AXIALLY SYMMETRIC CAVITATIONAL FLOW

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P. R. GARABEDIAN, H. LEWY, AND M. SCHIFFER

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TABLE OF CONTENTS

1. Introduction ........................................ 1
2. Formulation of the physical problem ............ 4
3. The minimum problem ................................ 8
4. A lemma on symmetrization ........................ 12
5. Existence of an extremal configuration .......... 21
6. Interior variations .................................. 24
7. Generalized boundary conditions .................. 30
8. Analyticity of the free boundary ................. 36
9. Solution of the functional equation .............. 44
10. Uniqueness .......................................... 51
11. The case of an infinite cavity ..................... 57
12. Summary and discussion of results .............. 62
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P. R. Garabedian, H. Lewy, and M. Schiffer

1. Introduction.

The mathematical theory of cavitational flow dates back to the early work of Helmholtz and Kirchhoff on free streamlines. There is an extensive literature on the subject for the case of the plane irrotational flow of an incompressible fluid. In addition to existence and uniqueness theorems, for plane cavitational flow many explicit examples are known and much qualitative information is available. However, for axially symmetric flow, there is almost no mathematical work on cavitation.

Cavitational flow occurs physically when an object, such as a torpedo, travels so fast through water that the fluid pressure falls to vapor pressure, thus causing the water to boil and to form a cavity of steam following the object. When a torpedo enters the water from the air, it is also possible for a pocket of air to follow, and thus the cavity in some cases consists partly of air and partly of steam. Typical of such a situation is that the object, or torpedo, possesses axial symmetry; if the object moves in the direction of the axis of symmetry and if the fluid fills the entire space, it can be shown that the flow will be axially symmetric as well.

A common procedure among the engineers in order to apply the highly developed mathematical theory of plane flow past free streamlines to the actual physical problem at hand has been to rotate the plane configuration about an axis of symmetry and estimate in this way the shape of the steam pocket and the drag on the object. Although obviously unsatisfactory to the mathematician, this approach has led to results which are in good agreement with experimental data [15].
The object of the present paper is to lay the foundation for an analysis and construction of axially symmetric cavitational flows. The development centers about the proof of existence and uniqueness of cavitational flow past a given rotationally symmetric body with prescribed cavitation constant, or with prescribed circle of separation. The method applies for various models of interest to the experimentalist, but it is carried out in complete detail only for the simplest case in which its main features can be clearly brought to light.

Our existence proof depends on a reformulation of the cavitational flow problem as a free boundary extremal problem in the calculus of variations. This variational formulation of the hydrodynamic question appears to have been given for the first time by Riabouchinsky [17]. A later discussion of the variational problem, which combined it with Dirichlet's principle, was given by Friedrichs [3], who deduced from it the local uniqueness of the flow from a nozzle. An existence theorem for plane cavitating flow based on the above variational theory has been given quite recently [1, 4, 5].

Basic for the present treatment of the variational problem is a lemma on the behavior of the virtual mass of an object under symmetrization which was first stated in [5] and later developed and generalized by Payne and Weinstein [14]. This lemma is an outgrowth of the original studies on symmetrization given by Pólya and Szegö [16]. It is used here in order to deduce the existence of a rectifiable solution of the extremal problem for cavity flow.

A preliminary analysis of the solution of the extremal problem is given in this report by means of variational techniques which were developed originally for applications in the theory of conformal mapping [18], but which have already proved quite useful in studying physical problems [5, 19]. The main idea of the technique as it is used here is to make a variation of the independent variables by means of
certain explicit conformal mappings. This method has occasionally been described
as the method of interior variations because it involves mapping a boundary curve
by a transformation which is regular on the curve, but has a pole in its interior.

One of the main difficulties encountered in generalizing the variational
proof of existence of plane cavitating flow given in [5] to the case of axially
symmetric flows is to show that the extremal free boundary which we obtain is an
analytic curve. This difficulty has been overcome in the present paper by performing
an analytic continuation of the stream function into the complex domain of the
independent variables [9]; this allows us in turn to express the values of a certain
complex analytic function on the free boundary in purely geometrical terms. The
condition that an analytic function with these prescribed complex values on the free
boundary does exist at all may be transformed into a functional equation connected
with the geometry of the boundary. The solution of this functional equation yields
the desired analyticity of the free boundary. Similar demonstrations of the analy-
ticity of free boundaries for minimal surfaces and of free surfaces for plane flows
in a gravity field have been given already [10, 11], and these earlier results were
applied in a similar way to obtain the existence of plane cavitating flow in a
gravity field [5].

Together with the existence of axially symmetric cavitational flow, we give a
brief treatment of the uniqueness of the free boundary and the continuous dependence
of the free surface on the point of separation and the cavitation parameter. Our
results here are merely extensions of developments due to Lavrentieff [7] and Gilbarg
[6], which were applied similarly in the existence proofs for plane free boundary
flows given in [5]. The basic tool is a powerful comparison lemma which Lavrentieff
[7] appears to have been the first to apply to cavities.
In summary, the principal new ideas appearing in this paper center about the formulation of the hydrodynamical problem as a problem in the calculus of variations, the analysis of this extremal problem by symmetrization and interior variations, the discussion of analyticity of the free surface by means of a functional equation developed in the complex domain, and the treatment of uniqueness by a comparison theorem parallel to Schwarz's lemma. The results of the paper are not only significant through their explicit reference to current hydrodynamical questions, but also they constitute a contribution to the study of free boundary extremal problems in the calculus of variations which arise in the theory of partial differential equations of elliptic type in two independent variables.

2. Formulation of the physical problem.

Let there be given in the \((x,y)\)-plane a simple curve \(C\) with the following properties. It is symmetric in the \(y\)-axis; it rises from a point \((-k_0,0)\) on the negative \(x\)-axis monotonically to a point \((-k,h)\) in the second quadrant; then it remains horizontal up to the point \((k,h)\); and thereafter it descends on a symmetric arc to the point \((k_0,0)\) on the positive \(x\)-axis. The arc \(C\) can be represented by giving \(y\) as a positive non-increasing function of \(|x|\).

We denote by \(B\) the object bounded by \(C\) and the \(x\)-axis. By \(L\) we denote a set of arcs outside \(B\) which lies in the half-strip \(-k \leq x \leq k, y > 0\) and which bounds there, together with \(C\), a set of regions \(W\), which we term the cavity. We define the region \(D\) of the upper half-plane \(y > 0\) as the complement of the closure of \(B + W\) in the space \(y > 0\), and we require it to be simply-connected.

In the three-dimensional space with cylindrical coordinates \(\Theta, y\) and \(x\) \((y = \text{distance from } x\)-axis\) we introduce the axially symmetric domains \(B', W'\)
and \( D' \) obtained by rotating \( B', W \) and \( D \) about the \( x' \)-axis. These bodies are separated from one another by surfaces of revolution \( C' \) and \( L' \) generated by rotation of \( C \) and \( L \) about the \( x' \)-axis. We shall be concerned here with the steady axially symmetric irrotational flow in \( D' \) of an incompressible fluid past the object \( B' + W' \). We assume the constant density of the fluid to be \( \rho = 1 \). This flow has a velocity potential \( \varphi = \varphi(x,y) \) and a stream function \( \psi = \psi(x,y) \) which satisfy the generalized Cauchy-Riemann equations

\[
\frac{\partial \varphi}{\partial x} = \frac{1}{y} \frac{\partial \psi}{\partial y}, \quad \frac{\partial \psi}{\partial y} = -\frac{1}{y} \frac{\partial \varphi}{\partial x}
\]

in \( D \) and which fulfill on those portions of \( L \) and \( C \) which generate the surface of \( D' \) the equivalent boundary conditions

\[
\psi = 0
\]

and

\[
\frac{\partial \varphi}{\partial n} = 0
\]

where \( n \) denotes the outer normal of \( C \) and \( L \) with respect to the flow region \( D' \).

Since \( \varphi \) and \( \psi \) do not depend on the angle \( \Theta \), it will suffice in our discussions to treat the system of partial differential equations (2.1) in the plane region \( D \) rather than in the actual flow space \( D' \). Thus we shall make very little explicit reference to the three-dimensional geometry of \( B', W', D', C', \) and \( L' \), and shall deal rather with a mathematical problem in two independent variables \( x \) and \( y \).
We can eliminate $\varphi$ in (2.1) to obtain for $\psi$ the elliptic partial differential equation

\[
(2.4) \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{y} \frac{\partial \psi}{\partial y},
\]

while elimination of $\psi$ yields

\[
(2.5) \quad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{1}{y} \frac{\partial \varphi}{\partial y} = 0.
\]

Equation (2.5) merely states that $\varphi$ is an axially symmetric harmonic function in three-dimensional space. It is well known that any solution of (2.4) or (2.5) in a region of the half-plane $y > 0$ cannot assume its maximum or minimum value at an interior point of the region, unless it is identically constant. We shall refer to this statement as the maximum principle.

We assume that our flow past $B' + W'$ behaves at infinity like a uniform flow in the horizontal direction with velocity $v$. Thus at large distances, $\varphi$ and $\psi$ have convergent expansions of the form

\[
(2.6) \quad \psi = \frac{y^2}{2} - \frac{ay^2}{r^3} + \ldots,
\]

\[
(2.7) \quad \varphi = x + \frac{ax}{r^3} + \ldots,
\]

where $r^2 = x^2 + y^2$. Thus we imagine the fluid to be in motion with the object at rest.

The physical problem for which we have developed these preliminaries arises when the portion $W'$ of the object $B' + W'$ consists of a cavity of steam and
air generated by the rapid motion of the flow. While the shape of the body \( B' \), which we may describe as the projectile, is presumed known, the shape of the pocket of steam \( W' \) is not known. On the other hand, we may assume that the pressure \( p \) of the gas within \( W' \) is constant, since this gas in \( W' \) has an inertia which is negligible relative to that of the water in \( D' \), so that the kinetic energy of its motion has a negligible influence on the pressure in \( W' \) [15]. Thus on the surface \( L' \) between \( W' \) and \( D' \) the pressure \( p \) must be constant.

Since Bernoulli's law

\[
\frac{1}{2}(\nabla \varphi)^2 + p = \text{const.}
\]  

holds throughout the flow, we deduce that the velocity \( |\nabla \varphi| \) of the fluid on the surface \( L' \) is constant. Thus on \( L \) we can write

\[
(\nabla \varphi)^2 = \lambda
\]

or, equally well, by (2.1),

\[
(\nabla \psi)^2 = \lambda \gamma^2
\]

We shall call the number \( \lambda \) the cavitation constant of the flow; it is a linear function of the difference between the pressure at infinity and the pressure on the free surface \( L' \).

The essential difficulty in determining the flow just described lies in the fact that the shape of the cavity \( W \), or, better, the shape of the curves \( L \), is unknown. Thus we call \( L \) the free boundary of the flow. In order to find the free boundary, we must make use of the additional boundary condition (2.10).

The problem of constructing the free boundary \( L \) for a given fixed boundary \( C \) will be studied in the remainder of this paper. We shall find that for each
value of the cavitation constant $\lambda$ in a certain interval it is possible to construct a free surface $L$ and a corresponding cavitation flow of the type described above. Alternatively, it will turn out that one can prescribe the point of separation of $L$ from $C$.

3. The minimum problem.

Our mathematical theory is based on an alternative formulation of the classical boundary value problem embodied in (2.2), (2.4), (2.6), and (2.10) as a variational problem [3, 5, 17]. We state the variational problem in this section and present a heuristic reasoning which indicates its relation to the boundary value problem.

A number of formal identities will prove to be quite useful. Let $K$ denote the curve made up of arcs of $C$ and $L$ which bounds together with the $x$-axis the flow region $D$. If

$$\Psi = \frac{\alpha y^2}{x^3} + \ldots$$

is any solution of (2.4) in $D$ which is regular and vanishes on the $x$-axis, then by Green's theorem we have

$$\int_K \Psi \frac{\partial \Psi}{\partial n} \frac{1}{y} \, ds + \int_{S_r} \left\{ \Psi \frac{\partial \Psi}{\partial n} - \psi \frac{\partial \Psi}{\partial n} \right\} \frac{ds}{y} = 0 ,$$

where $s$ is the arc length and $n$ is the outer normal along $K$ and $S_r$ with respect to the finite domain enclosed by $K$, $S_r$, and the $x$-axis, and where $S_r$ represents the arc of a large semi-circle of radius $r$ in the upper half-plane. Letting $r$ approach infinity, one finds in view of the normalization (2.6) of $\Psi$ and putting $y = r \sin \varphi$. 
\[
\lim_{r \to \infty} \int_{S_r} \left\{ \Psi \frac{\partial \Psi}{\partial n} - \psi \frac{\partial \Phi}{\partial n} \right\} \frac{ds}{y} \\
= \lim_{r \to \infty} \int_0^\pi \left[ \frac{\alpha y^2}{r^3} \cdot y \sin \varphi + \frac{1}{2} y^2 \cdot \frac{\alpha y^2}{r^4} \right] \frac{rd\varphi}{y} \\
= \frac{3}{2} \alpha \int_0^\pi \sin^3 \varphi \, d\varphi = 2 \alpha.
\]

Thus

\[(3.1) \quad 2 \alpha + \int_K \Psi \frac{\partial \psi}{\partial n} \frac{ds}{y} = 0.\]

We define the virtual mass \( M \) of the axially symmetric flow past \( B + W \) by the formula

\[M = \iint_D \left( \nabla \psi - \nabla \frac{y^2}{2} \right)^2 \frac{dx \, dy}{y},\]

which is equivalent by (2.1) to

\[M = \iint_D \left( \nabla \varphi - \nabla x \right)^2 y \, dx \, dy.\]

Clearly, \( 2 \Pi M \) represents the kinetic energy of the fluid motion induced if the body \( B + W' \) moves with velocity 1 against the fluid at rest. Since \( \psi = 0 \) on \( K \), Green's theorem yields for this quantity the expression

\[M = -\frac{1}{2} \int_K y^2 \frac{\partial \psi}{\partial n} \frac{ds}{y} + \frac{1}{2} \int_K y^2 \frac{\partial y}{\partial n} \, ds.\]
The second integral on the right is proportional to the volume

\[ 2 \pi V = \pi \int_y^x y^2 \, dx \]

enclosed by the surface of revolution generated by rotating \( K \) about the \( x \)-axis. This is merely the volume of \( B' + W' \). Setting \( \Psi = \psi - y^2/2 \) in (3.1) and using the development (2.6), we obtain for the first integral on the right

\[ \int_y^x y^2 \frac{\partial \psi}{\partial n} \frac{ds}{y} = -4a \]

since \( \psi \) vanishes on \( K \). Hence

\[ 2a = M + V \]  \hspace{1cm} (3.2)

an identity which expresses the coefficient \( a \) in the expansion (2.6) in terms of the mass and the virtual mass of the object \( B + W \).

