ESTIMATION OF PARAMETERS OF VELOCITY DISTRIBUTIONS: II. ESTIMATING GAMMA SCALE AND SHAPE PARAMETERS

BY

STEPHEN PORTNOY

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1. Introduction

This paper is a continuation of an earlier paper of the author (7) considering the estimation of velocity distributions arising in the study of traffic flow. As in the earlier paper, we will restrict consideration to the case of moderate or moderately small sample sizes (i.e., less than 100 observations). As remarked in (7), this situation can arise, even when there is a large amount of data, when we are not willing to assume that all the data belongs to the same distribution; that is, when the velocity distribution changes during the day or from day to day. Thus, large sample methods of estimating arbitrary distributions are not applicable, and we must assume a parametric model for the velocity distribution and consider the estimation of unknown parameters.

In the previous paper (7), comments of Ashton (2) and others were used to justify consideration of the gamma (or Erlang) family of distributions. In the earlier paper we assumed that the shape parameter was known. Here we will assume that both shape and scale are unknown and consider ways of estimating them. In particular, a gamma distribution with unknown scale, \( \theta \), and shape, \( \rho \), denoted by \( \Gamma(\rho, \theta) \), has density

\[
(1.1) \quad f(x) = \frac{x^{\rho-1}e^{-x/\theta}}{\theta^\rho \Gamma(\rho)} \quad \text{for} \quad x > 0; \quad \theta > 0, \quad \rho > 0.
\]

where \( \Gamma(\rho) \) is the gamma function (see (1), p. 255). On fitting a gamma
distribution to a set of observations, the method of moments is often used. Note that

\[(1.2) \quad \mathbb{E}(\rho, \theta)(X) = \rho \theta \quad \text{and} \quad \text{Var}(\rho, \theta)(X) = \rho \theta^2.\]

Thus, if \( \bar{X} \) is the observed mean and \( \frac{S^2}{\bar{X}} \) is the observed variance, then one can take for estimates of \( \theta \) and \( \rho \),

\[(1.3) \quad \hat{\theta} = \frac{S^2}{\bar{X}} \quad \text{and} \quad \hat{\rho} = \frac{\bar{X}^2}{S^2}.\]

Drew (3) suggests exactly this method for fitting a gamma distribution to a particular set of observations of time headways between vehicles on a highway. However, it is well known that the method of moments is generally highly inefficient compared to the maximum likelihood method. Thus, in this paper we will first discuss fitting a gamma distribution by finding the maximum likelihood estimators for \( \rho \) and \( \theta \). We will also use methods of the author ((6) and (7)) to construct formal Bayes estimators of \( \rho \) and \( \theta \) which in certain cases will be shown to be substantial improvements over the maximum likelihood estimators (and, hence, very substantial improvements over the method of moments estimators).

In section 2 we will first discuss some distribution theory necessary for carrying out later computations. In section 3 the maximum likelihood estimators will be described. Section 4 is an extremely technical section concerning the measurement of error in the particular problem of fitting gamma distributions to observed data. It may easily be omitted if one is willing to accept the loss functions used in the following section. In section 5 a family of formal Bayes estimators are constructed, and in
section 6 (using tables in the Appendix) it is shown that they are quite good in terms of expected error, that they have certain optimality properties, and that they should definitely be used when the sample size is about 50 or less. These estimators can be described as follows: Let \( S \) be the sum of the observations and let \( T \) be the product of the observations. Define

\[
W = \log\left(\frac{S^n}{nT}\right).
\]

(Note: here and throughout this paper, "log" denotes the natural logarithm; that is, "log\(_e\).") Then the formal Bayes estimator are given by

\[
\hat{p} = \frac{1}{W} \left\{ \frac{n-5}{2} + \frac{n-1}{2} P_1(aW) \right\}
\]

\[
\hat{\sigma} = \frac{S W}{n} / \left\{ \frac{n-1}{2} + \frac{n-1}{2} P_2(aW) \right\},
\]

where \( P_1 \) and \( P_2 \) can be expressed in terms of incomplete gamma functions (see (1) and (5)):

\[
P_1(x) = \frac{\int_0^x u^{n-3} e^{-u} \, du}{\int_0^x u^{n-3} e^{-u} \, du + x \int_0^\infty u^{n-2} e^{-u} \, du},
\]

\[
P_2(x) = \frac{\int_0^x u^{n-2} e^{-u} \, du}{\int_0^x u^{n-2} e^{-u} \, du + x \int_0^\infty u^{n-3} e^{-u} \, du},
\]

and where \( a \) is a certain constant chosen according to one's a priori opinions about \( p \). For ease of calculation, one should choose \( a = 0 \), for which \( P_1(aW) = 0 \) and \( P_2(aW) = 0 \). This simple choice of a still
gives substantial improvement over the maximum likelihood estimator of \( \rho \) and some improvement over that of \( \theta \). Even more improvement can be obtained by choosing \( a = 1 \). Furthermore, if one is willing to assume, \textit{a priori}, that \( \rho \) is greater than some fixed constant, then even more substantial improvement can be obtained by choosing \( a \) in accordance with Table III. Further conclusions are discussed in section 6.

We conclude this section by remarking that the estimator \( \hat{\rho} \) given in (1.5) can be used in conjunction with the methods of (7) to fit gamma distributions to pairs of velocity distributions with one larger than the other (on the average). This entails merely estimating \( \rho_1 \) and \( \rho_2 \) separately using (1.5) here and inserting these values in formulas (3.2) and (3.3) in (7).

2. Some Distribution Theory

Consider a sample of \( n \) observations from the gamma distribution with unknown shape and scale described in section 1. We can write the joint density as

\[
P(\theta, \rho)(x_1, \ldots, x_n) = \frac{n^{-\sum x_i/\theta} \prod x_i^{\rho-1}}{\theta^{n\rho} \Gamma^n(\rho)}
\]

where

\[
s = \sum_{i=1}^{n} x_i \quad \text{and} \quad w = \log \left( \frac{s}{\prod_{i=1}^{n} x_i} \right).
\]

It follows that the pair of random variables \((s, w)\) is sufficient for
Furthermore, the joint density of \((S, W)\) is given by

\[
 p_{(\theta, \rho)}(s, w) = \frac{s^{n-1} e^{-s/\theta}}{\theta^{n\theta} \Gamma(n\theta)} \cdot \frac{\Gamma(n\theta)e^{-n\rho} \log n}{\Gamma^n(\rho)} \cdot e^{-\rho w} J_n(s, w)
\]

where \(J_n(s, w)\) is essentially the integrated Jacobian of the transformation given by (2.2). Now, it is well known that, for the gamma distribution, the sum \(S\) is independent of any scale invariant statistic. Thus, \(S\) and \(W\) are independent, and the function \(J_n(s, w)\) is a function of \(w\) alone. Since the marginal distribution of \(S\) is \(\Gamma(n\theta, \theta)\), it follows that the distribution of \(W\) is that of the exponential family distribution with density

\[
 p_{\rho}(w) = e^{-\chi(\rho) - \rho w} h_n(w)
\]

for some function \(h_n(w)\) and where

\[
 \chi(\rho) = n \log \Gamma(\rho) - \log \Gamma(n\rho) + n\rho \log n.
\]

Although the function \(h_n(w)\) cannot be directly determined, the distribution of \(W\) is characterized by the function \(\chi(\rho)\). In particular, \(W\) has moment generating function \(e^{\chi(\rho) - \chi(\rho+t)}\) from which any moment can be calculated. Actually, we will later need to calculate rather general expectations with respect to the distribution of \(W\). For this we will require some expression for \(h_n(w)\). To do this we will find an asymptotic expansion for \(e^{\chi(\rho)}\) and use the Tauberian theorem (e.g., see (4)) to infer a power series development for \(h_n(w)\) about zero. Since we consider \(\rho\) rather large, the distribution of \(W\) will be concentrated near zero and this method should be valid for our purposes. Actually, we shall find that, by using enough terms, we can get the desired accuracy
so long as \( \rho \geq 3 \).

