A DEVELOPMENT AND EXPOSITION OF THE RELATIONSHIP BETWEEN
CONTINUOUS AND DISCRETE MODELS OF SINGLE
LANE TRAFFIC FLOW

BY

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0. Introduction.

During the last fifteen years or so two approaches to the study of
traffic flow have received considerable attention in the literature.
They are:

(i) The continuum model, or hydrodynamic analogy, which attempts
to study traffic flow as though it could be approximated by a continuous
fluid; and

(ii) The discrete, or microscopic approach, known under various
names such as car-following, or follow-the-leader theory, which focuses
attention on a particular driver and studies his responses to stimuli,
namely, the manoeuvres of the vehicle immediately in front of
him. The prevalent point of view seems to have been that (i) is a
macropscopic approach and (ii) is a microscopic approach and for this
reason the two theories are looking at different aspects of the problem.

In this report we give an elucidation of both of these theories
primarily to show that for single lane traffic flow the two theories
are not as dissimilar as they appear on the surface; in fact they are so
similar that they can be thought of as approximations to one another at
the microscopic level.
It is necessary in the exposition that follows to employ notation immediately. The reader is asked to bear with this. This seemed the best way of bringing out the salient points in an instructive manner.

1. The Continuous Fluid Analogy.

1.1 Notational Introduction.

In what follows

\[ x = \text{position along the road} \]
\[ t = \text{time} \]

and \( q, k, s, v \) are functions of the pair \((x, t)\) with the following interpretations that are standard in the literature:

\[ q = \text{flow} \]
\[ k = \text{concentration} \]
\[ s = \text{spacing} \]
\[ v = \text{velocity}. \]

If the units of \( x, t \) are \( L, T \) respectively, then those of \( q, k, s, v \) are \( T^{-1}, L^{-1}, L, LT^{-1} \) respectively.

The function \( s \) is introduced purely for convenience in relating the fluid analogy to microscopic theories of traffic flow. It is defined by the relation

\[ sk = 1. \]

Whereas \( k \) is analogous to the mass density of the physical situation, \( s \) is analogous to specific volume. In the continuous fluid analogy for single lane traffic flow the situation is idealized by postulating
an absolutely continuous distribution of cars along the road. The
density \( k(\cdot, t) \) of this distribution at time epoch \( t \) is assumed
to be such that \( \frac{\partial k}{\partial t}, \frac{\partial k}{\partial x} \) exist. These assumptions enable us to express
the principle of "conservation of vehicles" (analogous to the principle
of conservation of mass in fluid mechanics) in the form of the equation
of continuity

\[
\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1.4}
\]

In addition the assumption of a continuous mass distribution enables one
to derive the well-known relation

\[
q = kv \tag{1.5}
\]

which we will also assume.

For any function \( \phi = \phi(x, t) \) of the pair \( (x, t) \), \( \dot{\phi} \) will denote
the convective time derivative. That is,

\[
\dot{\phi} = \frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x} \tag{1.6}
\]

Intuitively \( \dot{\phi} \) is the rate of change of \( \phi \) as seen by an observed
traveling through the region of the fluid with the same velocity as the
fluid itself. In terms of traffic \( \dot{\phi} = \dot{\phi}(x, t) \) is the rate of change
of \( \phi \) as seen by a driver whose position at time \( t \) is \( x \). For example,

\[
\ddot{v}[\dot{x}(t), t] = \ddot{v}(t) = \dddot{x}(t) \tag{1.7}
\]

is the acceleration of the vehicle whose space-time path is \( \dot{x}(\cdot) \).

In general we write

\[
D_\phi(\phi) = \frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} \tag{1.8}
\]
for the rate of change of \( \phi \) as seen by an observer traveling through
the traffic fluid in such a way that his position at time epoch \( t \) is
\( \varphi(t) \). For any time function \( \varphi(\cdot) \), \( \dot{\varphi} \) will denote the derivative of \( \varphi \),
as in (1.7) above, with \( \varphi = v \).