Let us now shift the curves \( L \) to a new position \( L^* \) by displacing them a small distance \( \delta n \) along their outer normals. This shift results in a corresponding variation into new domains \( D^* \) and \( K^* \) of \( D \) and \( K \). In \( D^* \), there is a varied axially symmetric flow with a stream function

\[ \psi^* = \frac{y^2}{2} - \frac{a^* y^2}{x^3} + \ldots \]

and a virtual mass \( M^* \), and \( K^* \) generates a surface of revolution with varied volume \( 2 \pi V^* \). With accuracy of the first order in the magnitude of \( \delta n \), we have immediately

\[ \delta V = V^* - V = - \int_L \delta n y \, ds \]  \hspace{1cm} (3.3)

On the other hand, setting \( \Psi = \psi^* - \psi \) in (3.1), we find
\[ 2(\text{a}^* - \text{a}) = \int_K \Psi * \frac{\partial \Psi}{\partial n} \frac{ds}{y} \]

On \( K \) one has with accuracy of the first order in \( \delta n \)

\[ \Psi^* = -\frac{\partial \Psi^*}{\partial n} \delta n = -\frac{\partial \Psi}{\partial n} \delta n \]

by Taylor's theorem, since \( \Psi^* \) vanishes on \( K^* \). Thus the variation of the coefficient \( a \) is \([19]\)

\[ \delta a = a^* - a = -\frac{1}{2} \int_L (\frac{\partial \Psi}{\partial n})^2 \delta n \frac{ds}{y} \]

Combining (3.3) and (3.4) we obtain by (3.2) the formula for the variation of energy \( M \)

\[ \delta M = M^* - M = \int_L \left\{ 1 - \frac{1}{y^2} (\frac{\partial \Psi}{\partial n})^2 \right\} \delta n y \, ds \]

If we write the additional boundary condition (2.10) which holds along the constant pressure free boundary for a cavitation flow in the form

\[ \frac{1}{y^2} (\frac{\partial \Psi}{\partial n})^2 = \lambda \]

we see by (3.3) and (3.4) that the expression

\[ 2a - \lambda V \]

is stationary for shifts of the free boundary,

\[ 2 \delta a - \lambda \delta V = 0 \]

Thus it is natural in order to show the existence of cavitation axial symmetric flows to attempt to choose the free curves \( L \) bounding \( W \) in such a way that
We shall show in the following, indeed, that for each suitable positive value of $\lambda$, the extremal problem (3.7) has a solution with $L$ constrained to lie in the strip $-k \leq x \leq k$, $y > 0$ and outside $B$, and we shall prove that this solution yields a cavitational flow in which $L$ appears as the free surface.

4. A lemma on symmetrization.

Our treatment of the minimum problem (3.7) requires an analysis of the behavior of the virtual mass $M$ of an arbitrary axially symmetric object under symmetrization in the $y$-axis and in the $x$-axis [5, 14].

To begin with, we shall suppose that the curve $K$ bounding $D$ is an analytic arc with at most a finite number of points of inflection. In the expression for $M$ as an energy integral we have a factor $y^{-1}$ which leads to difficulties for the symmetrization process. In order to remove it, we make the substitution $u = \psi y^{-1}$, and we obtain

$$M = \iint_D \left\{ y(\nabla u - \nabla \frac{x}{2})^2 + \frac{1}{y} (u - \frac{x}{2})^2 + \frac{\partial}{\partial y} (u - \frac{x}{2})^2 \right\} dx \, dy$$

$$= \iint_D \left\{ y(\nabla u - \nabla \frac{x}{2})^2 + \frac{1}{y} (u - \frac{x}{2})^2 \right\} dx \, dy - \frac{1}{4} \int_K y^2 dx$$

or, since $u - \frac{x}{2} \to 0$ with $y \to \infty$,

$$M = \iint_D \left\{ y \left( \frac{\partial u}{\partial x} \right)^2 + y \left( \frac{\partial u}{\partial y} - \frac{1}{2} \right)^2 + \frac{1}{y} (u - \frac{x}{2})^2 \right\} dx \, dy - \frac{1}{2} V$$

(4.1)

By the maximum principle, $\psi > 0$ in $D$ and therefore also $u > 0$ there. Furthermore, by (2.4) we have
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{y} \frac{\partial u}{\partial y} = \frac{u}{y^2}.
\]

It is convenient to define \( u = 0 \) in the region \( B + W \) lying below \( K \).

We now set \( t = y^2/2 \) and consider \( u \) as a function of \( x \) and \( t \).

We propose to symmetrize \( u \) in the \( x \)-axis, and to show that (4.1) does not increase under the symmetrization, whereas \( V \) is unchanged. To define the symmetrization, we note that for each fixed \( x \) and for each fixed positive number \( \rho \), there is a finite odd number \( m \) of values of \( t \), say \( t_1 < t_2 < \ldots < t_m \), such that

\[
(4.3) \quad u(x, t_i) = \rho , \quad i = 1, \ldots, m .
\]

The number \( m \) is always finite because the surface \( z = u(x, t) \) is piecewise analytic for all finite values of \( x \) and \( t > 0 \) and behaves at infinity like \((t/2)^{1/2}\). We set

\[
(4.4) \quad T = t_1 - t_2 + t_3 - \ldots + t_m .
\]

It is easily seen that \( T = T(\rho, x) \) is a continuous function of \( \rho \) for \( 0 < \rho < \infty \) and that it varies monotonically from a value \( T_0(x) \geq 0 \) to infinity if \( \rho \) increases from zero to infinity. In the domain \( D^* \) lying above the curve \( t = T_0(x) \) we then define the symmetrized function \( U(x, T) \) by the formula

\[
(4.5) \quad U(x, T) = \rho ,
\]

and we define \( U(x, T) = 0 \) elsewhere.

The surface \( z = U(x, t) \) is the ordinary Steiner symmetrization of the surface \( z = u(x, t) \) in the \((x, z)\)-plane. The number \( V = \int y^2 \, dx \) represents in the
(x,t)-metric the area of the flat portion of the \( u \)-surface, that is, of the region \( u(x,t) = 0 \). The area of the corresponding plane section of the surface \( U \) has not been changed by the above process of symmetrization. We proceed to show that each of the three terms appearing in the integral (4.1) decreases, or, at least, does not increase, when we replace \( u \) by \( U \) and \( D \) by the region \( D^* \) in which \( U > 0 \).

For each fixed \( x \) we can replace \( u \) and \( U \) as functions of \( t \) and \( T \) by \( t \) and \( T \) as functions of \( z \), the coordinate in the direction of \( u \) and \( U \). The contribution of the first term in the integral (4.1) extended over the vertical line corresponding to this value of \( x \) alone is then of the form

\[
\int y \left( \frac{\partial u}{\partial x} \right)^2 \, dy = \int \left( \frac{\partial u}{\partial x} \right)^2 \, dt
\]

\[
= \int_0^\infty \sum_{i=1}^{m(z)} \left[ \left( \frac{\partial t_i}{\partial x} \right)^2 / \left| \frac{\partial t_i}{\partial z} \right| \right] \, dz.
\]

Here \( m(z) \) is an odd integer which may change for a denumerable set of values \( z \), for which the curve \( u(x,t) = z \) (\( x \) fixed) possesses a horizontal tangent. These exceptional points do not affect the value of the integral and may be disregarded. But by Schwarz's inequality,

\[
\left( \sum_{i=1}^{m} (-1)^{i-1} \left( \frac{\partial t_i}{\partial x} \right)^2 \right) \leq \left( \sum_{i=1}^{m} \left[ \left( \frac{\partial t_i}{\partial x} \right)^2 / \left| \frac{\partial t_i}{\partial z} \right| \right] \right) \left( \sum_{i=1}^{m} (-1)^{i-1} \left( \frac{\partial t_i}{\partial z} \right)^2 \right),
\]

since by the definition of the values \( t_i(z) \)

\[
(-1)^{i-1} \frac{\partial t_i}{\partial z} \geq 0, \quad i = 1, \ldots, m.
\]

Hence by (4.4)
\[ \int_0^\infty \left[ \left( \frac{\partial T}{\partial x} \right)^2 / \frac{\partial T}{\partial z} \right] \, dz \leq \sum_{i=1}^m \int_0^\infty \left[ \frac{\partial t_i}{\partial x} \right]^2 / \left| \frac{\partial t_i}{\partial z} \right| \, dz . \]

Integration of this inequality with respect to \( x \) yields finally, after a return to the original independent variables,

\[ \int_D \int y \left( \frac{\partial U}{\partial x} \right)^2 \, dx \, dy \leq \int_D y \left( \frac{\partial U}{\partial x} \right)^2 \, dx \, dy . \]  

The later terms in the integrand of (4.1) can be integrated over any finite vertical segments, and since for large \( y \), or \( t \), the surface \( u \) is asymptotic to the cylinder \( z = y/2 \) and does not change under symmetrization, it will suffice to show that for these segments

\[ \int y \left( \frac{\partial u}{\partial y} \right)^2 \, dy' = \int \left( \frac{\partial u}{\partial t} \right)^2 \, t \, dt \]

is not increased by the symmetrization (4.3), (4.4), (4.5). We have

\[ \int \left( \frac{\partial u}{\partial t} \right)^2 \, t \, dt = \int \sum_{i=1}^m \left[ t_i / \left| \frac{\partial t_i}{\partial z} \right| \right] \, dz , \]

whence the inequality

\[ \sum_{i=1}^m (-1)^{i-1} t_i \leq \left( \sum_{i=1}^m t_i / \left| \frac{\partial t_i}{\partial z} \right| \right) \left( \sum_{i=1}^m (-1)^{i-1} \frac{\partial t_i}{\partial z} \right) , \]

trivial because \[ (-1)^{i-1} \frac{\partial t_i}{\partial z} = \left| \frac{\partial t_i}{\partial z} \right| \], yields

\[ \int \left[ T / \left| \frac{\partial T}{\partial z} \right| \right] \, dz \leq \int \sum_{i=1}^m \left[ t_i / \left| \frac{\partial t_i}{\partial z} \right| \right] \, dz . \]

Integrating with respect to \( x \) and returning to \( x \) and \( y \) as independent variables, we find
(4.7) \[ \iiint y \left( \frac{\partial u}{\partial y} \right)^2 \, dx \, dy \leq \iint y \left( \frac{\partial u}{\partial y} \right)^2 \, dx \, dy \]

where the regions of integration are suitable bounded subsets of \( D^* \) and \( D \) outside of which \( U \equiv u \).

We note that

(4.8) \[ \int \left\{ y \frac{\partial u}{\partial y} + u \right\} dy = \int \left\{ y \frac{\partial U}{\partial y} + U \right\} dy \]

when the integrals are extended over any vertical lines from a position where \( u \) and \( U \) vanish up to a height beyond which \( u \equiv U \). Also

(4.9) \[ \int \frac{u^2}{y} \, dy \geq \int \frac{U^2}{y} \, dy \]

over any such lines, since the area under the curve \( z = u^2(y) \) is moved in the symmetrization away from the \( z \)-axis for each fixed \( x \) and is diminished.

Integrating (4.8) and (4.9) with respect to \( x \) and combining with (4.6) and (4.7), we obtain

(4.10) \[ \iiint_{D^*} \left\{ y \left( \frac{\partial U}{\partial y} \right)^2 + y \left( \frac{\partial U}{\partial y} - \frac{1}{2} \right)^2 + \frac{1}{y}(U - \frac{x}{2})^2 \right\} \, dx \, dy \leq \]

\[ \iint_{D} \left\{ y \left( \frac{\partial u}{\partial y} \right)^2 + y \left( \frac{\partial u}{\partial y} - \frac{1}{2} \right)^2 + \frac{1}{y}(u - \frac{x}{2})^2 \right\} \, dx \, dy \]

The object \( B \) lies in the domain where \( U \equiv 0 \), since the curve \( C \) is monotonic in each quadrant of the half-plane \( y > 0 \). The domain \( U \equiv 0 \) with \( B \) cut out will be defined to be the symmetrized cavity \( W^* \). The volume \( V^* \) of \( B + W^* \) is equal to \( V \). We denote by

\[ \psi^* = \frac{\psi^2}{2} - \frac{a^* y^2}{r^3} + \ldots \]

the stream function of the flow through the symmetrized region \( D^* \) and set
\( u^* = \psi^* y^{-\frac{1}{2}} \). The function \( u^*(x,y) \) satisfies the differential equation (4.2) in the flow region \( D^* \), vanishes on the boundary of \( D^* \) and behaves at infinity like \( \frac{X}{2} \), that is, like \( U \). Hence by Dirichlet's principle for (4.2) we have

\[
\iint_{D^*} \left\{ y\left(\frac{\partial u^*}{\partial x}\right)^2 + y\left(\frac{\partial u^*}{\partial y} - \frac{1}{2}\right)^2 + \frac{1}{y}(u^* - \frac{X}{2})^2 \right\} \, dx \, dy \leq \\
\iint_{D^*} \left\{ y\left(\frac{\partial U}{\partial x}\right)^2 + y\left(\frac{\partial U}{\partial y} - \frac{1}{2}\right)^2 + \frac{1}{y}(U - \frac{X}{2})^2 \right\} \, dx \, dy .
\]

Thus, by (4.10), the virtual mass \( M^* \) of the symmetrized object \( B + W^* \) is smaller than the original virtual mass \( M \),

\[
(4.12) \quad M^* \leq M.
\]

Therefore also

\[
(4.13) \quad a^* \leq a , \quad V^* = V .
\]

The inequalities (4.12) and (4.13) can be derived for Steiner symmetrization of \( B + W \) in the \( y \)-axis by a similar procedure. In this case, for each fixed \( y \) and positive \( \rho \) there are values \( x_1, \ldots, x_m \), finite in number, such that

\[
(4.14) \quad u(x_i, y) = \rho , \quad i = 1, \ldots, m .
\]