To find an asymptotic series for \( e^{X(\rho)} \), we use what is essentially Stirling's approximation for the gamma function (see (1), p. 257, for this approximation):

\[
\log \Gamma(x) - (x - \frac{1}{2})\log x - x + \frac{1}{2} \log (2\pi) + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}},
\]

where \( B_n \) are the Bernoulli numbers ((1), p. 810). From (2.5) and (2.6) we have (for \( c_n \), an appropriate constant),

\[
\chi(x) \sim c_n - \frac{n-1}{2} \log \rho - \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)\rho^{2m-1}} (n-\frac{1}{n})^{2m-1}. \tag{2.7}
\]

Continuing formally, one can use the power series expansion for the exponential function to obtain

\[
e^{-X(\rho)} \sim c_n \rho^{\frac{n-1}{2}} (1 + \sum_{m=1}^{\infty} d_m(n)\left(\frac{1}{\rho}\right)^m), \tag{2.8}
\]

where \( d_m(n) \) are appropriate coefficients. We can easily obtain

\[d_1(n) = \frac{n^2-1}{12} n \quad \text{and} \quad d_2(n) = \left(\frac{n^2-1}{2}\right)^2 \frac{n}{288} \quad \text{other} \quad d_m(n) \quad \text{were calculated on the computer only as needed.}
\]

Therefore, from the Tauberian theorem (4) it follows that the cumulative distribution function, \( \mathbb{H}_n(w) \), corresponding to \( h_n(w) \), can be expressed in the following expansion:

\[
\mathbb{H}_n(w) \sim c_n \left\{ \frac{w^{n-1}}{\Gamma\left(\frac{n+1}{2}\right)} + \sum_{m=1}^{\infty} d_m(n) \frac{w^{n-1}}{\Gamma\left(\frac{n+1}{2} + m\right)} \right\}. \tag{2.9}
\]

Hence, since \( h_n(w) = \frac{d}{dw} \mathbb{H}_n(w) \), we have for the density of \( W \):
(2.10) \( p_\rho(w) = e^{-\lambda(\rho) - \rho w} \sum h_n(w) e^{-\lambda(\rho)} \left\{ \frac{n-3}{\gamma^{n-1}/2} e^{-\rho w} + \sum_{m=1}^{\infty} \frac{n-3 + m e^{-\rho w}}{\gamma^{n-1}/2 + m} \right\} \)

\[
= c^*(\rho) \left\{ \frac{n-3}{\rho \gamma^{n-1}/2} e^{-\rho w} + \sum_{m=1}^{\infty} \frac{d_m(n)}{\rho^m} \left[ \frac{(n-1+m)(n-3+m)}{\rho \gamma^{n-1}/2 + m} e^{-\rho w} \right] \right\} 
\]

where \( c^*(\rho) = (1 + \sum_{m=1}^{\infty} \frac{d_m(n)}{\rho^m})^{-1}. \)

That is, the density of \( W \) is a weighted sum of gamma densities (corresponding to the distributions \( \Gamma(n-1/2 + m, \frac{1}{\rho}) \) for \( m = 0, 1, 2, \ldots \)) with weights summing to one. If \( d_m(n) \) were positive we could think of \( W \) as a mixture of gamma variables. However, \( d_m(n) \) may be negative for \( m \geq 4. \) Nonetheless, the expression (2.10) can be shown to be a legitimate approximation for \( \rho \geq 3 \) and can be used to calculate expectations as integrals over \( 0 < w < \infty. \) We will later find that the first term of (2.10) generally gives a fairly decent approximation; that is, many calculations will be done asymptotically where we take \( W \) to have a \( \Gamma(n-1/2, 1/\rho) \) distribution.

As an indication of how good the approximation (2.10) may be, the following is a list of the coefficients \( c^*(\rho) \) and \( c^*(\rho)d_m(n) \) of the first few successive terms in (2.10) for a few values of \( n \) and \( \rho: \)

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( n )</th>
<th>( c^*(\rho) )</th>
<th>( c^*(\rho)d_1(n) )</th>
<th>( c^*(\rho)d_2(n) )</th>
<th>( c^*(\rho)d_3(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>10</td>
<td>.96754</td>
<td>.03193</td>
<td>.00053</td>
<td>.00000</td>
</tr>
<tr>
<td>25</td>
<td>25</td>
<td>.92017</td>
<td>.07656</td>
<td>.00318</td>
<td>.00008</td>
</tr>
<tr>
<td>10</td>
<td>9</td>
<td>.92627</td>
<td>.06861</td>
<td>.00508</td>
<td>.00004</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>.88302</td>
<td>.10989</td>
<td>.00684</td>
<td>.00025</td>
</tr>
</tbody>
</table>
3. The Maximum Likelihood Estimates

From equation (2.1) we can write the logarithm of the likelihood function as

\[
L(\rho, \theta) = n \rho \log S - S/\theta - n \log \prod_{i=1}^{n} X_i - n \rho \log \theta - n \log \Gamma(\rho)
\]

where, as before,

\[
S = \sum_{i=1}^{n} X_i \quad \text{and} \quad W = \log \left( \frac{S^n}{\pi^{n} \prod_{i=1}^{n} X_i} \right).
\]

Setting partial derivatives to zero to find the maximum, we have

\[
0 = \frac{\partial}{\partial \rho} L(\rho, \theta) = n \log S - W - n \log \theta - n \log \hat{\theta} - n \psi(\hat{\theta})
\]

\[
0 = \frac{\partial}{\partial \theta} L(\rho, \theta) = \frac{S}{\theta^2} - \frac{n \hat{\theta}}{\theta}
\]

where \( \psi(x) = \frac{d}{dx} \log \Gamma(x) = \Gamma'(x)/\Gamma(x) \) (see (1), p. 258). From (3.4),

\[
\hat{\theta} = \frac{S}{n \hat{\rho}} = \frac{\bar{X}}{\hat{\rho}}.
\]

Substituting this into (3.3) we find

\[
0 = n \log S - W - n \log S + n \log n + n \log \hat{\rho} - n \psi(\hat{\rho}),
\]

from which it follows that \( \hat{\rho} \) must satisfy

\[
\frac{W}{n} = \log \hat{\rho} - \psi(\hat{\rho}).
\]

Again, this equation cannot be directly inverted to find \( \hat{\rho} \). If one really wanted to find \( \hat{\rho} \) for a number of observed values of \( W \), it would probably be best to plot the function \( \log \hat{\rho} - \psi(\hat{\rho}) \) and find \( \hat{\rho} \) graphically. However, since we are primarily interested in the case of \( \rho \) large
(and, hence, $\hat{\rho}$ probably large) we can proceed as follows. First note that
the asymptotic expansion for $\psi(x)$ is given by the following expression
(see (1), p. 259):

\[
\psi(x) = \ln x - \frac{1}{2x} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n} \frac{1}{2mx^2}
\]

where the $B_{2n}$ are the Bernoulli numbers mentioned before. Thus, (3.7)
leads to the approximation

\[
\frac{w}{n} - \frac{1}{2\rho} + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m2^{2m}} = \sum_{k=1}^{\infty} c_k \left( \frac{1}{\rho} \right)^k
\]

for some constants $c_k$.

