\]

\[
E(t^2) = \frac{1 + \frac{2a}{1-b} + \frac{k_1 + k_2}{1-b} \left( \frac{1}{1-b^2} \right) + \frac{1 + a}{1-b^2}}{(1-p_1)^2} + \frac{1 + \frac{a}{1-b}}{1-p_1^3}
\]
References