We set

\[
(4.15) \quad X = x_1 - x_2 + \ldots + x_m
\]

and define the symmetrized surface \( z = U(X, y) \) by the formula

\[
(4.16) \quad U(X, y) = \rho .
\]

Again we can use the coordinate \( z \) in the direction of \( u \) and \( U \) as independent variable, and we obtain in view of the trivial inequality \( \sum \alpha_i \cdot \sum \frac{1}{\alpha_i} \geq 1 \) valid for \( \alpha_i > 0 \).
\begin{align}
(4.17) \quad y \int \left( \frac{\partial U}{\partial x} \right)^2 \, dx &= y \int \frac{dz}{\frac{\partial x}{\partial z}} = y \int \frac{dz}{\sum_{i=1}^{m} \left| \frac{\partial x_i}{\partial z} \right|} \\
\int \sum_{i=1}^{m} y \frac{dz}{\left| \frac{\partial x_i}{\partial z} \right|} &= y \int \left( \frac{\partial u}{\partial x} \right)^2 \, dx,
\end{align}

and
\begin{align}
(4.18) \quad \frac{1}{y} \int (u - \frac{x}{2})^2 \, dx &= \frac{1}{y} \int (u - \frac{x}{2})^2 \, dx.
\end{align}

Since \( y \) is fixed in (4.14), (4.15) and (4.16), we see that the surface
\[ v(x,y) = u(x,y) - \frac{y}{2} \]
is symmetrized by these formulas to yield the surface
\[ V(X,y) = U(X,y) - \frac{y}{2} \]

We express now \( X = \Omega(y,z) \), where \( z = V(X,y) \). Similarly, we use \( z = v(x,y) \)
in order to express \( x = w(y,z) \). Then, we find by Schwarz's inequality
\begin{align}
(4.19) \quad y \int \left( \frac{\partial U}{\partial y} - \frac{1}{2} \right)^2 \, dx &= y \int \left( \frac{\partial V}{\partial y} \right)^2 \, dx = y \int \left[ \left( \frac{\partial x}{\partial y} \right)^2 / \left| \frac{\partial x}{\partial z} \right| \right] \, dz \\
&= y \int \left[ \left( \sum_{i=1}^{m} (-1)^i \frac{\partial x_i}{\partial y} \right)^2 / \sum_{i=1}^{m} \left| \frac{\partial x_i}{\partial z} \right| \right] \, dz \\
&\leq y \int \sum_{i=1}^{m} \left[ \left( \frac{\partial x_i}{\partial y} \right)^2 / \left| \frac{\partial x_i}{\partial z} \right| \right] \, dz = y \int \left( \frac{\partial u}{\partial y} \right)^2 \, dx \\
&= y \int \left( \frac{\partial u}{\partial y} - \frac{1}{2} \right)^2 \, dx.
\end{align}
We integrate (4.17), (4.18) and (4.19) with respect to \( y \) and obtain (4.10) once more, with the function \( U \) and the domain \( D^* \) now interpreted as the symmetrizations of \( u \) and \( D \) in the \( y \)-axis. By Dirichlet's principle for (4.2) we find that (4.11) is again valid, and this yields (4.12) and (4.13) for symmetrization of \( B+W \) in the \( y \)-axis, as was desired.

Thus far we assumed that the curve \( K \) bounding \( D \) is an analytic arc and has, therefore, finitely many inflection points. In order to remove this restriction, we will approximate a more general curve \( K \) from within \( D \) by analytic arcs and derive (4.13) by passage to the limit as the approximation becomes arbitrarily fine. The analytic arcs approximating \( K \) may be taken explicitly to be level curves of the Green's function for Laplace's equation in the domain consisting of \( D \) and its reflection in the \( x \)-axis.

In order to carry out the passage to the limit, we will use the following remarks. Let \( D_1 \) be a subdomain of \( D \) bounded by an arc \( K_1 \) and the two infinite segments of the \( x \)-axis. Let \( \psi_1(x,y) \) be the stream function belonging to \( D_1 \) with the development

\[
(4.20) \quad \psi_1(x,y) = \frac{x^2}{2} - \frac{a_1}{r^3} y^2 + \ldots
\]

at infinity. Since \( \psi_1(x,y) \) is a solution of (2.4), the difference function \( \psi(x,y) - \psi_1(x,y) \) represents a solution of (2.4) which is finite and regular everywhere in \( D_1 \). This function vanishes at infinity and is non-negative on \( K_1 \); hence, we have by the maximum principle

\[
(4.21) \quad \psi_1(x,y) \leq \psi(x,y) \quad \text{in} \quad D_1.
\]

Since near infinity

\[
(4.22) \quad \psi(x,y) - \psi_1(x,y) = \frac{(a_1 - a) y^2}{r^3} + \ldots
\]
we can conclude from (4.21) also

\[(4.23) \quad a \leq a_1.\]

Let now \(D_n\) be a sequence of such subdomains of \(D\) which converge increasingly to \(D\). From the preceding remarks it follows that the corresponding stream functions \(\psi_n(x,y)\) converge. Since for \(n > k\) the maximum of \(\psi - \psi_n\) in \(D_k\) is assumed on \(K_k\) and is \(\leq \psi\), and since \(\psi\) tends uniformly to zero on \(K_k\) as \(k \to \infty\), it follows that for a fixed \(D_{k_0}\), the convergence of the \(\psi_n\) is uniform and toward \(\psi\). Consequently, the coefficients \(a_n\) converge to \(a\).

Let \(K\) be an arbitrary curve in the half-plane \(y \geq 0\) which bounds a flow region \(D\). We can certainly find an analytic curve \(K_1\) in \(D\) which determines a subregion \(D_1 \subset D\) such that, for \(\epsilon > 0\) arbitrarily given, we have \(a_1 < a_1 + \epsilon\). If we now symmetrize \(D\) and \(D_1\) into two new regions \(D^*\) and \(D_1^*\), it can easily be seen that \(D_1^* \subset D^*\). Since we proved already that symmetrization decreases the coefficient \(a_1\) of a domain with analytic boundary curves and because of (4.23), we have the chain of inequalities

\[(4.24) \quad a^* \leq a_1^* \leq a_1 < a_1 + \epsilon.\]

Since this is true for every choice of \(\epsilon > 0\), we have

\[(4.13) \quad a^* \leq a\]

and have now proved that symmetrization diminishes the virtual mass for a general type of boundary curves \(K\).

Let us recall finally some properties of the level curves \(\psi = \text{const.}\) for a symmetrized body \(B + W\). Let us suppose that the boundary curve \(K\) of \(B + W\) is an analytic arc and that it is symmetrized in the \(y\)-axis. We clearly have
\( \psi(x, y) = \psi(-x, y) \) and hence \( \frac{\partial \psi}{\partial x} = 0 \) on the segment of the \( y \)-axis in \( D \). On the \( x \)-axis in \( D \) we also have \( \frac{\partial \psi}{\partial x} = 0 \), while on the part of the curve \( K \) which lies in the quadrant \( x > 0, y > 0 \) we obviously have \( \frac{\partial \psi}{\partial x} > 0 \). Hence, applying the minimum principle to the function \( \frac{\partial \psi}{\partial x} \), which is a solution of (2.4), we find \( \frac{\partial \psi}{\partial x} > 0 \) in the whole common part of \( D \) and the first quadrant. The same inequality must also be valid if \( K \) is a curve of general type, since it can be approximated arbitrarily by symmetrized analytic curves and since \( \frac{\partial \psi_n}{\partial x} \rightarrow \frac{\partial \psi}{\partial x} \) uniformly in each closed subdomain of the region considered. Thus, we find that the level curves \( \psi = \text{const.} \) of an object \( B + W \) symmetrized in the \( y \)-axis are themselves symmetrized in the \( y \)-axis.

If \( B + W \) is symmetrized in the \( x \)-axis, we consider the function \( \frac{\partial \varphi}{\partial x} = \frac{1}{y} \frac{\partial \psi}{\partial y} \). This function is non-negative on the boundary of \( D \), satisfies (2.5) and, hence, by the minimum principle we have \( \frac{\partial \psi}{\partial y} > 0 \) in \( D \). This shows that now all level curves are symmetrized in the \( x \)-axis. Thus the level curves \( \psi = \text{const.} \) for symmetrized bodies \( B + W \) are monotonically rising in the second quadrant and monotonically descending in the first quadrant, a fact which is of considerable use in the applications.

5. Existence of an extremal configuration.

For a given curve \( C \) of the type introduced in Section 1 enclosing a given object \( B \), we seek to find curves \( L \) enclosing with \( C \) a shape \( W \) lying outside \( B \), but lying within the strip \( -k \leq x \leq k \), which possess the extremal property (3.7). We show in this section that rectifiable extremal curves of the desired type exist for each positive value of \( \lambda \).

If, for a given \( \lambda \), such an extremal curve could not be found, we would be able to select a sequence of curves \( L_n \) in the strip \( -k \leq x \leq k \) and a sequence of
flow regions $D_n$ for which the corresponding expressions

$$2a_n - \lambda V_n$$

approach their greatest lower bound. We shall call such a sequence a minimal sequence. The object $B^+W_n$ complementary to $D_n$ can be assumed to be symmetrized in the $x$-axis and in the $y$-axis without loss of generality. For, if this were not the case, we could replace the minimal sequence $B^+W_n$ by the sequence of objects obtained by symmetrizing them in the $x$-axis and $y$-axis. By Section 4, this would not increase the numbers $a_n$ and $V_n$, and would thus leave us still with a minimal sequence. It is important to note here that symmetrization in the $x$-axis and $y$-axis, as described in Section 4, leaves the fixed object $B$ invariant.

Symmetrized objects $B^+W_n$ are intersected by each vertical or horizontal line in at most one segment. Hence the curves $L_n$ and the related curves $K_n$ bounding $D_n$ are monotonic in each quadrant. If we consider a coordinate system $\Sigma$, $\mathcal{C}$ obtained by rotation of the $(x,y)$-plane through $45^\circ$, we find that the representation

$$\mathcal{C} = h_n(\sigma)$$

of the arc of $K_n$ in the first quadrant $x > 0$, $y > 0$ satisfies the Lipschitz condition

$$|h_n(\sigma_2) - h_n(\sigma_1)| \leq |\sigma_2 - \sigma_1|.$$ 

Thus the functions $h_n$ are equicontinuous; we will show in the next paragraph that the curves $K_n$ have bounded ordinates and, consequently, bounded values $h_n(\sigma)$. Hence we can select among them a uniformly convergent subsequence. Let us assume that the original sequence is uniformly convergent, with no loss of generality. It
follows that the curves $K_n$ and the regions $D_n$ converge to a limit curve $K$ and a limit domain $D$, and it follows that the object $B + W_n$ converges to a limiting object $B + W$.

We wish to show that $W$ does not extend to infinity. To see this, we let

$$\psi_o = \frac{y^2}{2} - \frac{a_o y^2}{r^3} + \ldots$$

be the stream function for the flow past the unit disc $x = 0, 0 \leq y \leq 1$. This flow has a positive virtual mass and hence $a_o > 0$. On the other hand, from (3.4) we verify the well-known fact that the coefficient $a_n$ decreases as the object $B + W_n$ diminishes. Thus if $R_n$ denotes the ordinate of the highest point on $K_n$, we find that $a_n$ is larger than the corresponding coefficient for the axially symmetric flow past a disc of radius $R_n$, since such a disc is contained within $B + W_n$. The stream function for the flow past the disc of radius $R_n$ is easily derived from that for the unit disc, and is

$$R_n^2 \psi_o \left( \frac{x}{R}, \frac{y}{R} \right) = \frac{y^2}{2} - \frac{R_n^3 a_o y^2}{r^3} + \ldots;$$

therefore

$$a_n \geq R_n^3 a_o \ldots \tag{5.1}$$

For the volume $V_n$ we have obviously

$$V_n \leq 2k R_n^2 \ldots \tag{5.2}$$

and hence $a_n$ increases more rapidly than $V_n$ as $R_n$ tends to infinity. We conclude that $R_n$ is bounded for a minimal sequence for (3.7), and hence $W$ must be bounded.

Thus in order to show that for each positive $\lambda$ there exists a set of
monotonic extremal curves \( L \) in the strip \(-k \leq x \leq k\) bounding a finite cavity \( \mathbb{W}_0 \) which is a solution of the minimum problem (3.7), we have only to prove that the stream functions \( \psi_n \) for the flows in \( D_n \) converge to the stream function \( \psi \) of the flow in \( D \). Since the level curves of \( \psi_n \) descend in the first quadrant, \( \partial \psi_n / \partial y > 0 \) there. Hence by (2.4), \( \psi_n \) is subharmonic in the first quadrant. In the neighborhood of a point \( x_0, y_0 \) on \( K_n \), \( \psi_n \) must therefore be dominated by the positive harmonic function

\[
\text{Im} \left\{ [i(x-x_0)-(y-y_0)]^{1/2} \right\},
\]

since \( D_n \) does not intersect the vertical segment \( 0 \leq y \leq y_0, x = x_0 \). A similar statement can be made concerning \( \psi \), and we conclude by the maximum principle that if \( K_n \) has a maximal distance \( \varepsilon \) from \( K \), then

\[
| \psi - \psi_n | \leq A \varepsilon^{1/2},
\]

for a fixed positive number \( A \). The desired convergence of \( \psi_n \) to \( \psi \) follows, and also we find that \( a_n \to a \).

6. Interior variations.

Once in possession of the rectifiable extremal curves \( L \) for (3.7) we can proceed to apply variational methods in order to show that they satisfy the constant pressure condition (2.10).

Let \( z_0 = x_0 + iy_0 \) be an interior point of an arc of \( L \), and let \( \rho > 0 \) be so small that the circle

\[
|z - z_0| \leq 2 \rho
\]

(6.1) does not intersect \( B \). Suppose that \( F(z, \overline{z}) \) is a suitably differentiable,
complex-valued function of $x$ and $y$ which vanishes outside the circle (6.1). We make for sufficiently small complex values of the parameter $\epsilon$ the one-to-one transformation of coordinates

$$z^* = z + \epsilon F(z, \overline{z})$$

This transforms $L$ in the neighborhood of $z_0$ into a new free curve $L^*$ and induces at the same time a variation

$$\psi^{**}(x^*, y^*) = \psi(x, y)$$

of the function $\psi$, with $z = x + iy$, $z^* = x^* + iy^*$. The curve $L^*$ bounds a varied flow region $D^*$ with virtual mass $M^*$, and it bounds a varied object $B + W^*$ with volume $V^*$.