An equation of the form
\[
\frac{\partial \phi}{\partial t} + V \frac{\partial \phi}{\partial x} = 0 ,
\]
(1.9)
where \( V = V(x,t) \), will be taken to indicate a wave-phenomenon. This
is because an observer traveling through the fluid along a path
\( \varphi = \varphi(t) \) where \( \varphi \) is a solution of the first order ordinary differential equation
\[
\dot{\varphi} = V(\varphi,t)
\]
(1.10)
will observe no change in \( \phi \) for equations (1.9) and (1.10) imply that
\( D(\phi) = 0 \). Here of course the wave velocity \( V = V(x,t) \) need not be
\( \varphi \) constant.

We remark that by virtue of equation (1.9), the velocity at which
contours of constant \( \phi \) move is in general
\[
V_\phi = - \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial t} .
\]
(1.11)

1.2 The Equation of Continuity.

The following two assumptions are sufficient for the derivation of
the wave phenomenon in the fluid analogy.

(1) The equation of continuity
\[
\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0 .
\]
(1.12)
(ii) The relation
\[ q = kv. \] (1.13)
Later we will give the model more structure by introducing two additional assumptions: (a) an equation of state; and (b) a dynamical assumption.

An alternative form for the equation of continuity that is sometimes more convenient and easier to interpret is derived as follows. Substituting the relation (1.13) into (1.12) we have

\[ 0 = \frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = \frac{\partial k}{\partial t} + \frac{\partial}{\partial x} (kv) \]

\[ = \frac{\partial k}{\partial t} + v \frac{\partial k}{\partial x} + k \frac{\partial v}{\partial x} \] (1.14)

\[ = k + k \frac{\partial v}{\partial x}. \]

Thus the equation of continuity can be written in the appealing forms

\[ \dot{k} = -k \frac{\partial v}{\partial x} \] (1.15)

\[ \frac{\dot{k}}{k} = -\frac{\partial v}{\partial x}. \]

Introducing the function \( s = 1/k \) [equation (1.3)] we can rewrite (1.15) in the forms:

\[ \dot{s} = s \frac{\partial v}{\partial x} \] (1.15a)

\[ \frac{\dot{s}}{s} = \frac{\partial v}{\partial x}. \]

These last forms will be convenient for comparing the continuous fluid model with the discrete or microscopic approach to traffic flow.
It will be observed that, by virtue of the relations \( q = kv \) and \( sk = l \), it is only necessary to work with two of the four functions \( q, k, v, \) and \( s \). If we work with (1.12) we are focusing our attention on \( k \) and \( q \), with (1.15) we are working with \( k \) and \( v \), and with (1.15a) we are working with \( s \) and \( v \).

Since it is the pair \((s, v)\) that has an analogue in the discrete model, for purposes of comparing the two models it is often convenient to work with (1.15a).

1.3 Kinematic Waves in the Continuous Fluid Analogy.

In view of Section 1.2 we can rewrite the basic assumptions for the continuous fluid model in the form

\[
k + k \frac{\partial v}{\partial x} = 0 ;
\]

\[
q = kv.
\]  

(1.16)

This form is convenient for discussing the phenomena of velocity and concentration waves.

Consider an observer moving through the traffic fluid along a space-time path \( \phi_v = \phi_v(t) \) in such a way that the fluid velocity \( v[\phi_v(t), t] \) that he observes remains constant. That is, \( \phi_v(\cdot) \) is such that

\[
v[\phi_v(t), t] = v_0 ,
\]  

(1.17)

where \( v_0 \) is constant. Differentiating (1.17) with respect to time we obtain

\[
\frac{\partial v}{\partial t} + \phi_v \frac{\partial v}{\partial x} = 0 .
\]  

(1.18)
From (1.18) we see that the wave velocity $V_v$ for velocity waves is given by

$$V_v = \phi_v = -\frac{\partial v}{\partial t} \frac{\partial v}{\partial x} ,$$

(1.19)

[See equations (1.9) and (1.10)]. Using equations (1.16) and (1.18) we can write

$$V_v = -\frac{\partial v}{\partial t} = k \frac{\partial v}{\partial t} ,$$

(1.20)

and using equation (1.6) with $\Phi = v$ we have

$$V_v = k \frac{\partial v}{\partial t} = \frac{k}{k} \left[ \frac{\partial v}{\partial t} \right] ,$$

(1.21)