Reverting this series, we can obtain the approximation

\[
\frac{1}{\rho} \sim \sum_{k=1}^{\infty} d_k \left( \frac{w}{n} \right)^k
\]

for some determinable constants $d_k$, the first few of which are $d_1 = 2$,
$d_2 = -\frac{2}{3}$, and $d_3 = \frac{4}{9}$. We can now find $\hat{\rho}$ directly from $\frac{1}{\rho}$
or we can formally invert the series (4.0) to obtain another series,

\[
\hat{\rho} - \sum_{k=-1}^{\infty} e_k \left( \frac{w}{n} \right)^k = \frac{n}{2W} + \frac{1}{6} - \frac{1}{18} \frac{W}{n} + \sum_{k=2}^{\infty} e_k \left( \frac{w}{n} \right)^k
\]

for some determinable constants $e_k$.

The coefficients $e_k$ were calculated on the computer and (3.11) was
then used to obtain the expected errors listed in the Appendix. Note that
for $\rho$ large, $W$ is likely to be small and, approximately, $\hat{\rho}$ will be $\frac{n}{2W}$.

4. Appropriate Loss Functions

In the next section we will discuss the estimation of $\rho$ and $\rho$ by
means of the methods of formal Bayes estimation described in (6) and (7).
To use these methods, however, the loss function (specifying the loss incurred by estimating \( \hat{\rho} \) and \( \hat{\theta} \) when \( \rho \) and \( \theta \) are the true parameters) must be given. This section will discuss several alternative loss functions and compare them to find an appropriate one.

One natural possibility, of course, is to consider the estimation of \( \rho \) and \( \theta \) with normalized squared error as loss; that is, to take the loss functions:

\[
L_1(\rho, \hat{\rho}) = \frac{1}{\sigma^2} (\rho - \hat{\rho})^2
\]

\[
\tilde{L}_1(\theta, \hat{\theta}) = \frac{1}{\theta^2} (\theta - \hat{\theta})^2.
\]

However, the parameters \( \rho \) and \( \theta \) have no real intrinsic meaning; \( \theta \) is a scale parameter and \( \rho \) is a shape parameter, but it might be just as reasonable to estimate \( \frac{1}{\theta} \) and \( \frac{1}{\rho} \) as it is to estimate \( \theta \) and \( \rho \).

We are really interested in fitting distributions; that is, in choosing \( \hat{\rho} \) and \( \hat{\theta} \) so that \( p_{\hat{\rho}, \hat{\theta}}(x) \) is as close as possible (in some sense) to \( p_{\rho, \theta}(x) \) (where \( p_{\rho, \theta}(x) \) is given by (2.1)). One measure of the "distance" between two distributions is the Kullback-Leibler distance defined for two probability densities \( p_1(x) \) and \( p_2(x) \) by

\[
I(p_1, p_2) = E_{p_1}(X) \log \frac{p_1(X)}{p_2(X)}
\]

We will now use this distance to construct several loss functions on \( \rho \) (and later \( \theta \)) and compare these loss functions by considering the best asymptotic estimators as \( \rho \to \infty \). The comparisons will be carried out as follows:
Any smooth estimator, \( \hat{\theta} \), of \( \theta \) (including both the maximum likelihood estimator and the formal Bayes estimators we will later construct) will be approximately \( \frac{c}{W} \) for some constant \( c \) as \( \theta \to \infty \). For \( \rho \to \infty \), the maximum likelihood estimator acts like \( \frac{n}{2W} \). For normalized squared error loss \( 4.1 \), the asymptotic risk (i.e., the expected loss with respect to the asymptotic distribution \( W \sim \Gamma \left( \frac{n-1}{2}, \frac{1}{\rho} \right) \)) becomes

\[
(4.4) \quad R(c) = \frac{1}{\rho^2} E \left( \frac{c}{W} - \rho \right)^2 = E \left( \frac{c}{\rho W} - 1 \right)^2.
\]

To find the minimizing constant \( c \), we set the derivative equal to zero:

\[
(4.5) \quad \frac{d}{dc} R(c) = 2c E \left( \frac{1}{\rho W} \right)^2 - 2E \left( \frac{1}{\rho W} \right) = 0.
\]

That is, since \( \rho W \sim \Gamma(\frac{n-1}{2}, 1) \),

\[
(4.6) \quad c_1 \left( \frac{4}{(n-3)(n-5)} \right) = \frac{2}{(n-3)}, \quad \text{or} \quad c_1 = \frac{n-5}{2}.
\]

Therefore, for squared error loss \( 4.1 \), the best asymptotic estimator has risk

\[
(4.7) \quad R^*_1 = \frac{(n-5)^2}{(n-3)(n-5)} - \frac{2(n-5)}{n-3} + \frac{2}{n-3},
\]

while the maximum likelihood estimator has asymptotic risk

\[
(4.8) \quad R_{\text{m.l.e.}} = \frac{n^2}{(n-3)(n-5)} - \frac{2n}{n-3} + \frac{2n^2 + 15}{(n-3)(n-5)};
\]

which is substantially larger, even for \( n = 25 \). Considerations and comparisons of this sort will allow us to compare various loss functions.

We now consider the Kullback-Leibler distance \( 4.3 \) for the distribution \( 2.1 \):
(4.9) \[ I(\rho, \theta; \hat{\rho}, \hat{\theta}) = E(\rho, \theta) \left\{ np \log \frac{S}{\theta} - n \hat{\theta} \log \frac{S}{\hat{\theta}} - (\rho-\hat{\rho})(\hat{\theta}-\theta) \frac{1}{\hat{\theta}} \right\} \]

Now note that, since \( S \sim \Gamma(np, \theta) \), \( \frac{S}{\theta} \sim \Gamma(np, 1) \); so (see (1))

(4.10) \[ E(\rho, \theta) \log \frac{S}{\theta} = \psi(np) \] and \( E(\rho, \theta)S = np\theta \).

Also, using basic properties of exponential families of distributions, we have

(4.11) \[ E(\rho, \theta)W = E \hat{W} = \chi'(\theta) = n \psi(np) - n \psi(\rho) - n \log n. \]

Therefore,

(4.12) \[ I(\rho, \theta; \hat{\rho}, \hat{\theta}) = (\rho-\hat{\rho}) \psi(np) - \hat{\theta} \log \frac{\theta}{\hat{\theta}} - (\rho-\hat{\rho})(\psi(np)-\psi(\rho)) \]

\[ - \rho(1 - \frac{\theta}{\hat{\theta}}) - \log \frac{\Gamma(\rho)}{\Gamma(\hat{\rho})} \]

\[ = (\rho-\hat{\rho}) \psi(\rho) - \log \frac{\Gamma(\rho)}{\Gamma(\hat{\rho})} \; \hat{\theta} \log \frac{\theta}{\hat{\theta}} + \rho(\frac{\theta}{\hat{\theta}} - 1). \]

Since we want to estimate \( \rho \) and \( \theta \) separately we can construct a loss function for \( \rho \) by taking \( \theta = \hat{\theta} \) in (4.12) to obtain:

(4.13) \[ L_2(\rho, \hat{\rho}) = (\rho-\hat{\rho}) \psi(\rho) - \log \frac{\Gamma(\rho)}{\Gamma(\hat{\rho})}. \]

Using Stirling's approximations (1), we have, for \( \rho \) and \( \hat{\rho} \) large,