If $\psi^*$ denotes the stream function of the normalized axially symmetric flow in $D^*$, then by Dirichlet's principle

$$M^* = \iint_{D^*} (\nabla \psi^* - \nabla \frac{y^2}{2})^2 \frac{dx \, dy}{y}$$

$$\leq \iint_{D^*} (\nabla \psi^{**} - \nabla \frac{y^2}{2})^2 \frac{dx \, dy}{y}$$

where $\psi^{**} = \psi^{**}(x, y)$. But by (3.7) and (3.2)

$$M + V - \lambda V \leq M^* + V^* - \lambda V^*$$

whence

$$\lambda - 1 \left\{ \iint_{W^*} y \, dx \, dy - \iint_{W} y \, dx \, dy \right\} \leq$$

$$\iint_{D^*} (\nabla \psi^{**} - \nabla \frac{y^2}{2})^2 \frac{dx \, dy}{y} - \iint_{D} (\nabla \psi - \nabla \frac{y^2}{2})^2 \frac{dx \, dy}{y}. $$
We introduce the symbols
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]
and calculate the first order term in \(\epsilon\) in the inequality (6.4), noting that the terms of order zero cancel. Since \(F\) vanishes outside the circle of radius \(2\) about \(z_0\), the right-hand side of (6.4) can be written, after an integration by parts,
\[
\iint_{\Omega^*} \left[ (\nabla \psi^*)^2 + y^2 \right] \frac{dx \, dy}{y} = \iint_{\Omega} \left[ (\nabla \psi)^2 + y^2 \right] \frac{dx \, dy}{y},
\]
where \(\Omega\) is the intersection of \(D\) with the circle (6.1) and \(\Omega^*\) is the image of \(\Omega\) by the transformation (6.2). We find easily
\[
\frac{\partial (x^*, y^*)}{\partial (x, y)} = 1 + 2 \Re \left\{ \epsilon \frac{\partial F}{\partial \bar{z}} \right\} + O(\epsilon^2),
\]
\[
(\nabla \psi^*)^2 = 4 \left| \frac{\partial \psi^*}{\partial z^*} \right|^2 = 4 \left| \frac{\partial \psi}{\partial z} \right|^2 + 3 \Re \left\{ \epsilon \frac{\partial F}{\partial z} \left( \frac{\partial \psi}{\partial z} \right)^2 \right\} + O(\epsilon^2),
\]
\[
y^* = y + \Re \left\{ \frac{\epsilon F}{i} \right\},
\]
where \(O(\epsilon^2)\) denotes terms of the second order in \(\epsilon\), that is, terms such that \(O(\epsilon^2)/|\epsilon|^2\) remains bounded as \(\epsilon \to 0\). Hence, if we replace in the integrals over \(D^*\) and \(W^*\) the variables \(x, y\) by \(x^*(x, y)\) and \(y^*(x, y)\) and integrate over \(D\) and \(W\) respectively, (6.4) can be written
\[(\lambda - 1) \text{Re} \left\{ \varepsilon \iint_{W} \left\{ \frac{F}{i} + 2y \frac{\partial F}{\partial z} \right\} \, dx \, dy \right\} \leq 0 \]

\[
\text{Re} \left\{ \varepsilon \iint_{\Omega} \left[ \frac{F}{i} + 2y \frac{\partial F}{\partial z} \right] \, dx \, dy \right\} - 8 \text{Re} \left\{ \varepsilon \iint_{\Omega} \frac{\partial \varphi}{\partial z} \left( \frac{\partial \varphi}{\partial z} \right)^2 \, \frac{dx \, dv}{y^2} \right\}
- 4 \text{Re} \left\{ \varepsilon \iint_{\Omega} \left| \frac{\partial \varphi}{\partial z} \right|^2 \, \frac{dx \, dv}{y^2} \right\} + o(\varepsilon^2) \]

The circle (6.1) is transformed into itself by (6.2), and therefore the volume generated by rotating it about the x-axis is unchanged, whence

\[
\iint_{W} \left\{ \frac{F}{i} + 2y \frac{\partial F}{\partial z} \right\} \, dx \, dy + \iint_{\Omega} \left\{ \frac{F}{i} + 2y \frac{\partial F}{\partial z} \right\} \, dx \, dy = 0
\]

Thus, finally,

\[
(6.5) \quad \text{Re} \left\{ \varepsilon \lambda \iint_{W} \left[ \frac{F}{i} + 2y \frac{\partial F}{\partial z} \right] \, dx \, dy + 4 \varepsilon \iint_{\Omega} \frac{\partial \varphi}{\partial z} \left| \frac{\partial \varphi}{\partial z} \right|^2 \, \frac{dx \, dv}{y^2} \right. 
\left. + 8 \varepsilon \iint_{\Omega} \frac{\partial F}{\partial z} \left( \frac{\partial \varphi}{\partial z} \right)^2 \, \frac{dx \, dv}{y} \right\} + o(\varepsilon^2) \leq 0
\]

for all sufficiently small \(\varepsilon\).

Since \(\varepsilon\) is arbitrary in (6.5), we conclude in the usual way that the following variational identity holds

\[
(6.6) \quad \lambda \iint_{W} \left[ \frac{F}{i} + 2y \frac{\partial F}{\partial z} \right] \, dx \, dy + 4 \iint_{\Omega} \frac{F}{i} \frac{\partial \varphi}{\partial z} \frac{\partial \varphi}{\partial z} \, \frac{dx \, dv}{y^2} 
+ 8 \iint_{\Omega} \frac{\partial F}{\partial z} \left( \frac{\partial \varphi}{\partial z} \right)^2 \, \frac{dx \, dv}{y} = 0
\]

We now specialize (6.2) and (6.6) so as to obtain a more useful formula.
We let $\omega$ be a function of $|z - z_0|$ which is 1 for $|z - z_0| \leq \rho$, which vanishes for $|z - z_0| \geq 2 \rho$, and which has continuous derivatives of all orders. We let $t$ be a point with $|t - z_0| < \rho$, and for $t$ in $W$ we set

$$F(z, \bar{z}) = \frac{1}{z - t} \omega.$$

For $t$ in $D$, we denote by $\eta$ a positive number so small that the circle $|z - t| \leq \eta$ is contained in $D$ and in the circle $|z - z_0| \leq \rho$. We set

$$F(z, \bar{z}) = \frac{1}{z - t} \omega \quad \text{for } |z - t| \geq \eta,$$

as before, but for $|z - t| < \eta$, we define

$$F(z, \bar{z}) = \frac{z - t}{\eta^2}.$$

The function $F$ so defined is continuous, and (6.6) can be applied.

For $t$ in $W$, we obtain by (6.6)

$$\chi \iint_{W_0} \left[ \frac{1}{z - t} - \frac{2\psi}{(z - t)^2} \right] dx \, dy + 4 \oint_{\Omega_0} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial \bar{z}} \frac{dx \, dy}{iy^2 (z - t)}$$

$$= Q(t),$$

where $Q(t)$ is an analytic function in the circle $|t - z_0| < \rho$ which is defined by a definite integral over the ring $\rho \leq |z - z_0| \leq 2 \rho$, where $W_0$ and $\Omega_0$ are the intersections of $W$ and $D$ with $|z - z_0| < \rho$, and where the improper integral over $W_0$ is taken in the sense of the Cauchy principal value. The Cauchy principal value of a double improper integral of an integrand with point singularity at $t$ is taken here to mean the limit of the same integral over the domain obtained
by elimination of a circle about \( t \) with radius approaching zero. For \( t \) in \( D \), we have

\[
\lambda \iint_{W_0} \left[ \frac{-i}{z-t} - \frac{2y}{(z-t)^2} \right] \, dx \, dy + 4 \iint_{\Omega_1} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial z} \frac{dx \, dy}{iy^2(z-t)} + 4 \iint_{\Omega_2} \left\{ \frac{z-t}{i \eta^2} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial z} \frac{1}{y^2} + \frac{2}{\eta^2} \left( \frac{\partial \psi}{\partial z} \right)^2 \frac{1}{y} \right\} \, dx \, dy = Q(t)
\]

where \( \Omega_2 \) is the circle \( |z-t| \leq \eta \) and where \( \Omega_1 = \Omega_0 - \Omega_2 \). Letting \( \eta \to 0 \), we find for \( t \) in \( D \)

(6.10) \[
\lambda \iint_{W_0} \left[ \frac{-i}{z-t} - \frac{2y}{(z-t)^2} \right] \, dx \, dy + 4 \iint_{\Omega_0} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial z} \frac{dx \, dy}{iy^2(z-t)} + 8 \pi \frac{1}{y} \left( \frac{\partial \psi}{\partial z} \right)^2 \bigg|_{z=t} = Q(t)
\]

If \( \lambda \) denotes the arc of \( L \) inside the circle \( |z-z_0| \leq \rho \), then by Green's theorem, for \( t \) in either \( \Omega_0 \) or \( W_0 \),

\[
\lambda \iint_{W_0} \left[ \frac{-i}{z-t} - \frac{2y}{(z-t)^2} \right] \, dx \, dy = \lambda \iint_{W_0} \frac{\partial}{\partial z} \frac{2y}{z-t} \, dx \, dy
\]

\[
= \frac{\lambda}{2i} \int_{\ell} \frac{2y \, dz}{z-t} + Q_1(t)
\]

where \( Q_1(t) \) is an analytic function of \( t \) in the circle \( |t-z_0| < \rho \) which is represented simply by the line integral on the right extended over arcs of \( |z-z_0| = \rho \) bounding \( W_0 \), and where the integration over \( \ell \) is carried out in a direction such that \( D \) lies on the left. We denote by \( S(t) \) the function which is \( 1 \) inside \( D \) and \( 0 \) outside \( D \), and we set
\[ Q_0(t) = iQ_1(t) - iQ_2(t), \] which is an analytic function of \( t \) throughout the circle \( |t - z_0| < \rho \). The variational identities (6.9) and (6.10) can now be combined in the one formula

\[
(6.11) \quad \lambda \int \frac{\bar{\psi}}{z - t} + 4 \int \int_{\Omega_0} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial \bar{z}} \frac{dx \, dy}{y^2 (z - t)} - \frac{16 \pi}{t - t} (\frac{\partial \psi}{\partial t})^2 \xi (t) = Q_0(t),
\]
valid for all complex values of \( t \) in \( |t - z_0| < \rho \) which do not lie on \( \ell \).

This formula will be the basis of our future considerations.

7. Generalized boundary conditions.

Our next objective is to obtain from (6.11) relations for the velocity \( \partial \psi / \partial t \) as \( t \) approaches the arc \( \ell \) of the free boundary. In order to do this we must still rearrange the formula further by application of Green's theorem.

We will have to consider integrals of the form

\[
(7.1) \quad \int \int_{\Omega_0} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial \bar{z}} A(z, \bar{z}) \frac{dx \, dy}{z - t}, \quad \int \int_{\Omega_0} \frac{\partial \psi}{\partial z} \psi A(z, \bar{z}) \frac{dx \, dy}{z - t}.
\]

with analytic coefficients \( A(z, \bar{z}) \) and to study their discontinuity character as the argument point \( t \) crosses the free boundary arc \( \ell \). Since we may develop \( A(z, \bar{z}) \) into a power series in \( z - t \) and \( \bar{z} - \bar{t} \) near \( t \) and since integrals of the form
are continuous by the Osgood-Lebesgue theorem as $t$ crosses $L$, it is sufficient to study the character of the integrals (7.1) when $A(z,\bar{z})$ is replaced by the constant term $A(t,\bar{t})$ under the integral sign. We find easily

\begin{equation}
\iint_{\Omega_o} -\frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial \bar{z}} \frac{z}{z-t} \, dx \, dy, \quad \iint_{\Omega_o} \psi \frac{\partial \psi}{\partial z} \frac{z}{z-t} \, dx \, dy,
\end{equation}

\begin{equation}
\iint_{\Omega_o} \psi \frac{\partial \psi}{\partial z} \frac{z}{z-t} \, dx \, dy
\end{equation}

\begin{equation}
= \frac{1}{4} \iint_{\Omega_o} \psi^2 \frac{z}{z-t} \, dz - \frac{1}{2} \int_\delta (t) \Pi \psi(t)^2,
\end{equation}

where the line integral is extended over the boundary arcs of $\Omega_o$. Using (2.4), we have

\begin{equation}
\iint_{\Omega_o} \frac{\partial \psi}{\partial z} \psi \frac{dx \, dy}{z-t} = \frac{1}{2i} \int_\gamma \psi \frac{\partial \psi}{\partial z} \frac{z}{z-t} \, dz - \iint_{\Omega_o} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial \bar{z}} \frac{z}{z-t} \, dx \, dy,
\end{equation}

\begin{equation}
= \frac{1}{4} \iint_{\Omega_o} \psi \frac{\partial \psi}{\partial y} \frac{1}{y} \frac{z}{z-t} \, dx \, dy.
\end{equation}

Finally, using again the differential equation (2.4) satisfied by $\psi$, we find

\begin{equation}
\iint_{\Omega_o} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial \bar{z}} \frac{dx \, dy}{z-t} = \frac{1}{2i} \int_\gamma \psi \frac{\partial \psi}{\partial z} \frac{dz}{z-t} - \int_\delta (t) \Pi \psi(t) \frac{\partial \psi}{\partial z},
\end{equation}

\begin{equation}
- \frac{1}{4} \iint_{\Omega_o} \psi \frac{\partial \psi}{\partial y} \frac{dx \, dy}{y(z-t)}.
\end{equation}
Since
\[
\frac{\partial \psi}{\partial y} = i \left[ \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial \bar{z}} \right]
\]
we may again reduce the integral on the right to the integrals (7.1') and (7.1'').