Using (1.16) again in (1.20) we finally obtain

$$V_v = \frac{k}{k} \left[ \frac{\partial v}{\partial t} \right] = \frac{k}{k} \left[ \frac{\partial v}{\partial t} \right]$$

$$= \left( \frac{k \partial v}{k} \right) = \left( \frac{\partial v}{k} \right) = \frac{\partial v}{k} .$$

(1.22)

The velocity of the wave relative to the medium has the form, introducing $s = 1/k$,

$$V_v - v = k \frac{\partial v}{k} = -\frac{\partial v}{s} .$$

(1.23)

In summary we can write

$$V_v = v + k \frac{\partial v}{k} = \frac{\partial v}{s} = v - \frac{s v}{s} .$$

(1.24)

In the case where the flow is such that an equation of state of the form
\[ q = Q(k) = k\alpha(k) \]
\[ v = \alpha(k) \]

is assumed to hold at all times during the flow, we can write our
expression for the wave-velocity of a velocity wave in the form

\[ V_v = \frac{\dot{q}}{k} = \frac{\partial q}{\partial k} = Q'(k) \]

\[ = \alpha(k) + k\alpha'(k) = v + k \frac{dv}{dk} . \]

This is the form that has been given in the literature for the wave-
velocity of a concentration wave under the assumption (1.25). It is
clear that if \( \alpha \) in (1.25) is a one-one function the first equation in
(1.25) implies that the contours of equal velocity coincide with the
contours of equal concentration, so that then the velocity and concen-
tration waves coincide. When this is so the velocity waves we are dis-
cussing here coincide with the concentration waves discussed under the
assumption \( q = Q(k) \) in the literature. The discussion here, however,
is independent of this assumption and the wave velocity \( V_v = \dot{q}/k \) we
have obtained gives the velocity of velocity (not concentration) waves
even without the assumption \( q = Q(k) \).

It turns out in fact that a necessary and sufficient condition for
the velocity and concentration wave velocities to be equal is that \( k \)
and \( v \) be functionally dependent. To see this consider the phenomenon
of concentration waves. By analogy with (1.17) we consider a path \( \phi_k \)
such that

\[ k[\phi_k(t), t] = k_0 , \]
and differentiate (1.27) to obtain

\[ \frac{\partial k}{\partial t} + \phi_k \frac{\partial k}{\partial x} = 0. \tag{1.28} \]

Then, as in (1.19),

\[ V_k = \phi_k = -\frac{\partial k/\partial t}{\partial k/\partial x} \tag{1.29} \]

is the velocity of a concentration wave. Similarly the velocity of a flow wave is given by

\[ V_q = \phi_q = -\frac{\partial q/\partial t}{\partial q/\partial x}. \tag{1.30} \]

Suppose now the velocities of concentration and velocity waves coincide. Then we have

\[ V_v = V_k \]

\[ \Leftrightarrow -\frac{\partial v/\partial t}{\partial v/\partial x} = -\frac{\partial k/\partial t}{\partial k/\partial x} \]

\[ \Leftrightarrow \frac{\partial v}{\partial t} \frac{\partial k}{\partial x} - \frac{\partial v}{\partial x} \frac{\partial k}{\partial t} = 0 \]

\[ \Leftrightarrow \frac{\partial (v,k)}{\partial (x,t)} = 0. \tag{1.31} \]

That is, the determinant of the Jacobian matrix

\[ J = \begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial t} \\ \frac{\partial k}{\partial x} & \frac{\partial k}{\partial t} \end{bmatrix} \tag{1.32} \]

vanishes identically and this is equivalent to the existence of a functional relationship (an equation of state) \( F(v,k) = 0 \) relating \( v \) and \( k \), as we have asserted.
We should point out that the preceding argument applies not only to the pair \((v,k)\) but to any pair \((f,g)\) of functions defined on the space time domain of \((x,t)\). For example if acceleration and flow wave velocities always coincide, then these two characteristics are functionally dependent, and conversely.