(4.14) \[ L_2(\rho, \hat{\rho}) \sim (\rho-\hat{\rho}) + \hat{\rho} \log \frac{\hat{\rho}}{\rho}. \]

To find the best asymptotic estimation of the form \( \frac{c}{W} \) for this loss, we want to minimize

(4.15) \[ R(c) = E(\rho - \frac{c}{W} + \frac{c}{W} \log \frac{c}{\rho W}). \]
Setting the derivative to zero (and using the asymptotic distribution of $W$),

\begin{align*}
(4.16) \quad 0 = \rho \mathbb{E}( \frac{1}{\rho W} + \frac{1}{\rho W} \log c - \frac{1}{\rho W} \log (\rho W) + \frac{1}{\rho W}) \nonumber \\
&= \frac{2}{n-3} \log c - \frac{2}{n-3} \psi(\frac{n-3}{2}).
\end{align*}

This can be solved for $c$ in terms of the $\psi$ function, or for $n$ moderately large we can use the approximation $\psi(x) \sim \log x - \frac{1}{2x}$ to obtain (with an error of order $1/n$)

\begin{equation}
(4.17) \quad c_2 \approx \frac{n-4}{2}.
\end{equation}

One criticism of the Kullback-Leibler distance is that it is not symmetric. Thus, one could consider a symmetrized version of $I(\rho, \theta; \hat{\rho}, \hat{\theta})$ given by

\begin{align*}
(4.18) \quad I^*(\rho, \theta; \hat{\rho}, \hat{\theta}) = I(\rho, \theta; \hat{\rho}, \hat{\theta}) + I(\hat{\rho}, \hat{\theta}; \rho \theta) = (\rho - \hat{\rho})(\psi(\rho) - \psi(\hat{\theta})) + (\rho - \hat{\rho}) \log \frac{\theta}{\hat{\theta}} + \rho(\frac{\theta}{\hat{\theta}} - 1) + \rho(\frac{\rho}{\hat{\rho}} - 1).
\end{align*}

This leads to the loss function for $\rho$ given by

\begin{equation}
(4.19) \quad L_2(\rho, \theta) = (\rho - \hat{\rho})(\psi(\rho) - \psi(\hat{\theta})) - (\rho - \hat{\rho}) \log \frac{\rho}{\hat{\rho}}.
\end{equation}

As before, we can find the best asymptotic estimator of the form $\hat{\theta}_2^\circ$.

For this loss (given by (4.19)), the optimum $c$ satisfies

\begin{equation}
(4.20) \quad \log c_2 = \frac{n-3}{2c_2} = \psi\left(\frac{n-3}{2}\right) - 1.
\end{equation}

For moderately large $n$, we can find that

\begin{equation}
(4.21) \quad c_2 \approx \frac{n-7/2}{2} = \frac{n-4}{2} + \frac{1}{2}
\end{equation}
with an error of the order of $\frac{1}{n}$.

Alternately, we can directly consider the marginal distributions of $W$ parameterized by $\rho$ (given by (2.4)). Using general properties of exponential families and (2.4), we can find that

\begin{align*}
I(\rho, \rho) & \equiv \mathbb{E}_{\rho} \frac{p(\rho)}{p(\rho)} = x(\hat{\rho}) - x(\rho) + (\rho - \hat{\rho})x'(\rho) \\
& = n \log \Gamma(\hat{\rho}) \log \Gamma(n\hat{\rho}) + n \log \Gamma(\rho) - \log \Gamma(n\rho) \\
& \quad + n(\rho - \hat{\rho})(\psi(\rho) - \psi(\hat{\rho})).
\end{align*}

Using the approximation given by (2.7) and (3.8),

\begin{equation}
I(\rho, \hat{\rho}) = \left( \frac{n-1}{2} \right) (-\log \frac{\hat{\rho}}{\rho} + \frac{\hat{\rho}}{\rho} -1).
\end{equation}

Thus, we consider the loss function

\begin{equation}
L_4(\rho, \hat{\rho}) = -\log \frac{\hat{\rho}}{\rho} + \frac{\hat{\rho}}{\rho} - 1,
\end{equation}

which leads to a best asymptotic estimator with

\begin{equation}
c_4 = \frac{n^3}{2}.
\end{equation}

Similar to the previous case, the symmetrized version, $I^*(\rho, \hat{\rho}) = I(\rho, \hat{\rho}) + I(\rho, \hat{\rho})$ leads to a loss function

\begin{equation}
L_5(\rho, \hat{\rho}) = \frac{\hat{\rho}}{\rho} + \frac{\hat{\rho}}{\rho} - 2.
\end{equation}

Here, the best asymptotic estimator $\frac{c}{W}$ has $c$ satisfying

\begin{equation}
c^2 = 5 = \frac{(n-3)(n-1)}{4};
\end{equation}

that is, it has $c_5 \approx \frac{n-2}{2}$ (with error of order $\frac{1}{n}$).

To compare these loss functions we will list the asymptotic risks of the various best asymptotic estimators for several values of $n$. This
is done in Table I in the Appendix where the indicated expectation is with respect to the asymptotic distribution $W \sim \Gamma\left(\frac{n-1}{2}, \frac{1}{\rho}\right)$, and where $\frac{c_i}{W}$ is the best asymptotic estimator with respect to the loss $L_i$. This table shows that any of the estimators, $\frac{c_i}{W}$ for $i = 1, 2, 3, 4, 5$, does definitely better than the maximum likelihood estimator. Furthermore, it indicates that the estimator $\frac{c_4}{W} = \frac{n-3}{2W}$ does over all about as well as any other estimator. Since this corresponds to the loss function $L_4$, which (we will later see) is easy to deal with computationally, we will restrict ourselves to consideration of $L_4$ in subsequent sections. Note that all of the loss functions considered ($L_1$ through $L_5$) are convex in $\hat{\theta}$ and the basic difference between them is the rate at which they go to infinity as $\rho$ and $\hat{\theta}$ approach zero and infinity.

We now consider loss functions for estimating $\theta$. We already have the squared error loss function, $\tilde{L}_1$, given by (4.2). As before, we will consider best asymptotic estimators to compare various loss functions. Estimators of $\theta$ will be asymptotically $\hat{\theta} = cSW$ for some constant $c$. For squared error we have

\begin{equation}
\text{(4.28)} \quad E \tilde{L}_1(\theta, cSW) = E \frac{c^2SW^2}{\sigma^2} - 2E \frac{cSW}{\theta} + 1 \\
= c^2n\rho(n+1)EW^2 - 2cn\rho EW + 1 \\
= \frac{c^2n\rho(n+1)(n-1)(n+1)}{4\rho^2} - cn(n-1) + 1 \\
\approx \frac{1}{4} c^2n^2(n-1)(n+1) - cn(n-1) + 1,
\end{equation}

for which the minimizing constant $c$ is

\begin{equation}
\text{(4.29)} \quad c_1 = \frac{2}{n(n+1)}.
\end{equation}
As before, we can construct an alternate loss function by setting \( \rho = \hat{\rho} \) in the Kullback-Leibler "distance" (equation (4.12)), and obtain

\[
(4.30) \quad \tilde{L}_2(\theta, \hat{\theta}) = \frac{\theta}{\hat{\theta}} - \log \frac{\theta}{\hat{\theta}} - 1.
\]

For this loss function one can calculate that the best asymptotic constant \( c \) is

\[
(4.31) \quad c_2 = \frac{2}{n(n-3)}.
\]