In all these transformations line integrals taken over the boundary of \( \Omega \) appear. In order to clarify their character, we make the following observations. On each level curve of \( \psi \), the normal derivative of \( \psi \) is of one sign, since symmetrization showed the level curves of \( \psi \) to be monotonic in each quadrant. Hence
\[
\int \left| \frac{\partial \psi}{\partial z} \right| \, dz = \int \left| \frac{\partial \psi}{\partial \bar{z}} \right| \, dz
\]
can be estimated on each level curve in terms of the increment of \( \varphi \), according to (2.1). Therefore the gradient of \( \psi \) has a uniformly bounded integral along such curves. Thus, if we evaluate the integrals (7.1) over a subregion of \( \Omega \) and let this subregion expand to fill \( \Omega \), we find that the line integrals occurring in the above transformations will approach zero for arcs tending to \( \ell \), and they will as a consequence represent in the limit continuous functions in the circle \( |t - z_0| < \rho \), since \( \psi = 0 \) on \( \ell \).

Applying all these considerations to the particular integral which occurs as second term on the left in (6.11), we find
\[
(7.2) \quad 4 \iint_{\Omega_0} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial \bar{z}} \frac{dx \, dy}{y^2 (z - t)} = \frac{16 \pi \delta(t)}{(t - \bar{t})^2} \psi(t) \frac{\partial \psi}{\partial t} - \frac{4 \pi \delta(t)}{(t - \bar{t})^3} \psi(t)^2
\]
\[+ Q_2(t),\]
where \( Q_2(t) \) is a continuous function of \( t \) in the entire circle \( |t - z_0| < \rho \).

We can, therefore, replace (6.11) by the formula
\begin{align}
(7.3) & \quad \lambda \int \frac{\psi \, dz}{z-t} + \mathcal{S}(t) \left\{ - \frac{16 \pi}{t-t} \left( \frac{\partial \psi}{\partial t} \right)^2 + \frac{16 \pi}{(t-t)^2} \psi \frac{\partial \psi}{\partial t} - \right. \\
& \quad \left. \frac{4 \pi}{(t-t)^3} \psi^2 \right\} = q(t),
\end{align}

in which \( q(t) \) is a continuous function of \( t \) in the circle \(|t-z_0| < \rho\).

Since \( \mathcal{S}(t) \) vanishes for \( t \) in \( W \) and has the value 1 for \( t \) in \( D \), the bracket in (7.3) must have a limit as \( t \) approaches \( L \) which is equal to the jump of the integral in (7.3) as \( t \) crosses \( L \). We proceed to show that this jump exists almost everywhere along \( L \).

On \( L \), we define a function \( A(z) \) by the relation

\[ A(z) = \int_{z_0}^{z} y \, dz, \]

where the integration is to be carried out along \( L \). We find that almost everywhere on \( L \) the derivative

\begin{align}
(7.3') & \quad \dot{z} = \frac{\partial z}{\partial s} \\
(7.4) & \quad \frac{\partial A}{\partial s} = y \, z
\end{align}

exists and since \( L \) consists of monotonic arcs. Let \( z_1 \) be a point on \( L \) satisfying (7.3') and (7.4) and let \( z_1 \) denote the tangent vector to \( L \) at that point. We know that \( L \) ascends monotonically in the second quadrant and descends monotonically in the first quadrant. Thus in each case points \( z \) on a line making an angle of 45° with the horizontal and intercepting \( L \) at \( z_1 = x_1 + iy_1 \), lie in \( \Omega_0 \) for \( y > y_1 \) and in \( W_0 \) for \( y < y_1 \). Denote by \( t \) a point in \( \Omega_0 \) which lies on such a line through \( z_1 \), and denote by

\[ w = 2z_1 - t \]

the point in \( W_0 \) on the same line and at an equal distance.
\[ \varepsilon = |t - z_1| \]

from \( z_1 \). We wish to prove the jump condition.

\[
(7.5) \quad \lim_{t \to z_1} \left\{ \int_{l} \frac{y_1 dz}{z-t} - \int_{l} \frac{y_1^2 dz}{z-w} \right\} = 2 \pi i y_1 \bar{z}_1^2 ,
\]

where \( z_1 = x_1 + iy_1 \). By Cauchy's formula, we have immediately

\[
(7.5) \quad \lim_{t \to z_1} \left\{ \int_{l} \frac{y_1 \bar{z}_1^2}{z-t} - \int_{l} \frac{y_1 \bar{z}_1^2}{z-w} \right\} = 2 \pi i y_1 \bar{z}_1^2 .
\]

Therefore an integration by parts shows that \((7.5)\) is equivalent to

\[
(7.6) \quad \lim_{t \to z_1} \left\{ \int_{l} \left[ \frac{A(z) - A(z_1)}{z - z_1} - y_1 \bar{z}_1^2 \right] \frac{(w-t)(z-z_1)^2}{(z-t)^2(z-w)^2} \right\} = 0 .
\]

We introduce a coordinate system \( \sigma, \tau \) with its origin at \( z_1 \), and with the \( \sigma \)-axis inclined at \(-45^\circ\) with the \( x \)-axis if \( x_1 > 0 \), and at \(+45^\circ\) with the \( x \)-axis if \( x_1 < 0 \). If \( z \) and \( \sigma + i \tau \) are corresponding points of \( l \), and if \( |\sigma| \leq 2 \varepsilon \), then \( |z-t| \geq 2^{1/2} \varepsilon \), \( |z-w| \geq 2^{1/2} \varepsilon \), \( |z-z_1| \leq 2^{1/2} \sigma \) and \( |dz| \leq 2^{1/2} d \sigma \), while if \( |\sigma| \geq \varepsilon \), then \( |z-t| \geq \sigma \), \( |z-w| \geq \sigma \), \( |z-z_1| \leq 2^{1/2} \sigma \) and \( |dz| \leq 2^{1/2} |d \sigma| \).

Therefore the integral \((7.6)\) is in absolute value smaller than a fixed constant times

\[
(7.7) \quad \varepsilon^{-3} \int_{-2 \varepsilon}^{2 \varepsilon} \left| \frac{A(z) - A(z_1)}{z - z_1} - y_1 \bar{z}_1^2 \right| \sigma^2 d \sigma +
\]

\[
\varepsilon \left\{ \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left[ \frac{A(z) - A(z_1)}{z - z_1} - y_1 \bar{z}_1^2 \right] \frac{d \sigma}{\sigma^2} \right\} +
\]

\[
+ \varepsilon \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left[ \frac{A(z) - A(z_1)}{z - z_1} - y_1 \bar{z}_1^2 \right] \frac{d \sigma}{\sigma^2} = I_1 + I_2 + I_3 .
\]
where the last integral \( I_3 \) is extended over the projection of \( l \) on the 
\( \sigma \)-axis outside the interval \(-\sqrt{\varepsilon} \leq \sigma \leq \sqrt{\varepsilon}\). The first two integrals, 
\( I_1 \) and \( I_2 \), approach zero as \( \varepsilon \to 0 \) because

\[
\lim_{z \to z_1} \left\{ \frac{A(z) - A(z_1)}{z - z_1} - y_1 \frac{z_2}{z_1} \right\} = 0
\]

by (7.3') and (7.4). The last integral \( I_3 \to 0 \) as \( \varepsilon \to 0 \) because the difference quotient

\[
\frac{A(z) - A(z_1)}{z - z_1}
\]

is bounded. Hence (7.5) is proved and it follows from (7.3) and the continuity of 
\( q(t) \) that

\[
(7.8) \quad \lim_{z \to z_1} \left\{ -\frac{4}{y} \left( \frac{\partial \Psi}{\partial z} \right)^2 + \frac{2}{y} \frac{\partial \Psi}{\partial z} + \frac{1}{4y^3} \psi^2 \right\} = \lambda y_1 \frac{z_2}{z_1}
\]

for almost all \( z_1 \) on \( l \), as \( z \) approaches \( z_1 \) from within \( D \) on a 
line inclined \( 45^\circ \) to the \( x \)-axis.

The difference quotient

\[
\frac{A(z) - A(z_1)}{z - z_1}
\]

is bounded for all \( z_1 \) on \( l \), even when (7.3') and (7.4) are not fulfilled 
at \( z_1 \). But then an estimate similar to that of the left side integral in (7.5) 
which leads to the integrals (7.7), shows in the general case that for each \( \varepsilon > 0 \) 
the bracket on the left in (7.8) is bounded when \( |x| > \varepsilon \) and \( z \) is near \( L \).

This calculation is again based on (7.3), and the steps involve merely replacing 
\( y_1 \frac{z_2}{z_1} \) by 0 in previous formulas.

We conclude, since the bracket in (7.8) is a quadratic in \( \partial \Psi / \partial z \), that 
\( \partial \Psi / \partial z \) is bounded in \( \Omega_0 \), if \( \Omega_0 \) does not intersect the \( y \)-axis. Finally,
since \( \psi \to 0 \) as \( z \to l \), we deduce from (7.8) the generalized boundary condition

\[
\lim_{z \to l} \frac{2}{y} \frac{\partial \psi}{\partial z} = -i \sqrt{\lambda} \frac{z}{2}
\]

or

\[
\lim_{z \to l} 2 \frac{\partial \phi}{\partial z} = \sqrt{\lambda} \frac{z}{2}
\]

for \( z \) approaching almost all points on \( l \) along lines inclined at \( 45^\circ \).

This result, combined with the boundedness of the derivatives \( \partial \psi/\partial x \) and \( \partial \psi/\partial y \), is a substitute for the constant pressure condition (2.10) for the arc \( l \) of the free boundary.

The generalized boundary condition can be formulated also by stating that the normal derivative

\[
- \frac{1}{y} \frac{\partial \psi}{\partial n} \rightarrow \sqrt{\lambda}
\]

almost everywhere along any curve which approaches \( l \) in a suitable way, for example, on an infinitesimal translation of \( l \). By integrating (7.11) and applying Lebesgue's convergence theorem to the formula

\[
\psi = - \int \frac{\partial \psi}{\partial n} \frac{ds}{y}
\]

we find that \( \psi \) has on \( l \) the continuous boundary values

\[
\psi \equiv \sqrt{\lambda} s
\]

8. Analyticity of the free boundary.

On the free surface \( L \) we have by (2.2) and (7.12) the overdetermined analytic boundary condition.
where \( s \) is arc length. We shall derive in this section from the known condition on the function \( \varphi + i \psi \) an analytic boundary condition, more complicated than the previous one, but which involves only an analytic function regular in the flow region. We shall be able to apply the new condition to obtain from its overdetermined character the analyticity of the free boundary.

We set \( z = x + i y \), \( \bar{z} = x - i y \), \( \zeta = \xi + i \eta \), \( \bar{\zeta} = \xi - i \eta \), and we introduce the fundamental solution of (2.4),

\[
S(z, \bar{z}; \zeta, \bar{\zeta}) = A(z, \bar{z}; \zeta, \bar{\zeta}) \log(z - \zeta)(\bar{z} - \bar{\zeta}) + B(z, \bar{z}; \zeta, \bar{\zeta}),
\]

defined by the following properties. \( A \) and \( B \) are regular functions of \( x, y \) and \( \xi, \eta \) in the half-plane \( y > 0 \) or \( \eta > 0 \), and they are real for real values of \( x, y, \xi, \eta \). \( S \) satisfies with respect to \( x, y \) the equation (2.4),

\[
\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = \frac{1}{y} \frac{\partial S}{\partial y},
\]

and with respect to \( \xi, \eta \) it satisfies the adjoint equation

\[
\frac{\partial^2 S}{\partial \xi^2} + \frac{1}{\eta} \frac{\partial S}{\partial \eta} + \frac{\partial^2 S}{\partial \eta^2} = \frac{S}{\eta^2}.
\]

For \( z = \zeta \) we have

\[
A(z, \bar{z}; z, \bar{z}) \equiv 1,
\]

and \( S \) is symmetric according to the rule

\[
\sqrt{\frac{\zeta - \bar{\zeta}}{z - \bar{z}}} S(z, \bar{z}; \zeta, \bar{\zeta}) = \sqrt{\frac{\bar{z} - \bar{\zeta}}{\bar{z} - \bar{\zeta}}} S(\zeta, \bar{\zeta}; z, \bar{z}),
\]

since the equation (2.4) becomes self-adjoint if we introduce \( \psi y^{-1/2} \) as new dependent variable.
The function \( A(z, \bar{z}; \xi, \bar{\xi}) \) is merely the Riemann function for the hyperbolic equation

\[
(8.1) \quad \frac{\partial^2 \psi}{\partial z \partial \bar{z}} + \frac{1}{2(z - \bar{z})} \frac{\partial \psi}{\partial z} - \frac{1}{2(z - \bar{z})} \frac{\partial \psi}{\partial \bar{z}} = 0
\]
equivalent to (2.4), and it has the explicit representation [2]

\[
(8.2) \quad A(z, \bar{z}; \xi, \bar{\xi}) = \frac{(z - \xi)^{1/2}(\xi - \bar{z})^{1/2}}{\bar{z} - \xi} \, _{2}F_{1}[\frac{1}{2}, \frac{1}{2}, 1, \frac{(z - \xi)(\bar{z} - \bar{\xi})}{(z - \bar{z})(\bar{z} - \xi)}],
\]
where \( F \) is the hypergeometric series.

If \( \Gamma \) is any closed curve in \( D \), we have by Green’s formula

\[
(8.3) \quad \oint_{\Gamma} \left\{ \psi \frac{\partial S}{\partial n} - S \frac{\partial \psi}{\partial n} \right\} \, ds = 0,
\]
for points \( \xi \) outside the curve \( \Gamma \). If we let one arc of \( \Gamma \) approach \( \mathcal{L} \), while restricting the remainder of \( \Gamma \) to lie in \( D \) but outside the circle \( |z - z_o| = \rho \) enclosing \( \mathcal{L} \), we obtain from (8.3) by the boundary conditions (2.2) and (7.11)

\[
(8.4) \quad \int_{\mathcal{L}} S(z, \bar{z}; \xi, \bar{\xi}) \, ds = P(\xi, \bar{\xi}),
\]
for \( \xi \) in \( \mathbb{W}_o \), where \( P(\xi, \bar{\xi}) \) is a regular analytic function of \( \xi \) and \( \eta \) in \( \mathbb{W}_o \). This follows from an application of Green’s theorem, and \( P \) merely stands for the integral (8.3) extended over the arcs of \( \Gamma \) lying in \( D \) after the limit process.

