APPLICATION OF CERTAIN BIRTH-DEATH AND DIFFUSION PROCESSES IN TRAFFIC FLOW

by

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For the linear growth birth-death process with parameters \( \lambda_n = n\lambda, \mu_n = n\mu \), Puri (1966, 1968) has investigated the joint distribution of the number \( X(t) \) of survivors in the process and the associated integral \( Y(t) = \int_0^t X(\tau)d\tau \). In particular, he has obtained limiting results as \( t \to \infty \). Recently one of us (McNeil (1969)) has derived the distribution of the integral functional \( W_n = \int_0^{T_n} g(X(\tau))d\tau \), where \( T_n \) is the first passage time to the origin in a general birth-death process with \( X(0) = x \) and \( g(\cdot) \) is an arbitrary function. Functionals of the form \( W_n \) arise naturally in traffic and storage theory; for example \( W_n \) may represent the total cost of a traffic jam, or the cost of storing a commodity until expiration of the stock. Moments of such functionals were found in the case of \( M/G/1 \) and \( GI/M/1 \) queues by Gaver (1969) and Daley (1969).

Our object in the present paper is to study the joint distributions of both \( \{X(t), Y(t)\} \) and \( \{T_n, W_n\} \) for general birth-death processes. We shall first consider the joint distribution function

\[
(1.1) \quad P_{X_n}(t,u) = \text{Pr}(X(t) = n, Y(t) = \int_0^t g(X(\tau))d\tau \leq u | X(0) = x)
\]

for a general time-homogeneous birth-death process \( X(t) \) with parameters
\( \lambda_n, \mu_n \). Note that \( Y(t) \) is here defined somewhat more generally than in Puri's work, where \( g(X) = X \); the initial population size \( x \) takes integral values \( 0 \leq x < \infty \).

Next we derive an explicit representation for the joint Laplace-Stieltjes transform

\[
\psi_x(\theta, \varphi) = E(e^{-\theta T_x + \varphi W_x}), \text{Re}(\theta), \text{Re}(\varphi) \geq 0
\]

of \( \{T_x, W_x\} \) in the birth-death process, when \( x > 0 \). Thence, using some results of Stone (1961), the study of (1.2) is extended to a family of diffusions which includes the classical Wiener process as a member.

In order to predict \( Y(t) \) from \( X(t) \), or alternatively \( W_x \) from \( T_x \), or simply to obtain a measure of the dependence of one variable on the other, when moments are infinite, the regression is of interest. Explicit formulas are given for the regression in special cases. Conditions are also obtained for the asymptotic joint normality of \( T_x \) and \( W_x \) when \( x \to \infty \). The paper concludes with an application to a car parking model.