2. The Discrete Fluid Model.

2.1 Notational Introduction.

In the discrete fluid model we represent single lane traffic flow by a sequence \(\{c_n: n=0, \pm 1, \pm 2, \ldots\}\) of cars and focus our attention on a single car \(c_n\) and its predecessor \(c_{n+1}\). It is assumed that of all the cars on the road the behavior of \(c_n\) is dependent only on the behavior of \(c_{n+1}\). That is, the driver of \(c_n\) is assumed to be directly affected only by the behavior of the vehicle \(c_{n+1}\) immediately preceding him. We adopt the following notation:

\[
\begin{align*}
x_n &= \text{position of } c_n \\
v_n &= \dot{x}_n = \text{velocity of } c_n \\
v_n &= \ddot{x}_n = \text{acceleration of } c_n \\
s_n &= \Delta x_n = x_{n+1} - x_n \\
s_n &= \Delta v_n = \dot{x}_n = v_{n+1} - v_n
\end{align*}
\]

(2.1)

The symbols \(x_n, v_n, s_n, \) etc. represent time functions and as in the continuous model \((\cdot)\) denote time derivative.
2.2 The Equation of Continuity.

In the discrete model the equation of continuity of the continuum model is replaced with the relation

\[ \dot{s}_n = v_{n+1} - v_n. \]  

(2.2)

The validity of (2.2) is clear since

\[ \dot{s}_n = (x_{n+1} - x_n) = \frac{\dot{x}_{n+1}}{s_{n+1}} - \frac{\dot{x}_n}{s_n} = \frac{v_{n+1}}{s_{n+1}} - \frac{v_n}{s_n}. \]

If we divide both sides of equation (2.2) by \( s_n \) we obtain

\[ \frac{\dot{s}_n}{s_n} = \frac{v_{n+1} - v_n}{s_n} = \frac{\Delta v_n}{\Delta x_n}. \]  

(2.3)

Dropping the subscript \( n \), we have

\[ \frac{\dot{s}}{s} = \frac{\Delta v}{\Delta x}. \]  

(2.4)

The analogous relation (1.15a) in the fluid model is

\[ \frac{\dot{s}}{s} = \frac{\partial v}{\partial x}. \]  

(2.5)

Thus it appears that adopting the continuum model amounts to postulating a "virtual" continuum of vehicles in such a way that, at least approximately, we have

\[ \frac{\partial v}{\partial x} = \frac{\Delta v}{\Delta x}, \]  

(2.6)

where the left-hand side of (2.6) refers to the "virtual" continuum and the right-hand side refers to the more realistic discrete situation.
The variables on the right-hand side are functions of the pair \((n,t)\)
whereas the variables on the left-hand side are functions of the pair
\((x,t)\); finite differences with respect to \(n\) are replaced with differentials. Convective time derivatives [denoted by \(\varphi \rightarrow \dot{\varphi}\)] are the same in the two models.

In this way we can look upon the continuum model as a convenient approximation to the more realistic discrete model provided that the traffic we are considering is dense enough so that headways remain small enough to justify linear approximations like (2.6).

It has usually been said that the continuous fluid analogy is a macroscopic model whereas the discrete model is microscopic. However, in view of the foregoing it also seems appropriate to view the continuum model as an approximation to the discrete model yet still on the microscopic level in the sense that to introduce it we conceptually interpolate a virtual distribution of mass between the cars on the road, so that the "dx" of infinitesimal calculus should be pictured as representing the spacing or headway between adjacent vehicles and not as a segment of roadway containing many cars, as it seems usually to have been visualized in the past.

2.5 Waves in the Discrete Fluid.

In view of the preceding discussion it should not be surprising that first order approximations give expressions for wave velocities in the discrete model that are analogous to those we obtained for the continuum model. We shall now show that this is indeed the case.
2.3.1 Velocity Waves.

Consider two successive vehicles \( c_n \) and \( c_{n+1} \) with velocities \( v_n \) and \( v_{n+1} \), and spacings \( s_n \) and \( s_{n+1} \) respectively. Let the time function \( \tau_n \) be defined by

\[
v_n(t+\tau_n) = v_{n+1}(t).
\]

(2.7)

More precisely \( \tau_n(t) \) is defined to be the smallest number \( \tau_n \) satisfying (2.7). Then \( t+\tau_n(t) \) is the earliest time at which the velocity \( v_{n+1}(t) \) experienced at time \( t \) by vehicle \( c_{n+1} \) is transmitted back to the vehicle \( c_n \). Assuming \( \tau_n(t) \) is small enough to justify a first order approximation we write

\[
v_n(t+\tau_n) \approx v_n(t) + \tau_n(t) \dot{v}_n(t).
\]

(2.8)

[Where \( \approx \) denotes a first order approximation.]