The identity (8.4) holds for real values of \( \xi, \eta \) in \( \mathbb{W}_o \), but by analytic continuation of \( S \) and \( P \) to complex values of \( \xi, \eta \), we see that it holds also for any values of the independent complex numbers \( \xi = \xi + i\eta \) and \( \xi^* = \xi - i\eta \) with \( \xi \) and \( \xi^* \) in \( \mathbb{W}_o \). Indeed, the functions
A = A(z, \bar{z}; \xi, \xi^*) and B(z, \bar{z}; \xi, \xi^*) are regular analytic functions of the independent complex numbers \( \xi \) and \( \xi^* \) in \( \mathbb{W}_0 \), and \( S \) and \( P \) are defined in terms of \( A \) and \( B \). Finally, we can let \( \bar{\xi} \rightarrow \bar{z}_0 \) for each fixed \( \bar{\xi} \) in \( \mathbb{W}_0 \) to obtain there the relation

\[
(3.5) \quad \int_{\mathcal{L}} S(z, \bar{z}; \xi, \bar{z}_0) \, ds = P(\xi, \bar{z}_0).
\]

Since \( S = A \log(z - \xi)(\bar{z} - \bar{z}_0) + B \), we have to state the branch of the logarithm used in the formula. For this purpose, we draw an arbitrary curve from \( \xi \) to \( \bar{z}_0 \) which lies in \( |z - \bar{z}_0| < \rho \) and which meets \( \mathcal{L} \) only at \( \bar{z}_0 \). Outside of this branch line the above logarithm is a single-valued function in the whole \( z \)-plane. We determine it in a unique way by requiring, for example, that its imaginary part at a given point \( z_\infty \in \mathcal{L} \) lie between \( 0 \) and \( 2\pi \) (the latter value excluded).

For \( \xi \) in the intersection \( \Omega_0 \) of \( D \) with the circle \( |z - \bar{z}_0| < \rho \), we define an analytic function \( F(\xi) \) by the formula

\[
F(\xi) = \int_{\mathcal{L}} S(z, \bar{z}; \xi, \bar{z}_0) \, ds - P(\xi, \bar{z}_0).
\]

We use again the above convention in the determination of \( S \). Using (3.5), we wish to find the boundary values of \( F(\xi) \) on \( \mathcal{L} \).

The function \( P(\xi, \bar{z}_0) \) and the integral

\[
\int_{\mathcal{L}} B(z, \bar{z}; \xi, \bar{z}_0) \, ds.
\]

are continuous in \( \xi \) across \( \mathcal{L} \). On the other hand, the expression

\[
\int_{\mathcal{L}} A(z, \bar{z}; \xi, \bar{z}_0) \log(z - \xi)(\bar{z} - \bar{z}_0) \, ds
\]

jumps by the integral along \( \mathcal{L} \)

\[
-2\pi i \int_{z_0}^{t} A(z, \bar{z}; t, \bar{z}_0) \, ds.
\]
as \( \zeta \) crosses \( \ell \) at the point \( t \), since \( A \) is continuous and the logarithm has a determination which shifts by \( 2\pi i \) on the arc of \( \ell \) between \( z_0 \) and \( t \). Hence \( F(\zeta) \) has on \( \ell \) the boundary values

\[
(8.6) \quad F(t) = -2\pi i \int_{z_0}^{t} A(z, \bar{z}; t, z_0) \, ds,
\]

where the integral is to be evaluated along \( \ell \). This is the overdetermined analytic boundary condition which we have been seeking. It gives the analytic function \( F \) explicitly along \( \ell \) as a geometric integral. Arc length \( s \) is assumed to increase with the region \( D \) on the left. While we may prescribe the real or the imaginary part of an analytic function quite arbitrarily along any given curve \( \ell \), it is not always possible to prescribe the whole analytic function along such an arc. Often such a boundary value problem is only solvable if the arc considered is analytic, and we shall show that the condition \((8.6)\) will just lead to this consequence.

It is worth remarking, for the sake of the interested reader, that the analytic function \( F \) represents merely the analytic continuation of the stream function \( \psi \) into the four-dimensional, complex domain of two complex variables \( x \) and \( y \) \([9]\). \( F(t) \) is, in fact, the stream function \( \psi \) on the characteristic plane for \((8.1)\) which passes through the point \( z_0 \) in the real domain. Thus \((8.6)\) is nothing more than the explicit representation of \( \psi \) along that characteristic in terms of the Riemann function \( A \), and it could have been foreseen that such a representation would hold, since by \((2.2)\) and \((7.11)\) the Cauchy data for \( \psi \) are given explicitly along the free curve \( \ell \).

For the applications, we wish to replace \((8.6)\) by the condition obtained formally by differentiating it with respect to \( t \). To do this, we differentiate \((8.5)\) with respect to \( \zeta \) and for \( \zeta \) in \( W_0 \) we obtain
\( (8.7) \quad \int_{\ell} S_\zeta (z, \bar{z}; \zeta, \bar{z}_0) \, ds = P'(\zeta, \bar{z}_0), \)

where

\[
S_\zeta = \frac{\partial S}{\partial \zeta} = \frac{A}{\zeta - z} + \frac{\partial A}{\partial \zeta} \log(z - \zeta)(\bar{z} - \bar{z}_0) + \frac{\partial B}{\partial \zeta}.
\]

Similarly, in \( \Omega_0 \)

\( (8.8) \quad F'(\zeta) = \int_{\ell} S_\zeta (z, \bar{z}; \zeta, \bar{z}_0) \, ds - P'(\zeta, \bar{z}_0). \)

The function \( P'(\zeta, \bar{z}_0) \) and the integral

\[
\int_{\ell} \frac{\partial B}{\partial \zeta} \, ds
\]

are continuous in \( \zeta \) across \( \ell \), and the integral

\[
\int_{\ell} \frac{\partial A}{\partial \zeta} \log(z - \zeta)(\bar{z} - \bar{z}_0) \, ds
\]

jumps by

\[-2 \pi i \int_{\zeta_0}^{t} A_t(z, \bar{z}; t, \bar{z}_0) \, ds\]

when \( \zeta \) crosses \( \ell \) at \( t \), where \( A_t = \partial A / \partial t \). On the other hand, we have by Green's theorem and \((7.10)\), together with \((2.5)\),

\( (8.9) \quad \int_{\ell} \frac{A(z, \bar{z}; \zeta, \bar{z}_0)}{z - \zeta} \, ds = \int_{\ell} \frac{A(z, \bar{z}; \zeta, \bar{z}_0)}{z - \zeta} z \, dz = \frac{2}{\sqrt{\lambda}} \int_{\ell} \frac{A(z, \bar{z}; \zeta, \bar{z}_0)}{z - \zeta} \frac{\partial \Phi}{\partial z} \, ds + \frac{A_{x'i}}{\sqrt{\lambda}} \int_{\Omega_0} \frac{\partial A}{\partial z} \frac{\partial \Phi}{\partial z} \frac{dx \, dy}{z - \zeta} - \frac{i}{\sqrt{\lambda}} \int_{\Omega_0} A \frac{\partial \Phi}{\partial y} \frac{dx \, dy}{z - \zeta} + \frac{4\pi i}{\sqrt{\lambda}} \delta(\zeta) A(\zeta, \bar{z}; \zeta, \bar{z}_0) \frac{\partial \Phi}{\partial \zeta} + P_1(\zeta), \)
where $P_1(\zeta)$ is a contour integral which is analytic for $|\zeta - z_0| < \rho$.

Therefore, by (7.10) and the boundedness of $\partial \phi / \partial z$,

$$
\int_{\ell} \frac{A(z, \bar{z}; \zeta, \bar{z}_0)}{\zeta - z} \, ds
$$

jumps at almost all points $t$ on $\ell$ by

$$
-2 \pi i A(t, \bar{t}; t, \bar{z}_0) \frac{\bar{t}}{t}.
$$

By comparison of (8.7) and (8.8), we now conclude that for almost all $t$ on $\ell$, $F'(t)$ has the boundary values

$$
(8.10) \quad F'(t) = -2 \pi i A(t, \bar{t}; t, \bar{z}_0) \frac{\bar{t}}{t} - 2 \pi i \int_{z_0}^{t} A(t, \bar{z}; t, \bar{z}_0) \, ds.
$$

Furthermore, (8.9) shows that $F'(t)$ is bounded in $\Omega_0$, because $\partial \phi / \partial t$ is bounded there, provided $\Omega_0$ does not intersect the $y$-axis. Formula (8.10) is seen to be merely the derivative of (8.6).

We prefer to rewrite (8.10) in the form

$$
(8.11) \quad \begin{cases}
\bar{t} = -\frac{F'(t)}{2 \pi i A(t, \bar{t}; t, \bar{z}_0)} - \int_{z_0}^{t} \frac{A(t, \bar{z}; t, \bar{z}_0)}{A(t, \bar{t}; t, \bar{z}_0)} \frac{\bar{z}}{z} \, dz,
\bar{t} = \bar{z}_0 + \int_{z_0}^{t} \frac{\bar{z}}{z} \, dz,
\end{cases}
$$

valid for almost all $t$ on $\ell$. We observe from (8.2) that

$$
(8.2') \quad A(t, \bar{t}; t, \bar{z}_0) = \sqrt{\frac{t - \bar{t}}{t - \bar{z}_0}}
$$

is an elementary function.

We introduce the system of two integral equations
\[
f(t) = -\frac{F'(t)}{2 \pi i A(t, g(t); t, z_0)} - \int_{z_0}^{t} \frac{A_z(z, g(z); t, z_0)}{A(t, g(t); t, z_0)} f(z) \, dz \]

(8.12)

\[
g(t) = \bar{z}_0 + \int_{z_0}^{t} f(z)^2 \, dz
\]

for the determination of the unknown analytic functions \( f(t) \) and \( g(t) \) in \( \Omega_0 + l \). These equations have a sense for analytic functions \( f(t) \) and \( g(t) \) in \( \Omega_0 \), since by Cauchy's theorem the integrals on the right are independent of path in \( \Omega_0 + l \). The objective now is to show that (8.12) has a unique solution \( f, g \), since by (8.11) this solution would then have to agree with the known solution \( \bar{z}, \bar{t} \) on \( l \). If this were the case, the analytic function \( g(t) \) would have the continuous boundary values

\[
g(t) = \bar{t}
\]

on \( l \), and thus we would have

\[
\bar{\mathcal{F}} = g(t) + t = \text{real}
\]

\[
\mathcal{F} = g(t) - t = \text{imaginary}
\]

there. Denoting by \( w(t) \) a conformal mapping of \( \Omega_0 \) onto the upper half-plane, we check by the Schwarz principle of reflection that \( \mathcal{F} \) and \( \mathcal{F} \) can be continued analytically across those segments of the real axis of the \( w \)-plane corresponding to \( l \). As a result,

\[
t(w) = \frac{\mathcal{F} - \mathcal{F}}{2}
\]

is analytic on those segments and \( l \) consists of analytic arcs.

In the next section, we prove that (8.12) has a unique, analytic solution \( f, g \) in a neighborhood of \( z_0 \) in \( \Omega_0 + l \) by successive approximations, and we thus obtain the analyticity of the free boundary \( L \).
The analyticity of $\psi$ on the analytic free boundary $L$ follows easily by the Cauchy-Kowalewski power series method. By this method, we can construct in the neighborhood of $L$ a solution $\psi_0$ of (2.4) satisfying the boundary conditions (2.2) and (2.10) which is analytic on $L$. If we express $\psi - \psi_0$ as an integral along $C$ and $L$ using the elementary fundamental solution $S$, the integral along $L$ will drop out, since $\psi - \psi_0 = 0$ and $\partial (\psi - \psi_0)/\partial n = 0$ there. Hence $\psi_0$ and $\psi - \psi_0$ are both analytic on $L$, and it follows that $\psi$ is analytic there.


The functional equations (8.12) differ from the non-linear integral equations usually arrived at in the theory of ordinary differential equations by Picard's method only in that the independent variable $t$ appears on the right, not only as an upper limit of integration, but also as an argument in the integrand and as an argument of $g$ on the right. However, the standard procedure for discussing such problems can still be carried through, as we shall proceed to show [12].

Let us suppose at first that $z_0$ does not lie on the $y$-axis. We set $f_0 = 0$ and

$$g_0 = z_0 + \int_{z_0}^{t} f_0(z)^2 \, dz = z_0,$$

and we define sequences of analytic functions $f_n(t)$ and $g_n(t)$ recursively by the formulas

$$f_{n+1}(t) = \frac{F'(t)}{2 \pi i A(t, g_n(t); t, z_0)} - \int_{z_0}^{t} \frac{A_t(z, g_n(z); t, z_0)}{A(t, g_n(t); t, z_0)} f_n(z) \, dz,$$
(9.2) \[ g_{n+1}(t) \equiv \bar{z}_o + \int_{\bar{z}_o}^t f_{n+1}(z)dz \]

wherever these integrals are defined. Let \( I \) denote an upper bound for

\[ f_1(t) = -\frac{F'(t)}{2 \pi i A(t, \bar{z}_o; t, \bar{z}_o)} \]

in a suitable neighborhood of \( t = z_o \); it is possible to find such a number because \( F'(t) \) is bounded near \( z_o \) and \( A(z_o, \bar{z}_o; z_o, \bar{z}_o) = 1 \). It is readily verified that there exists a positive constant \( \delta \) such that if

(9.2') \[ |f_n| \leq I + \delta, \quad |f_{n-1}| \leq I + \delta, \quad |g_n - \bar{z}_o| \leq \delta, \quad \]

\[ |g_{n-1} - \bar{z}_o| \leq \delta, \quad |z - z_o| \leq \delta, \quad |t - z_o| \leq \delta, \]

then we have

\[ |g_n(t) - g_{n-1}(t)| \leq \frac{N_1}{N_0} \int_{\bar{z}_o}^t \left| f_n(z) - f_{n-1}(z) \right|dz \]

and therefore, by the analyticity of \( A \),

\[ \left| \frac{F'(t)}{2 \pi i A(t, g_n(t); t, \bar{z}_o)} - \frac{F'(t)}{2 \pi i A(t, g_{n-1}(t); t, \bar{z}_o)} \right| \leq \]

\[ \leq \frac{N_2}{N_0} \int_{\bar{z}_o}^t \left| f_n - f_{n-1} \right|dz \]

for suitable positive constants \( N_1 \) and \( N_2 \). Furthermore, for small enough \( \delta > 0 \), there are positive numbers \( N_3, N_4 \) and \( N_5 \) such that in addition.
\[
\left| \frac{A_t(z, g_n(z); t, \overline{z}_o)}{A(t, g_n(t); t, \overline{z}_o)} f_n(z) - \frac{A_t(z, g_{n-1}(z); t, \overline{z}_o)}{A(t, g_{n-1}(t); t, \overline{z}_o)} f_{n-1}(z) \right| \leq N_3 \left| g_n(t) - g_{n-1}(t) \right| + N_4 \left| g_n(z) - g_{n-1}(z) \right| + N_5 \left| f_n(z) - f_{n-1}(z) \right|
\]

For paths from \( z_o \) to \( t \) which have uniformly bounded length and lie in \( \Omega_o + l \), these Lipschitz conditions yield according to (9.1) the inequality

\[
(9.3) \quad \left| f_{n+1}(t) - f_n(t) \right| \leq N \int_{z_o}^{t} \left| f_n(z) - f_{n-1}(z) \right| dz \leq I N s
\]

for a suitably large positive value of \( N \).