The symmetrized version of \( \tilde{L}_2 \) is

\[
(4.32) \quad \tilde{L}_3(\theta, \hat{\theta}) = \left( \frac{\theta - \hat{\theta}}{\theta \hat{\theta}} \right)^2
\]

for which the best asymptotic constant, \( c \), satisfies

\[
(4.33) \quad c_3 = \frac{4}{n^2(n-1)(n-3)}.
\]

In this case we will also find it convenient to consider the loss \( \tilde{L}_2(\hat{\theta}, \theta) \) as a separate function; that is, we also define

\[
(4.34) \quad \tilde{L}_4(\theta, \hat{\theta}) = \frac{\hat{\theta}}{\theta} - \log \frac{\theta}{\hat{\theta}} - 1.
\]

We will later see that \( \tilde{L}_4 \) leads to a particularly easy form for the formal Bayes estimator. For \( \tilde{L}_4 \), the best asymptotic constant is

\[
(4.35) \quad c_4 = \frac{2}{n(n-1)}.
\]

We also have the constant, \( c_5 \), corresponding to the maximum likelihood estimator. From (3.5) and (3.11) we see that

\[
(4.36) \quad c_5 = \frac{2}{n^2}.
\]

We now compare these loss functions and best asymptotic estimators as before. The results of the calculations of asymptotic risks for the
various estimators $c_iSW$ ($i = 1, 2, 3, 4, 5$) are given in Table II in the Appendix. Consulting Table II we see that $c_4SW$ does about as well overall as any other estimator. Since the corresponding loss $L_4$ will lead to a particularly reasonable formal Bayes estimator, we will henceforth restrict ourselves to $\hat{L}_4$ in subsequent sections. Note that the restriction to loss functions $L_4$ (for $\rho$) and $\tilde{L}_4$ (for $\theta$) appears particularly reasonable since they are both of the same form. Also note that $L_4$ and $\tilde{L}_4$ are strictly convex in the estimate, are zero when the estimate exactly equals the parameter, and are strictly increasing as the absolute difference between the estimate and the parameter increases.

5. Formal Bayes Estimators

In this section we will construct formal Bayes estimators of $\rho$ and $\theta$ which will later be shown to be particularly well suited for estimating these parameters when the sample size is moderate (say, about 50 or fewer observations). First consider the estimation of $\rho$. In section 4 the following loss function was justified:

\[(5.1) \quad L(\rho, \hat{\rho}) = \frac{\hat{\rho}}{\rho} - \log \frac{\hat{\rho}}{\rho} - 1.\]

Since this loss function is independent of $\theta$ (i.e., invariant under scale changes), we might expect to find good (invariant) estimators by considering prior measures with densities (over $\rho$ and $\theta$) of the form $\frac{1}{\theta} \pi(\rho)$, where $\pi(\rho)$ is integrable or a limit of integrable functions. In fact, results of the author (6) indicate that one should find nearly minimax procedures by taking $\pi(\rho)$ to act like $\frac{1}{\rho}$ both at $\rho = 0$ and $\rho = \infty$.

Now, to find the invariant formal Bayes estimator, $\hat{\rho}_B(\omega)$, of $\rho$ (using loss function (5.1) and distributions given by (2.1)) with respect
to the prior density \( \frac{1}{\theta} \pi(\theta) \), we must minimize (for each \( x \))

\[
R^*(\pi, x) = \int \int L(\theta, \phi(\pi(w))) \frac{1}{\theta} \pi(\theta) d\theta d\phi
\]

\[
= \int \int \left( \frac{\phi(\pi(w))}{\theta} - \log \frac{\phi(\pi(w))}{\theta} - 1 \right) \frac{\pi(\theta)}{\theta}^{n} \frac{e^{-s/\theta \cdot \pi(w) \cdot \log n}}{\Gamma(\pi(\theta))} \frac{1}{\theta} \pi(\theta) d\theta d\phi
\]

\[
= \left( \frac{1}{n} \right) \int \int \left( \frac{\phi(\pi(w))}{\theta} - \log \frac{\phi(\pi(w))}{\theta} - 1 \right) \frac{\Gamma(n)}{\Gamma(\pi(\theta))} e^{-\rho_\pi \cdot n \cdot \log n \pi(\theta)} d\theta d\phi
\]

\[
= \left( \frac{1}{n} \right) \int \int \left( \frac{\phi(\pi(w))}{\theta} - \log \frac{\phi(\pi(w))}{\theta} - 1 \right) \frac{\Gamma(n)}{\Gamma(\pi(\theta))} e^{-\rho_\pi \cdot n \cdot \log n \pi(\theta)} d\theta d\phi
\]

\[
+ \frac{1}{n} \int \left( \log \theta - 1 \right) \frac{\Gamma(n)}{\Gamma(\pi(\theta))} e^{-\rho_\pi \cdot n \cdot \log n \pi(\theta)} d\theta
\]

To choose \( \phi(\pi(w)) \) so as to minimize \( R^*(\pi, x) \) we must minimize the quantity in brackets. We thus find directly (differentiating with respect to \( \phi(\pi(w)) \) and setting the result equal to zero):

\[
\phi(\pi(w)) = \frac{\int \Gamma(n)}{\Gamma(\pi(\theta))} e^{-\rho_\pi \cdot n \cdot \log n \pi(\theta)} d\theta
\]

(5.3)

\[
\phi(\pi(w)) = \frac{\int \Gamma(n)}{\Gamma(\pi(\theta))} e^{-\rho_\pi \cdot n \cdot \log n \pi(\theta)} d\theta
\]

Thus, if we choose \( \pi(\theta) \) to be of the form

(5.4)

\[
\pi(\theta) = \frac{\Gamma(n)}{\Gamma(\theta)} e^{\rho_\pi \cdot n \cdot \log n \pi(\theta)},
\]
then for suitably chosen \( f(\rho) \), \( \phi_{\pi}(w) \) will be comparatively easy to calculate. Now, basic properties of the gamma function show that

\[ \Gamma(x) \sim \frac{1}{x} \quad \text{as} \quad x \to 0. \]

Therefore,

\[ (5.5) \quad \frac{\Gamma'(\rho)}{\Gamma(\rho)} \cdot e^{\rho n \log n} \sim \rho^{-(n-1)} \quad \text{as} \quad \rho \to 0. \]

Furthermore, using Stirling's approximation, we see that

\[ (5.6) \quad \frac{\Gamma'(\rho)}{\Gamma(\rho)} \cdot e^{\rho n \log n} \sim \rho^{\frac{n-1}{2}} \quad \text{as} \quad \rho \to \infty. \]

Since we want \( n(\rho) \) to act like \( \frac{1}{\rho} \), we are led to consider \( f(\rho) \) of the form

\[ (5.7) \quad f_{a}(\rho) = \begin{cases} \rho^{n-2} & \rho < a \\ \frac{n-1}{a^2} \rho^{n-3} & \rho \geq a \end{cases} \]

for any constant \( a, \ 0 \leq a < \infty \) (note: \( f_{a} \) is continuous). Thus, we can calculate the Bayes rule \( \phi_{a}(w) \) corresponding to (5.4) with \( f_{a}(\rho) \) given by (5.7) as follows:

\[ (5.8) \quad \phi_{a}(w) = \frac{\int_{\rho}^{\infty} e^{-\rho w} f_{a}(\rho) d\rho}{\int_{\rho}^{\infty} \frac{1}{\rho} e^{-\rho w} f_{a}(\rho) d\rho} \]

\[ = \frac{\int_{\rho}^{a} \rho^{n-2} e^{-\rho w} d\rho + a^2 \int_{a}^{\infty} \rho^{n-3} e^{-\rho w} d\rho}{\int_{\rho}^{a} \rho^{n-3} e^{-\rho w} d\rho + a^2 \int_{a}^{\infty} \rho^{n-2} e^{-\rho w} d\rho} \]

\[ = \frac{1}{w} \left\{ \left( \frac{w}{\rho} \right)^{n-2} e^{-\rho w} \right\} + \left( \frac{w}{\rho} \right)^{n-3} e^{-\rho w} \right\}} + \left( \frac{w}{\rho} \right)^{n-3} e^{-\rho w} \right\}} \right) \right) \].