Combining the expression (2.8) with equation (2.7) we have

\[
v_n + \tau_n \dot{v}_n \approx v_{n+1}.
\]

(2.9)

\[
\tau_n \approx \frac{v_{n+1} - v_n}{\dot{v}_n} = \frac{\Delta v_n}{\dot{v}_n}.
\]

(2.10)

Then the velocity \( \bar{v}_n \), relative to the traffic medium, with which the value \( v_{n+1}(t) \) is transmitted from \( c_{n+1} \) back to \( c_n \) is given by

\[
\bar{v}_n = -\frac{s_n}{\tau_n} = -\frac{s_n}{\tau_n} \frac{v_n}{v_{n+1} - v_n} = -s_n \frac{v_n}{\Delta v_n} = -\frac{\Delta v}{\Delta x_n} = -\frac{s_n}{s_n} \frac{\dot{v}_n}{v_n},
\]

(2.11)

and the absolute wave velocity is then given by
\[ V_n - v_n \approx \tilde{v}_n = v_n - s_n \frac{\dot{v}_n}{s_n}. \] (2.12)

We see that equations (2.11) and (2.12) are (except for the subscripts) identical with the continuum equations (1.23) and (1.24).

2.3.2 Concentration Waves.

If we use the same approach as above to approximately calculate the velocity of a concentration (or spacing) wave we obtain another interesting and analogous result.

Let \( \tau_n(t) \) be now defined by

\[ s_n(t + \tau_n) = s_{n+1}(t) \] (2.13)

where as before \( \tau_n(t) \) is the smallest \( \tau_n \) satisfying (2.13). We again assume that \( \tau_n(t) \) is small enough to justify a linear approximation and write

\[ s_n(t + \tau_n) \approx s_n(t) + \tau_n(t) \dot{s}_n(t). \] (2.14)

[As before \( \approx \) denotes a first order approximation]. Combining (2.13) and (2.14) we have

\[ s_n + \tau_n \dot{s}_n \approx s_{n+1} \] (2.15)

and

\[ \tau_n \approx \frac{s_{n+1} - s_n}{s_n}. \]

Then the absolute velocity \( V_n \) of the concentration wave is given by
\[ V_n \simeq v_n - \frac{s_n}{T_n} = v_n - \frac{s_n s_n}{s_{n+1} - s_n} = \frac{v_{n+1} - v_n}{s_{n+1} - s_n} \quad (2.17) \]

Since \( s_n = v_{n+1} - v_n \) [equation (2.2), the equation of continuity in the discrete fluid] we have

\[ V_n \simeq \frac{v_n s_{n+1} - v_n s_n - s_n (v_{n+1} - v_n)}{s_{n+1} - s_n} = \frac{v_n s_{n+1} - v_n l_{n+1} s_n}{s_{n+1} - s_n} \quad (2.18) \]

Let us introduce quantities analogous to those used in the continuum theory by writing

\[
\begin{align*}
k_n &= 1/s_n \\
q_n &= k_n v_n = v_n / s_n
\end{align*} \quad (2.19)
\]

Substituting these definitions into equation (2.18), our expression for the velocity of a concentration wave takes the form

\[ V_n \simeq \frac{v_n k_n - v_n l_{n+1} k_n}{k_n - k_{n+1}} = \frac{q_n - q_{n+1}}{k_n - k_{n+1}} \quad (2.20) \]

It is interesting to observe that this last expression is precisely the result that has been obtained in the literature for the velocity of a shock-wave (the propagation of a density discontinuity, that is, a spatial discontinuity in the concentration \( k \)) in the continuum model.