Let now \( \Gamma \) be any sufficiently short path from \( z_o \) to \( t \) in \( \Omega_o + l \). Then the conditions (9.2') will be fulfilled for \( n = 1 \), and hence by (9.3)

\[
\left| f_2(t) - f_1(t) \right| \leq N \int_{z_o}^{t} \left| f_1 - f_0 \right| dz \leq I N s
\]

where \( s \) is arc length from \( z_o \) to \( t \) along \( \Gamma \). For a short enough path \( \Gamma \), (9.2') is fulfilled also for \( n = 2 \), and therefore

\[
\left| f_3 - f_2 \right| \leq N \int_{z_o}^{t} I N s ds = I N^2 \frac{s^2}{2!}
\]

By the usual application of induction to (9.3), we obtain now in general

\[
(9.4) \quad \left| f_{n+1} - f_n \right| \leq I \frac{(N s)^n}{n!}
\]

Therefore in a sufficiently small circle about \( z_o \)

\[
\left| f_n(t) \right| \leq I e^{Ns}
\]

\[
\left| g_n(t) - \overline{z}_o \right| \leq I \frac{N}{N} \left\{ e^{Ns} - 1 \right\}
\]
which shows that for sufficiently short paths $\Gamma$ the iteration (9.1), (9.2) is defined.

We deduce now from (9.4) that

$$
\begin{align*}
\begin{cases}
  f(t) &= f_1(t) + \sum_{n=1}^{\infty} \left\{ f_{n+1}(t) - f_n(t) \right\}, \\
  g(t) &= g_0(t) + \sum_{n=0}^{\infty} \left\{ g_{n+1}(t) - g_n(t) \right\}
\end{cases}
\end{align*}
$$

converge uniformly in $\Omega_0$ and uniformly almost everywhere along $\mathcal{L}$ to a solution of (8.12), provided that $|t - z_0|$ is sufficiently small, say, $|t - z_0| < \delta$. By (9.1), the iterated analytic functions $f_n(t)$, $g_n(t)$ are independent of the defining path of integration from $z_0$ to $t$, and therefore the same is true of the solution $f, g$. Weierstrass's convergence theorem shows by the uniform convergence of the above series that the solution is analytic near $z_0$ in $\Omega_0$.

The solution is unique in the neighborhood of $z_0$. For if there were another bounded solution, $\tilde{f}(t)$, $\tilde{g}(t)$, of (8.12), then we could substitute $f, f, g, g$ for $f_{n-1}, f_n, g_{n-1}, g_n$ and $\tilde{f}, \tilde{f}, \tilde{g}, \tilde{g}$ for $f_n, f_{n+1}, g_n, g_{n+1}$ in (9.1) and (9.2), in that order, and we could apply (9.3) to obtain

$$
\begin{align*}
\begin{cases}
  \left| \tilde{f}(t) - f(t) \right| &\leq N \int_{z_0}^{t} \left| f - \tilde{f} \right| dz, \\
  \left| \tilde{g}(t) - g(t) \right| &\leq N_1 \int_{z_0}^{t} \left| f - \tilde{f} \right| dz
\end{cases}
\end{align*}
$$

(9.3')

for suitable large values of $N$ and $N_1$. Consider now an arbitrary path $\Gamma$ of integration $\Gamma$ from $z_0$ to $t$ and let
\[ m = \max \int_{z_0}^{\zeta} |f - f'| \, ds \]

for \( \zeta \) on \( \Gamma \) between \( z_0 \) and \( t \). Then by integration of (9.31) along \( \Gamma \) from \( z_0 \) to the point where the maximum is attained we find

\[ m \leq N \int_{z_0}^{t} |dz| . \]

For \( t \) sufficiently near \( z_0 \) along \( \Gamma \) this gives \( m = 0 \) and hence \( \tilde{f} = f \) almost everywhere. Consequently \( \tilde{g} = g \) also.

The function \( g(t) \), as an integral of the bounded function \( f^2 \), must be continuous. By the uniqueness of the solution of (8.12) just shown, \( g(t) \) must have on \( \mathcal{L} \) near \( z_0 \), the boundary values \( \tilde{r} \), since by (8.11), \( \tilde{r} \) and \( \tilde{t} \) solve (8.12) along \( \mathcal{L} \). Thus by the argument given at the end of Section 8, \( \mathcal{L} \) is near \( z_0 \) an analytic arc. Since \( z_0 \) can be taken as any interior point of the free boundary \( L \) not on the \( y \)-axis, this shows by the Heine-Borel theorem that the free boundary consists of analytic arcs.

The argument given thus far shows that the free boundary curve \( L \) is analytic in the neighborhood of each point \( z_0 \) not on the \( y \)-axis. In the proof, we have used the boundedness of the analytic function \( F'(z) \) for values of \( z \) near \( L \) and outside a neighborhood of the \( y \)-axis. We next show that \( L \) is analytic in the neighborhood of its intersection with the \( y \)-axis, which we denote again by \( z_0 \). The result depends on showing, by means of the Phragmén-Lindelöf principle [13], that \( F'(z) \) is bounded in \( D \) near \( z_0 \).

It is clear from (8.10) that \( F'(z) \) is analytic and bounded on those arcs of \( L \) for which \( x \neq 0 \), so that we may write \( |F'(z)| \leq T_1 \) there. We denote by \( M(r) \) the quantity
\[(9.5) \quad M(r) = \max_{z \in D} |F'(z)|, \quad \text{if} \quad |z - z_o| = r\]

and we find by the Phragmén-Lindelöf principle that if \( F'(z) \) is not bounded near \( z_o \), then

\[(9.6) \quad M(r) \geq T_2 \exp \left\{ T_3 r^{-1/2} \right\}, \]

where \( T_2 \) and \( T_3 \) are positive constants. By (8.8), \( F'(z) \) differs in \( D \) from

\[\int_{\mathcal{E}} \frac{A(t, t'; z, z_o)}{z - t} ds \]

by a term which is bounded near \( z_o \). On the other hand, there is a constant \( T_4 \) such that

\[\left| \int_{\mathcal{E}} \frac{A(t, t'; z, z_o)}{z - t} ds \right| \leq T_4 \int_{\mathcal{E}} \frac{dx + dy}{|z - t|}, \]

since \( \mathcal{E} \) is monotonic on each side of the \( y \)-axis. Thus if \( \varepsilon \) denotes the shortest distance from \( z \) to \( \mathcal{E} \), we have

\[(9.7) \quad |F'(z)| \leq \frac{T_5 \varepsilon}{\varepsilon} \]

near \( z_o \), for a suitable choice of \( T_5 \).

Let \( z \) be a point on \( |z - z_o| = r \) such that \( |F'(z)| = M(r) \). Then by (9.6) and (9.7),

\[(9.8) \quad \varepsilon \leq T_6 \exp \left\{ -T_3 r^{-1/2} \right\}, \quad T_6 T_2 = T_5.

Thus if \( |F'(z)| \) is unbounded near \( z_o \), the points on \( |z - z_o| = r \) where it is largest lie much closer to \( \mathcal{E} \) than to the summit \( z_o \).

We shall show that (9.8) is, in fact, impossible by estimating \( F'(z) \) in terms of \( \varepsilon / r \). Let \( z_1 \) be the point of \( \mathcal{E} \) nearest to \( z \) on the line...
through \(z\) inclined at an angle of \(45^\circ\) with the \(x\)-axis, and let \(w = 2z_1 - z\).

We choose \(\beta\) to be a closed contour about \(z\) which consists of an open arc of \(\ell\) including \(z_0\) and \(z_1\) together with a curve in \(D\) whose shortest distance from \(z\) is of at least the order of magnitude \(r\) and which is composed of two level curves \(\phi = \text{const.}\) and one level curve \(\psi = \text{const.}\).

We obtain in a manner similar to that leading from (8.3) to (8.8) the formula

\[
F'(z) = -\frac{1}{\sqrt{\lambda}} \int_{\beta} \left\{ \left[ S_z(t, \overline{t}; z, \overline{z}_0) - S_w(t, \overline{t}; w, \overline{z}_0) \right] \frac{\partial \psi}{\partial n} \frac{ds}{y} 
- \psi \frac{\partial}{\partial n} \left[ S_z - S_w \right] \frac{ds}{y} \right\} .
\]

This yields for \(|F'(z)|\) the estimate

\[
|F'(z)| \leq T_7 \int_{\beta} \frac{|w - z| |z - \phi|}{|w - t| |z - t|} + T_8 \int_{\beta} \frac{|w - z| ds}{|w - t|^2 |z - t|^2},
\]

whence by the monotonicity of \(\ell\) in each quadrant

\[
(9.9) \quad |F'(z)| \leq T_9 \frac{\epsilon}{r^4} + T_{10}.
\]

The proof is analogous to the estimate performed in (7.7).

Since \(z\) was chosen so that \(|F'(z)| = M(r)\), we have

\[
M(r) \leq T_9 \frac{\epsilon}{r^4} + T_{10}.
\]

Combined with (9.6) and (9.8), this gives

\[
T_2 \exp\left\{ T_3 r^{-1/2} \right\} \leq T_6 T_9 r^{-4} \exp\left\{- T_3 r^{-1/2} \right\} + T_{10}.
\]

For \(r \to 0\), we arrive thus at a contradiction. Hence \(F'(z)\) is, indeed, bounded in the neighborhood of \(z_0\) on the \(y\)-axis, and therefore the previous discussion of the functional equation (8.12) shows the analyticity of \(L\) near \(z_0\).

This completes the proof that the free boundary \(L\) is an analytic arc.
10. Uniqueness.

While so far we have obtained the existence of axially symmetric cavitations flows using the minimum problem (3.7) and have shown the analyticity of the free boundary using (8.12), we wish now to show unique, continuous dependence of the flow region $D$ on the cavitation constant $\lambda$. Basic for such a discussion will be the following lemma [5, 6, 7].

Let $\Omega$ be a region of the upper half-plane $y > 0$ bounded by a closed analytic curve $\Gamma$ lying in $y > 0$. Corresponding to a positive solution of (2.4), let $u = \psi y^{-1/2}$ be a positive solution of

$$
(10.1) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{3}{4y^2} u.
$$

If $u$ vanishes at a point $t$ on $\Gamma$, then at that point

$$
(10.2) \quad \frac{\partial u}{\partial n} < 0,
$$

where $n$ represents the outer normal of $\Gamma$.

For the proof, we set

$$
\gamma = \max \frac{3}{4y^2}
$$

for $z = x + iy$ in $\Omega + \Gamma$. We denote by $g(z, \xi)$ the Green's function of (10.1) in $\Omega$ with a positive logarithmic singularity at $\xi$ and zero boundary values and we denote by $G(z, \xi)$ the corresponding Green's function for

$$
\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \gamma G
$$

in $\Omega$. The functions $g$ and $G$ are positive, by the (generalized) maximum principle, and $g - G$ is continuous in $\Omega + \Gamma$. We apply Green's theorem to $g(z, w)$ and $G(z, \xi)$ in $\Omega$ to obtain
\[ G(w, \zeta) - g(\zeta, w) = \frac{1}{2\pi} \iint_{\Omega} \left( \Delta g + g \Delta G \right) \, dx \, dy \]
\[ - \frac{1}{2\pi} \int_{\Gamma} \left( g \frac{\partial g}{\partial n} - G \frac{\partial G}{\partial n} \right) \, ds \]
\[ = \frac{1}{2\pi} \iint_{\Omega} G \, g \left\{ \frac{3}{4y^2} - \frac{1}{Y} \right\} \, dx \, dy \leq 0, \]
whence by the symmetry of \( G(w, \zeta) = G(\zeta, w) \), or of \( g \), we have
\[ G(z, w) \leq g(z, w) \]
in \( \Omega \). This inequality is part of a general theory of the monotonic dependence of solutions of differential equations on the coefficients of the equation [1].

By the maximum principle, both \( G \) and \( g \) decrease as the domain \( \Omega \) diminishes. Thus \( G \), in particular, is larger than the Green's function \( G_0 \) of the same equation for a circle \( |z-w| < \epsilon \) contained in \( \Omega \) whose circumference \( |z-w| = \epsilon \) is tangent to \( \Gamma \) at \( t \). We consider the function \( u(z) = H g(z, w) \) in the domain \( \Omega' \) obtained from \( \Omega \) by removing the disk \( |z-w| < \frac{\epsilon}{2} \). Since \( u(z) > 0 \) on the circumference \( |z-w| = \frac{\epsilon}{2} \), we can choose \( H > 0 \) so small that this function is positive there. It is non-negative on \( \Gamma \) and is a solution of (10.1). Hence, by the minimum principle
\[ u(z) \geq H g(z, w) \]
in \( \Omega' \) and, in particular, in the intersection of \( |z-w| < \epsilon \) and \( |z-t| < \frac{\epsilon}{2} \). On the other hand,
\[ g(z, w) \geq G(z, w) \geq G_0(z, w) \]
there, and consequently at \( t \) on \( \Gamma \).
\[ \frac{\partial u}{\partial n} \leq H \frac{\partial g}{\partial n} \leq H \frac{\partial G}{\partial n} \leq H \frac{\partial \varrho}{\partial n} . \]

But \( G_0(z,w) \) can be calculated explicitly as a combination of Bessel functions of \( |z-w| \), and
\[ \frac{\partial G}{\partial n} < 0 \]
at \( t \). As a result, we obtain (10.2).