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Note that $\varphi_a(w)$ is now expressed in terms of the incomplete gamma function which is well tabulated (e.g., by Pearson (7)). However, integrating by parts, we can simplify (5.8) even further to obtain

\begin{equation}
(5.9) \quad \varphi_a(w) = \frac{1}{w} \left\{ \left( \frac{n-1}{2} \right) P_1(sw) + \frac{n-3}{2} \right\} = \frac{1}{w} \left\{ \left( \frac{n-3}{2} \right) [1-P_1(sw)] + (n-2)P_1(sw) \right\}
\end{equation}

where

\begin{equation}
(5.10) \quad P_1(x) = \frac{\int_{0}^{x} u^{n-3} e^{-u} du}{\int_{0}^{\infty} u^{n-1} e^{-u} du + x^2 \int_{x}^{\infty} u^{n-5} e^{-u} du}.
\end{equation}

Note that $P_1(x)$ is a monotonically increasing function going from zero at $x = 0$ to one at $x = \infty$. Thus, $\varphi_a(w)$ is $\frac{1}{w}$ multiplied by a coefficient which is a weighted average of $\left( \frac{n-2}{2} \right)$ and $(n-3)$. This estimator will be compared with the maximum likelihood estimator in the following section.

We now consider the estimation of $\theta$. In section 4 we justified the loss function

\begin{equation}
(5.11) \quad L(\theta, \hat{\theta}) = \hat{\theta} - \log \frac{\hat{\theta}}{\theta} - 1.
\end{equation}

Again this loss function is invariant under scale changes in $\theta$ and $\hat{\theta}$, and, hence, one would expect a prior measure with density $\frac{1}{\theta} \pi(\rho)$ to be reasonable. Actually, the same priors used before will again lead to a reasonable estimator. Thus, using $\pi(\rho)$ of form (5.4) with $f_a(\rho)$ given by (5.7) (as before), we can find the formal Bayes estimator $\varphi_a^*(s,w)$ (using reasoning in (5.2)) as follows:

\begin{equation}
(5.12) \quad \varphi_a^*(s,w) = \frac{\int \int p(\theta, \rho)(x) \frac{1}{\theta} \pi(\rho) d\theta d\rho}{\int \int \frac{1}{\theta} p(\theta, \rho)(x) \pi(\rho) d\theta d\rho}.
\end{equation}
\[
\int \frac{e^{-s/\theta} \log n}{(\Pi x_i)^{n\theta}} \int \frac{e^{n\theta} \Gamma_n(n)}{\theta^{n\theta}} \pi(\rho) d\theta d\rho \\
= \int \frac{e^{-s/\theta} \log n}{(\Pi x_i)^{n\theta}} \int \frac{e^{n\theta} \Gamma_n(n)}{\theta^{n\theta}} \pi(\rho) d\theta d\rho
\]

\[
= S \left\{ \int \frac{\Gamma(n\theta)}{\Pi \theta^{n\theta}} e^{-s/\theta} \log n \cdot \frac{\Gamma_n(n)}{\Gamma(n\theta)} e^{n\theta} \log n f_a(\rho) d\rho \right\} \\
= S \left\{ \int \frac{\Gamma(n\theta)}{\Pi \theta^{n\theta}} e^{-s/\theta} \log n \cdot \frac{\Gamma_n(n)}{\Gamma(n\theta)} e^{n\theta} \log n f_a(\rho) d\rho \right\}
\]

As in formulas (5.8) and (5.9), this can be expressed as follows:

\[
(5.13) \quad \theta_a^*(s, w) = \frac{sw}{n} \left\{ \int_0^{\infty} u^{-2} e^{-u} du + (sw)^2 \int_0^{\infty} u^{-2} e^{-u} du \right\} \\
= \frac{sw}{n} \left\{ \frac{n-1}{2} \mu_2(sw) + \frac{n-1}{2} \right\}
\]

where

\[
(5.14) \quad \mu_2(x) = \frac{\int_0^{\infty} u^{-2} e^{-u} du}{\int_0^{\infty} u^{-2} e^{-u} du + \frac{n-1}{2} \int_0^{\infty} u^{-2} e^{-u} du + \frac{n-1}{2} \int_0^{\infty} u^{-2} e^{-u} du}
\]

Note that \( \mu_2(x) \) behaves quite similarly to \( \mu_1(x) \) (defined in (5.10)), and as before the formal Bayes estimator is of the form of the asymptotic maximum likelihood estimator except that the coefficient in the denominator is not fixed but is a weighted combination of \( \frac{n-1}{2} \) and \( n-1 \).
6. **Conclusions**

In this section justification will be given for the statements made in section 1 that the formal Bayes estimators should definitely be preferred to the maximum likelihood or to the method of moments estimators.

The first point of comparison is that the formal Bayes estimators are easier to calculate than the maximum likelihood estimators. Although graphical methods or the use of computers can greatly simplify calculation of maximum likelihood estimators, the formal Bayes estimators are already expressed in terms of the incomplete gamma function which is well tabulated (e.g. see Pearson (5)) or generally available as a computer program. In fact, the formal Bayes estimators \( \varphi_0(w) \) for \( \rho \) (see 5.8) and \( \varphi^*(s,w) \) for \( \theta \) (see 5.13) are just \( \frac{n-1}{2w} \) and \( \frac{2sw}{n(n-1)} \) respectively (with \( s \) the sum of the observations and \( w \) given by (1.4)); and, hence, are particularly easy to calculate. Furthermore, since for \( \rho \) large \( w \) will tend to be small, \( P \) (see (5.10)) will be small and \( \varphi_a \) will be very nearly \( \varphi_0 \). Indication of this is given in table V in the appendix.

More important, of course, is the fact that the expected loss of the formal Bayes estimators is substantially smaller than that of the maximum likelihood estimators. The general order of this improvement, especially of \( \varphi_a \) and \( \varphi^* \) with a small, is indicated by the asymptotic results given in tables I and II. In particular, consider table I for the estimation of \( \rho \). For small \( n \) \( (n \leq 15) \) the formal Bayes estimators (asymptotically corresponding to \( c_4 \)) can be
substantially better than the maximum likelihood estimator; and this improvement is still definite until \( n = 35 \) or so. Furthermore, the improvement holds even if other loss functions are considered. For example, for squared error loss \( (L_2) \) the formal Bayes constant \( (c_4) \) appears to give even better improvement over the maximum likelihood constant \( (c_6) \) than for the loss \( L_4 \).