Moreover, writing

\[ V_n \simeq \frac{q_n - q_{n+1}}{k_n - k_{n+1}} = \frac{\Delta q_n}{\Delta k_n} = \frac{\Delta q_n / \Delta x_n}{\Delta k_n / \Delta x_n} \approx \frac{\partial q}{\partial x} \quad (2.21) \]

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we see that the last expression in (2.21) seems to be the analogue in the continuum model of the wave velocity we have just derived. This last expression is indeed the expression for the velocity of a concentration wave in the continuum model, for by the equation of continuity \( \frac{\partial q}{\partial x} = -\frac{\partial k}{\partial t} \) in the continuum model [equation 1.21] we have

\[ v \approx \frac{\partial q}{\partial x} = -\frac{\partial k}{\partial x} \cdot \frac{\partial k}{\partial t}. \]  \hspace{1cm} (2.22)

The last term in (2.22) is the expression we derived in section 1.3 for the velocity of a concentration wave in the continuum model [see equation (1.29)].

3. Dynamics of Traffic Flow.

3.1 Dynamics for the Continuum Model.

Thus far in our discussions we have been considering only the kinematics of flow in our models without considering the forces (whether physical or psychological) responsible for the motion.

Several authors have introduced dynamic assumptions of the form

\[ \dot{v} = \psi'(k) \frac{\partial k}{\partial x} = \frac{\partial}{\partial x} \psi(k), \]  \hspace{1cm} (3.1)

where \( \psi' \) is some specified continuous function and (') denotes derivative. The function \( \psi = \psi(k) = \psi(k(x,t)) \) defined on the space-time domain of \( (x,t) \) can be thought of as a sort of potential function for the force field governing the flow. As always \( \dot{v} \) represents the acceleration of the vehicle whose position at time \( t \) is \( x \). [See the discussion following equation (1.5)].]
In addition these authors assume the existence of an equation of state, that is, a functional relation between \( v \) and \( k \), or equivalently between \( q \) and \( k \), of the form

\[
q = Q(k) = k\alpha(k),
\]
\[
v = \alpha(k),
\]

(3.2)

where \( \alpha \) is a monotone decreasing continuously differentiable function but otherwise unspecified. Using the relations (3.2) the dynamic assumption can be rewritten as follows:

\[
\dot{v} = \psi(k) \frac{\partial k}{\partial x} = \psi'(k) \frac{\partial k}{\partial v} \frac{\partial v}{\partial x} = \frac{\psi'(k)}{\alpha'(k)} \frac{\partial v}{\partial x}.
\]

(3.3)

From this last form it is easy to derive the form of the unspecified function \( \alpha \) [except for a constant of integration] from that of the specified function \( \psi' \). For, using the equation of continuity in the form \( \partial v/\partial x = -\dot{k}/k \) in equation (3.2), we have

\[
\dot{v} = -\frac{\psi'(k)}{\alpha'(k)} \frac{\dot{k}}{k},
\]

(3.4)

and using the fact that \( \dot{v} = \alpha'(k) \dot{k} \) we have

\[
[\alpha'(k)]^2 = -\frac{\psi'(k)}{k},
\]

(3.5)

Since \( \alpha \) is a monotone decreasing function we can take the negative square root of both sides of (3.4) to obtain
\[ \alpha'(k) = -\sqrt{-\frac{\psi'(k)}{k}} . \]  

(3.6)

From (3.5) and (3.6) we see that there is a one-one correspondence between the functions \( \alpha' \) and \( \psi' \). So assuming a form for the potential function is essentially the same as assuming a form for the equation of state [except for constants of integration which are to be obtained from appropriate boundary conditions] since from (3.6) we can write

\[ \alpha(k) - \alpha(k_0) = -\int_{k_0}^{k} \frac{-\psi'(k)}{k} \, dk . \]  

(3.7)

An examination of the preceding paragraphs will indicate that what made the derivation of \( \alpha \) from \( \psi \) easy was the fact that by assuming the existence of an equation of state we were able to rewrite the dynamical assumption (3.1) in the form (3.3). This suggests considering dynamical assumptions of the form

\[ \dot{\psi} = \sigma(s,v) \frac{\partial \psi}{\partial x} . \]  

(3.8)

We have shifted to the variable \( s = 1/k \) for convenience in relating our results to those of the discrete case.