Table III gives a precise comparison of the estimators of \( \rho \). This table lists the expected loss \( (L_n) \) for the maximum likelihood estimator and for the formal Bayes estimators \( \varphi_a \) for several values of \( a \). First note that (as mentioned above) the expected loss for arbitrary values of \( \rho \) is quite close to the asymptotic values for the maximum likelihood estimator (for \( \rho \geq 3 \)) and for the Bayes estimator for small \( a \) (for \( \rho > 5 \) or so). More important, however, is comment that the formal Bayes estimators (for a strictly greater than zero) can do substantially better than their asymptotic values. This improvement (as a percent) persists even as \( n \) gets fairly large; for example, for \( n = 45 \) the best asymptotic estimator is only about a 10% improvement over the maximum likelihood estimator, but the formal Bayes estimator \( \varphi_a \) for \( a \geq 2 \) can have less than half the expected loss of the maximum likelihood estimator. The problem with choosing a too large is that for \( \rho \) small the expected loss can become far too large. Since \( \varphi_1 (a = 1) \) is (according to table III) uniformly better than \( \varphi_0 \) it can be safely used no matter what the true value of \( \rho \) is (so long as \( \rho \geq 1 \)). However, \( \varphi_1 \) really does not offer substantial improvement over \( \varphi_0 \) (especially for \( \rho \) moderate, say \( \rho \geq 20 \)), and, thus, if one is willing to assume that \( \rho \) is larger
than some preassigned constant, a large value of $a$ should be chosen. For example, if one is willing to assume $\rho > 8$ then $\varphi_a$ for $a = 10$ can probably be used and will still give some substantial improvement for $\rho = 20$ or $\rho = 25$ (especially for small $n$).

In considering estimation of $\theta$, Table II indicates that asymptotically the formal Bayes estimators offer little improvement over the maximum likelihood estimators. Nonetheless, use of the formal Bayes estimator $\varphi_a^*$ (see (4.13)) for $a = 0$ is probably worth using because of convenience; and when $n$ is small, Table IV indicates that one should still probably use $\varphi_a^*$ for $a = 1$ (or for $a$ larger if $\rho$ is assumed to be larger than some constant).

The formal Bayes estimators can also be justified on the basis that, for $a > 0$, they are admissible among scale invariant estimators (that is, there is no estimator of the form $f_1(w)$ for $\rho$ or $f_2(w)$ for $\theta$ that has uniformly smaller expected loss than $\varphi_a^*$ or $\varphi_a^*$ for $a > 0$).

A sketch of the proof of this fact is given in the Appendix. Although many estimators may be admissible and, in fact, the formal Bayes estimators considered here are probably not admissible among all estimators, they do at least have this optimality property not shared by the method of moments estimator or the maximum likelihood estimator. It is to be noted, in order to get a better idea of the comparisons between these estimators and of the behavior of the formal Bayes estimators, a large number of pseudo-random samples were drawn according to the distribution $f(\rho, \theta)$ and the various estimators were calculated. The results of a few of these calculations are listed in Table V and clearly indicate...
several comments. First, the maximum likelihood and method of moments estimates of $\rho$ tend to be over-estimates; and, since the Bayes estimates tend to be somewhat smaller, they tend to be definite improvements. This is indicated in most of the samples and in the selected samples 2, 4, 5, 7, and 8. The samples also indicate that the method of moments estimator is likely to be about as good as the maximum likelihood estimate. However, when they differ substantially (see samples 2 and 8), the method of methods estimate tends to be worse; so its expected loss is probably somewhat larger. Also note that the Bayes estimate for $a = 1$ is almost identical to that for $a = 0$ and it is not until $a = 10$ or so that there is any real difference. Note further, however, that the difference (due to taking $a > 0$) is appreciable only if $W$ is large (and, hence $\hat{\rho}$ small); and, thus, is almost sure to be an improvement. Similar comments also pertain to the estimation of $\theta$; although note that the Bayes estimates of $\theta$ seem to depend less on $a$ than those of $\rho$.

We conclude this section with a further remark about the method of moments estimators (see (1.3)). Actually, this method appears to be the one commonly used, probably because of the difficulty of calculating the maximum likelihood estimator. However, it is well known that the method of moments is generally inefficient compared to the maximum likelihood estimators, and probably substantially worse than any of the formal Bayes estimators. This fact is exemplified in table V. Thus if ease of calculation is desired, the formal Bayes estimators with $a = 0$
(that is, \( \hat{\rho} = \frac{n-3}{2W} \) and \( \hat{\theta} = \frac{2SW}{n(n-1)} \)) should definitely be preferred to the method of moments estimator. As remarked above, if ease of calculation is not important (or if \( n \) is small), the formal Bayes estimator with \( a = 1 \) or with \( a \) chosen according to tables III and IV) are to be preferred to the maximum likelihood estimator. Thus, strong recommendation of the formal Bayes estimators is clearly indicated.
Appendix

Tables I through IV list the expected loss, \( E(\rho, \theta)\hat{L}(\rho, \theta) \) and \( E(\rho, \theta)\tilde{L}(\rho, \theta) \) for various estimates \( \hat{\rho} \) of \( \rho \) and \( \hat{\theta} \) of \( \theta \). Table I gives the asymptotic expected loss \( E\rho_{j} L_{1}(\rho, \theta_{j}) \) of estimators \( \hat{\rho}_{j} = \frac{c_{j}}{W} \) for \( L_{1}, L_{2}, L_{3}, L_{4} \) and \( L_{5} \) given by (4.1), (4.13), (4.19), (4.24) and (4.26) respectively, and \( c_{1}, c_{2}, c_{3}, c_{4}, c_{5} \) and \( c_{6} \) given by (4.6), (4.17), (4.20), (4.25), (4.27), and \( c_{6} = \frac{n}{2W} \) respectively. The expectation is taken with respect to the asymptotic distribution of \( W \), \( W \sim \Gamma(\frac{n-1}{2}, \frac{1}{\rho}) \). Note that \( c_{1} \) is the best estimator of the form \( \frac{c}{W} \) for loss \( L_{1} \) (1-1,2,3,4,5) and \( c_{6} \) is the asymptotic maximum likelihood estimator. Similarly, table II gives the asymptotic expected loss \( E(\rho, \theta)\tilde{L}_{1}(\theta, \theta) \) of estimators \( \hat{\theta}_{j} = c_{j}SW \) for \( \tilde{L}_{1}, \tilde{L}_{2}, \tilde{L}_{3} \) and \( \tilde{L}_{4} \) given by (4.2), (4.30), (4.32), and (4.34) respectively, and \( c_{1}, c_{2}, c_{3}, c_{4}, c_{5} \) given by (4.29), (4.31), (4.33), and (4.36) respectively. Again, expectation is with respect to \( W \sim \Gamma(\frac{n-1}{2}, \frac{1}{\rho}) \) and \( S \sim \Gamma(np, \theta) \). Also, \( c_{1} \) is the best constant with respect to \( L_{1} \), and \( c_{5}SW \) is the asymptotic maximum likelihood estimator. Note that \( \tilde{L}_{1} \) and \( \tilde{L}_{1} \) correspond to squared error loss and \( \tilde{L}_{4} \) and \( \tilde{L}_{4} \) are the loss functions chosen in section 4.

Table III lists \( E\rho_{j} L_{4}(\rho, \theta) \) for \( L_{4} \) given by (4.24) and for the maximum likelihood estimator (see section 3) or the formal Bayes estimators, \( \varphi_{a} \) (see (5.9)), for several values of \( a \). The expectations are with respect to the true distribution given in section 2 for several values of the sample size, \( n \), and for several values of the parameter, \( \rho \). Since all estimators in table III depend only on \( W \), the expectations
do not depend on \( \theta \). Table IV lists \( E(\rho, \theta) \tilde{L}_4(\theta, \hat{\theta}) \) for \( \tilde{L}_4 \) given by (4.34) and for \( \hat{\theta} \) the maximum likelihood estimator (see section 3) or the formal Bayes estimators, \( \varphi_a^* \) (see (5.13)), for several values of \( a \). Again the distributions are the true ones (given in section 2) for several values of \( n \) (the sample size) and \( \rho \). Since all estimators in table IV are scale invariant, there is, again, no real dependence on \( \theta \).