Using the equation of continuity \( \partial \psi/\partial x = \dot{s}/s \) in (3.8) we get

\[ \dot{\psi} = \sigma(s,v) \frac{s}{\dot{s}} . \]  

(3.9)

This last equation is essentially a first order ordinary differential equation that can in general be valued to give

\[ v = \tilde{\alpha}(s) , \]  

(3.10)
where the function $\tilde{\alpha}$ is determined by $\sigma$ and appropriate boundary conditions, and

$$s\tilde{\alpha}'(s) = \sigma[s,\tilde{\alpha}(s)] . \quad (3.11)$$

Thus an assumption of the form (3.8) with $\sigma$ specified implies the existence of an equation of state and when combined with appropriate boundary conditions determines its form.

Conversely if we are given an equation of state $v = \tilde{\alpha}(s)$ that is assumed to hold for all $(x,t)$ we can differentiate it to obtain a dynamical equation

$$\dot{v} = \tilde{\alpha}'(s) \dot{s} \quad (3.12)$$

which is of the same form as (3.9) with $s\tilde{\alpha}'(s)$ playing the role of $\sigma(s,v)$. See equation (3.11).

In the next section we point out that there can be many different functions $\sigma$ that give rise to essentially the same $\alpha$. However, if we restrict ourselves to a sufficiently small class of $\sigma$'s it is possible that within this class there may be (with given boundary conditions) a one-one correspondence between $\sigma$'s and $\alpha$'s. For example, this is the case if we restrict ourselves to functions $\sigma$ of the form

$$\sigma(s,v) = Cs^{1-\alpha} \frac{\beta}{v} \quad (3.13)$$

$$-\infty < \alpha < \infty , \quad \beta > 1 , \quad C = \text{constant} .$$

Models of this form are studied extensively in [May, Hartmut, Keller].
3.2 Dynamics for the Discrete Model - Car Following.

In the discrete model, the so-called microscopic car-following or follow-the-leader approach, in the non-lagged case, the dynamic assumption of the continuum model is replaced by a psychological stimulus-response equation of the form

\[ \dot{v} = \lambda(s, v) \dot{s} \]  \hspace{1cm} (3.14)

where the response \( \dot{v} \) is the acceleration of one of the cars; the stimulus is the rate of change of \( s \), the spacing associated with this car, that is, the space headway between the car and its predecessor; and \( \lambda \) is a given sensitivity function.

Writing \( v' = \dot{v}/\dot{s} \) equation (3.14) takes the form

\[ v' = \lambda(s, v) . \hspace{1cm} (3.15) \]

Equation (3.15) is a standard first order ordinary differential equation which, when combined with appropriate boundary conditions [for example, \( s = L \Rightarrow v = 0 \), where \( L = \text{car-length} \)] can in general be solved to give

\[ v = \tilde{\alpha}(s) . \hspace{1cm} (3.16) \]

It is clear that in general there will not be a unique function giving rise to a given function \( \tilde{\alpha} \); any \( \lambda \) satisfying the relation

\[ \lambda[s, \tilde{\alpha}(s)] = \tilde{\alpha}'(s) \hspace{1cm} (3.17) \]

will suffice and to distinguish among such is impossible in a flow
situation where the relation \( v = \dot{\alpha}(s) \) is maintained throughout the flow as is typical of all non-lagged car-following theories — the equation of state is always satisfied even during transitions between steady states, and not only when the flow is in a steady state. This being the case it seems appropriate to replace equation (3.14) with the canonical form

\[
\dot{v} = \dot{\alpha}'(s) \dot{s}
\]  

(3.18)

obtained by differentiating the equation of state (3.16).

Finally, it is apparent that the present considerations become exactly analogous to those of the previous section in the discussion following equation (3.9) provided that we make the identification

\[
\lambda(s,v) = \frac{c(s,v)}{s}.
\]  

(3.19)

In particular equation (3.9) can be interpreted as a stimulus-response equation with exactly the same interpretation as equation (3.14).

That all these similarities we are presenting are not merely formal is apparent when we realize that the symbol "s" has essentially the same meaning in both the continuum and discrete models and that in both models the operation symbolized by \( \varphi \to \dot{\varphi} \) represents the operation of calculating the rate of change of \( \varphi \) as seen by an observer traveling with the same velocity as the fluid, namely a driver. This is why equation (3.9) can really be interpreted as a stimulus-response equation for a driver.
REFERENCES