Table V lists calculations applied to a number of samples drawn by a random number generator on the computer. The samples were of size \( n \) with each pseudo-random observation chosen according to the distribution \( \Gamma(\rho, \delta) \), for the listed values of \( n, \rho, \) and \( \delta \). This table lists the values of the sufficient statistics \( S \) and \( W \); the values of the method of moments, the maximum likelihood, and the formal Bayes estimators, \( \varphi_a^* \) (for \( a = 0, 1, 5, 10 \)) of \( \varphi_a \); and the values of the method of moments, maximum likelihood, and formal Bayes estimators, \( \varphi_a^* \) (for \( a = 0, 1, 5, 10 \)) of \( \theta \).

We conclude the appendix with a sketch of the proof that the formal Bayes estimators \( \varphi_a \) of \( \rho \) are admissible among invariant estimators for \( a > 0 \). A similar proof will work for the estimators \( \varphi_a^* \) of \( \theta \).

First, we reduce the problem by invariance to the following statistical decision problem: let \( W \) have distribution \( p_\rho(W) \) given in section 2. We will show that \( \varphi_a(W) \) (given by (5.9)) is an admissible estimate of \( \rho \) when the loss is given by

\[
(A.1) \quad L(\rho, \hat{\rho}) = \frac{\hat{\rho}}{\rho} - \log \frac{\hat{\rho}}{\rho} - 1.
\]
To do this, the following general sufficient condition for admissibility is used (for example, see (6)): \( \varphi_a \) is admissible if for every compact subset \( C \) of \( (0,\infty) \) there is a sequence of prior densities \( f_b(\rho) \) for which the expected loss of \( \varphi_a \) is integrable, which are greater than 1 on \( C \), and which satisfy

\[
(A.2) \quad \int_{\rho} E_p [L(\varphi_a, \rho) - L(\psi_b, \rho)] f_b(\rho) \, d\rho \to 0 \quad \text{as} \quad b \to 0
\]

where \( \psi_b \) is the Bayes estimator with respect to \( f_b \).

Let \( a \) be fixed, \( a > 0 \). We apply this theorem for each \( a \) by considering \( f_b = f(a,b) \) given by

\[
(A.3) \quad f(a,b)(\rho) = \begin{cases} 
\rho^{(n-2)+b} \left[ \frac{\Gamma^n(\rho)}{\Gamma(n\rho)} \right] e^{\rho \log n} & \rho \leq a \\
\frac{(n-1)}{2} \rho \rho^{(n-3)-b} \left[ \frac{\Gamma^n(\rho)}{\Gamma(n\rho)} \right] e^{\rho \log n} & \rho > a 
\end{cases}
\]

Note that \( f(a,0) \) is the formal prior which yields \( \varphi_a \). Thus, using the similarity between \( f(a,b) \) and \( f(a,0) \), we can find that the Bayes rule \( \psi_b \) (with respect to \( f(a,b) \)) is

\[
(A.4) \quad \psi_b(W) = \frac{1}{W} \left( \frac{1}{2} (n-1) + 2b \right) P_b(aW) + \left( \frac{3}{2} - 2b \right)
\]

where \( P_b \) is similar to \( P_1 \) given by (5.10). In fact, \( P_b = P_1 \).

We now evaluate (A.2). By Taylor's theorem (using (A.1)),
\[ L(\varphi_a, \rho) - L(\psi_b, \rho) = (\varphi_a - \psi_b)L'(\psi_b, \rho) + \frac{1}{\rho}(\varphi_a - \psi_b)^2L''(\psi_b, \rho) \]

where differentiation is with respect to the first argument and \( \psi^* \) is between \( \varphi_a \) and \( \psi_b \). Since (by definition of Bayes rule)

\[ \int L'(\psi_b, \rho) f_b(\rho) d\rho = 0 , \]

and since \( L''(\rho, \rho) = \frac{1}{\rho^2} \) (and both \( \varphi_a \) and \( \psi_b \) are greater than \( (\frac{n-2}{2})W \) for \( b < \frac{1}{4} \))

\[ |L(\varphi_a, \rho) - L(\psi_b, \rho)| \leq \frac{(n-2)^2}{8} W^2 (\varphi_a(W) - \psi_b(W))^2 . \]

Now, from (A.4), direct calculation yields

\[ |\varphi_a(W) - \psi_b(W)| \leq \frac{1}{W} (K_1 b + K_2 |P_b^*(aw) - P_0^*(aw)|) . \]

Furthermore, some calculation (similar to that in (6)) will show that

\[ |\frac{d}{db} P_b^*(aw)| \leq K_3 \]

where \( K_3 \) is independent of \( b \) and \( W \) so long as \( n \geq 4 \) and \( b \) is small enough (say less than \( \frac{1}{6} \)). (Note that \( n \geq 4 \) is needed in order that the expected loss of \( \varphi_a \) be finite.) Therefore,

\[ |P_b^*(aw) - P_0^*(aw)| \leq \left| \int_0^b \left( \frac{d}{db} P_b^*(aw) \right) db \right| \leq bK_3 . \]
So, inserting this into (A.6) (for $n \geq 4$ and $b < \frac{1}{6}$),

\begin{equation}
(A.10) \quad |L(\varphi_a, \rho) - L(\Psi_b, \rho)| \leq K_4 b^2
\end{equation}

for some constant $K_4$. Thus, (from (A.2)),

\begin{equation}
(A.11) \quad \int_\rho \left[ L(\varphi_a, \rho) - L(\Psi_b, \rho) \right] f_b(\rho) d\rho \leq K_4 \int b^2 f_b(\rho) d\rho.
\end{equation}

Now, if $C$ is an arbitrary compact subset of $(0, \infty)$, then $K_5 f_b(\rho) \geq 1$ for $\rho \in C$ (where $K_5$ depends on $C$). Furthermore, since $f_b(\rho)$ is integrable it is easy to check that the expected loss of $\varphi_a$ is integrable. Finally, since $f_b(\rho)$ is of the order of $\rho^{-1+b}$ at $\rho = 0$ and $\rho^{-1-b}$ at $\rho = \infty$,

\begin{equation}
(A.12) \quad \int f_b(\rho) d\rho \leq \frac{1}{b} K_6.
\end{equation}

Thus, (from (A.11),

\begin{equation}
(A.13) \quad \int \rho \left[ L(\varphi_a, \rho) - L(\Psi_b, \rho) \right] (K_5 f_b(\rho)) d\rho \leq b K_4 K_5 K_6 \to 0 \text{ as } b \to 0,
\end{equation}

and the hypotheses of the theorem are satisfied. Therefore, $\varphi_a$ is admissible in the reduced problem; that is, among invariant estimators (for $n \geq 4$).

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38
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TABEL V

Results of Some Pseudo-random Samples

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θ mom | 4.894| 1.300| 3.598| 3.284| 2.870| 4.608| 2.743| 2.097|
| mle  | 4.887| 1.399| 3.974| 3.437| 2.970| 5.135| 3.103| 2.012|
| a=0  | 5.292| 1.504| 4.294| 3.709| 3.120| 5.422| 3.512| 2.273|
| a=1  | 5.292| 1.504| 4.294| 3.709| 3.120| 5.422| 3.512| 2.273|
| a=5  | 5.264| 1.504| 4.287| 3.707| 3.119| 5.407| 3.496| 2.270|
| a=10 | 4.874| 1.503| 4.129| 3.641| 3.084| 4.865| 3.384| 2.244|
REFERENCES