It follows that any positive solution \( \psi \) of (2.4) in a region bounded by an analytic arc which vanishes at a point of that analytic arc must have there a negative outer normal derivative

(10.3) \[ \frac{\partial \psi}{\partial n} < 0 . \]

Let \( D_1 \) and \( D_2 \) be two extremal flow regions for (3.7) corresponding, respectively, to two different values \( \lambda_1 \) and \( \lambda_2 \) of the cavitation parameter \( \lambda \). Let us suppose that \( D_1 \neq D_2 \) and that we can magnify the region \( D_1 \) about the origin until it just includes \( D_2 \). We denote the magnification of \( D_1 \) by \( D_1^* \) and we let \( \alpha \leq 1 \) be the factor of magnification. If \( \psi_1, \psi_2, \) and \( \psi_1^* \) denote, respectively, the stream functions of the flows through \( D_1, D_2 \) and \( D_1^* \), normalized at infinity to behave like \( y^2/2 \), we have
\[ \psi_1^*(x^*, y^*) = \alpha^2 \psi_1(x^*/\alpha, y^*/\alpha) \]
in \( D_1^* \), and

(10.4) \[ \frac{1}{y^*} \frac{\partial \psi_1^*}{\partial n} = \frac{1}{y} \frac{\partial \psi_1}{\partial n} \]
at corresponding points \( x^*, y^* \) and \( x, y \) on the boundaries of \( D_1^* \) and \( D_1 \),
\[ x^* = \alpha x, \ y^* = \alpha y . \]
The free boundaries $L_2$ and $L_1^*$ of $D_2$ and $D_1^*$ must be tangent at some point $t$. This is a geometrical lemma which is based on the fact that $D_1$ and $D_2$ have an identical fixed boundary $C$ which consists within the strip $|x| \leq k$ of a horizontal segment. Since $D_1^*$ includes $D_2$, the solution $\psi_1^* - \psi_2$ of (2.4) in $D_2$ must be positive in $D_2$, by the maximum principle. Hence by (10.3)

$$\frac{1}{y} \frac{\partial \psi_1^*}{\partial n} < \frac{1}{y} \frac{\partial \psi_2}{\partial n}$$

at $z = t$. By (10.4) this yields the final inequality

(10.5) \quad \lambda_2 < \lambda_1 .

As a first application of (10.5), we show that the cavitation flow solving the extremal problem (3.7) for a given $\lambda$ is unique. If, indeed, there were two distinct flow regions $D_1$ and $D_2$ corresponding to the same value of $\lambda$, we could magnify one of them until it just contained the other, and by (10.5) this would give the absurd inequality $\lambda < \lambda$.

A second application of (10.5) shows that the extremal flow region $D$ for (3.7) diminishes steadily as $\lambda$ increases, or, in other words, that the cavity $W$ expands as $\lambda$ increases. For the proof, suppose first that of two cavitation flow regions $D_1$ and $D_2$ neither one contains the other. Then each one can be magnified to just include the other, and (10.5) shows that the parameters $\lambda_1$ and $\lambda_2$ for $D_1$ and $D_2$ satisfy both of the inequalities

$$\lambda_1 < \lambda_2 , \quad \lambda_2 < \lambda_1 ,$$

a manifest contradiction. Thus one region, say $D_2$, includes the other, say $D_1$. Hence when we magnify $D_1$ to include $D_2$, we obtain (10.5), and this proves the
desired monotonic dependence of \( D \) on \( \lambda \).

Finally, we wish to show that \( D \) and the free boundary \( L \) depend continuously on \( \lambda \). Suppose that \( D_n \) and \( L_n \) are the flow regions and free boundaries for a sequence \( \lambda_n \) of cavitation parameters which approach the limit \( \lambda \).

Since the flow region \( D \) is admissible for the extremal problem (3.7) corresponding to the value \( \lambda_n \) of the cavitation parameter, we have

\[
2a - \lambda_n V_n \geq 2a_n - \lambda_n V_n,
\]

in an obvious notation. The volume \( V_n \) remains bounded, by (5.2), and therefore \( \lambda_n \to \lambda \) implies

\[
\lim_{n \to \infty} \left\{ 2a_n - \lambda V_n \right\} \leq 2a - \lambda V.
\]

Hence the curves \( L_n \) are a minimal sequence for (3.7), and by Section 5 we can select from any subsequence of \( L_n \) a subsequence which converges to the unique solution \( L \) of (3.7). It follows that \( L_n \) converges to \( L \) and \( D_n \) converges to \( D \), and this means precisely that \( D \) depends continuously on \( \lambda \).

So far we have not discussed the possibility that the extremal curves \( L \) for (3.7) could consist in part of two segments along the vertical lines \( x = \pm k \). Such segments would have to be considered as part of the fixed boundary, since we could not apply variational arguments in order to show that (2.10) is satisfied on them. In order to show that in at least one case such segments do not occur, we observe that by the assumptions made in Section 2 the fixed boundary \( C \) of \( B \) consists in the strip \( -k \leq x \leq k \) of a horizontal segment. Let \( \psi_o \) be the stream function for the flow past \( B \) alone without a cavity, and let \( D_o \) be the flow region complementary to \( B \). For a given \( \lambda \), we magnify \( D_o \) to include the extremal flow region \( D \) and apply (10.3) to obtain
desired monotonic dependence of $D$ on $\lambda$.

Finally, we wish to show that $D$ and the free boundary $L$ depend continuously on $\lambda$. Suppose that $D_n$ and $L_n$ are the flow regions and free boundaries for a sequence $\lambda_n$ of cavitation parameters which approach the limit $\lambda$. Since the flow region $D$ is admissible for the extremal problem (3.7) corresponding to the value $\lambda_n$ of the cavitation parameter, we have

$$2a - \lambda_n V \geq 2a_n - \lambda_n V_n,$$

in an obvious notation. The volume $V_n$ remains bounded, by (5.2), and therefore $\lambda_n \to \lambda$ implies

$$\lim_{n \to \infty} \left\{ 2a_n - \lambda V_n \right\} \leq 2a - \lambda V.$$

Hence the curves $L_n$ are a minimal sequence for (3.7), and by Section 5 we can select from any subsequence of $L_n$ a subsequence which converges to the unique solution $L$ of (3.7). It follows that $L_n$ converges to $L$ and $D_n$ converges to $D$, and this means precisely that $D$ depends continuously on $\lambda$.

So far we have not discussed the possibility that the extremal curves $L$ for (3.7) could consist in part of two segments along the vertical lines $x = \pm k$. Such segments would have to be considered as part of the fixed boundary, since we could not apply variational arguments in order to show that (2.10) is satisfied on them. In order to show that in at least one case such segments do not occur, we observe that by the assumptions made in Section 2 the fixed boundary $C$ of $B$ consists in the strip $-k \leq x \leq k$ of a horizontal segment. Let $\Psi_0$ be the stream function for the flow past $B$ alone without a cavity, and let $D_0$ be the flow region complementary to $B$. For a given $\lambda$, we magnify $D_0$ to include the extremal flow region $D$ and apply (10.3) to obtain
where the term on the left is to be evaluated at the intersection of \( C \) with the \( y \)-axis. This inequality shows that for sufficiently small \( \lambda \) the extremal flow region \( D \) coincides with \( D_0 \).

On the other hand, in order to get an upper estimate on values of \( \lambda \) for which the extremal curve contains no vertical segments we proceed as follows. Let \( \gamma_t \) be a smooth arc consisting of two vertical segments \( x = t + \varepsilon \) and \( x = t - \varepsilon \), \( 0 \leq y \leq h \) joined at their upper end-points by the semi-circle \( S: (x-t)^2 + (y-h)^2 = \varepsilon^2 \) \( y \geq h \), with \( \varepsilon < k \). We consider the axially symmetric flow past the object belonging to \( \gamma_t \) and denote its stream function by \( \psi_t \). Let \( \mu(\varepsilon) \) be the minimum of the velocity of this flow on the semi-circle \( S \).

For sufficiently large values of \( \lambda \) we can obviously find values of \( t \) and \( \varepsilon \) such that the curve \( \gamma_t \) lies outside of the flow region \( D \). For such a value \( \varepsilon \), taken fixed, we can take \( t \) so large that \( \gamma_t \) just touches the boundary of \( D \). If the point of tangency lies on the free boundary \( L \), then by

\[
\mu(\varepsilon) = \frac{1}{y} \left| \frac{\partial \psi_t}{\partial n} \right| \geq \frac{1}{y} \left| \frac{\partial \psi}{\partial n} \right| = \lambda^{1/2}.
\]

Thus, if \( \lambda > \mu(\varepsilon)^2 \) then \( \gamma_t \) cannot touch the free boundary, which must, therefore, detach itself from the vertical segments on the line \( |x| = k \) above \( y = h \). This shows that for large \( \lambda \), \( L \) must consist of two vertical segments on \( x = \pm k \) together with a single free boundary arc joining the upper end-points of those segments.

We see that for small \( \lambda \), \( L \) collapses onto \( C \), while for large \( \lambda \), \( L \) lies entirely above \( C \). Since \( L \) depends continuously upon \( \lambda \), we
conclude that for one particular intermediate value of $\Lambda$, the free boundary $L$ consists of a single symmetric arc joining the intersection of $C$ with $x = -k$ to the intersection of $C$ with $x = +k$. Thus we obtain a model for an axially symmetric cavitational flow in which a prescribed nose (that portion of $C$ lying in the half-plane $x \leq -k$) is joined by a constant pressure free boundary $L$ to a tail consisting of the reflection of the nose in the $y$-axis.

As a particular case of the above theory, we deduce the existence of a cavitational flow past two symmetric discs perpendicular to the $x$-axis of arbitrary size, with a pocket of steam between the discs enclosed by a free surface $L$. As the radius of the discs increases for fixed distance between them, the cavitation parameter $\Lambda$ increases, and the cavity expands.

11. The case of an infinite cavity.

We have seen in the previous section that from a given nose generated by a monotonic curve we can obtain a cavitational flow with a free surface leading back to a tail which is the reflected image of the nose. While this model is satisfactory from the point of view of the experimentalist, it is nevertheless desirable to ask for a similar flow in which the tail is replaced by an infinite cavity. We shall find such a flow here by allowing the tail to tend towards infinity to the right and by performing a limit process on the finite cavity.

Let $D_n$ be a sequence of flow regions with the same nose $C$ (changing slightly our earlier notation, we denote now by $C$ the monotonic increasing arc of the obstacle in the second quadrant), with free boundaries $L_n$, and with tails $T_n$ which tend steadily towards infinity to the right. We denote by $\Lambda_n$ the corresponding cavitation parameters. For $n < m$, we can translate $D_n$ to the
right and magnify it until it just includes $D_m$, and, indeed, this can be done
in such a way that the free boundaries $L_n$ and $L_m$ just touch. Hence by
(10.5) we have

$$\lambda_n > \lambda_m,$$

and the numbers $\lambda_m$ decrease to a limit $\lambda$.

In the coordinate system $\sigma, \zeta$ obtained by a rotation of the usual system
through $45^\circ$, we find, as in Section 5, that the representations $\zeta = h_n(\sigma)$
of the monotonically ascending portions of the free curves $L_n$ form an
equicontinuous class of functions. We can select a subsequence of these functions
which converges uniformly in every bounded region, and we assume without loss of
generality that the original sequence has this property. Thus the curves $L_n$
approach a monotonic limit curve $L$ bounding a region $D$ which is the limit
of the flow regions $D_n$.

The stream functions $\psi_n$ are equicontinuous in every closed subdomain of
$D$, and hence a subsequence of them converges in $D$ to a limit $\psi$ which
represents the stream function of an axially symmetric flow through $D$. We
assume, again without loss of generality, that the original sequence $\psi_n$ converges
to $\psi$.

We wish to show that $L$, as the limit of free boundaries for the flows $\psi_n$
is a free boundary for the flow $\psi$. To show this, let $z_o$ be any point of
$L$ and let $F$ be a suitably differentiable function which vanishes outside a
circle $|z - z_o| \leq 2 \rho$ so small that it does not intersect the fixed nose $C$.

By (6.6) we obtain for each $n$ the identity
\begin{equation}
(11.2) \quad \lambda_n \iint_{W_n} \left[ \frac{F}{i} + 2y \frac{\partial F}{\partial z} \right] \, dx \, dy + 4 \iint_{D_n} \frac{F}{i} \frac{\partial \psi_n}{\partial z} \frac{\partial \psi_n}{\partial z} \, dx \, dy + \frac{8}{y^2} \iint_{D_n} \frac{\partial F}{\partial z} \left( \frac{\partial \psi_n}{\partial z} \right)^2 \, dx \, dy = 0,
\end{equation}

where \( W_n \) is the cavity complementary to \( D_n \). If we can show that \( \frac{\partial \psi_n}{\partial z} \) is bounded for \( |z-z_0| \leq 2R \) uniformly in \( n \), then we shall be able to pass to the limit in (11.2) as \( n \to \infty \) and \( \psi_n \to \psi \) to obtain by Lebesgue's convergence theorem

\begin{equation}
(11.3) \quad \lambda \iint_{W} \left[ \frac{F}{i} + 2y \frac{\partial F}{\partial z} \right] \, dx \, dy + 4 \iint_{D} \frac{F}{i} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial z} \, dx \, dy + \frac{8}{y^2} \iint_{D} \frac{\partial F}{\partial z} \left( \frac{\partial \psi}{\partial z} \right)^2 \, dx \, dy = 0,
\end{equation}

where \( W \) is the infinite cavity for the flow \( \psi \).

Let \( \tilde{\psi} \) be the stream function for the axially symmetric flow past the nose slit \( C \), normalized to behave like \( y^2/2 \) at infinity. Let \( \omega \) be the harmonic function in 3-dimensional space which is regular outside the surface generated by rotation of \( C \) about the \( x \)-axis and assumes on that surface the boundary values.
\[ \omega = \frac{1}{y^2} \left( \nabla \tilde{\varphi} \right)^2 \]

The functions \((\nabla \varphi_n)^2\) are subharmonic in the regions \(D_n^1\) of space obtained by rotating the regions \(D_n\) about the \(x\)-axis, and they satisfy on the nose

\[ (\nabla \varphi_n)^2 \leq \frac{1}{y^2} (\nabla \tilde{\varphi})^2 = \omega, \]

by (10.3). On \(L_n\),

\[ (\nabla \varphi_n)^2 = \lambda_n \leq \lambda_1, \]

and hence throughout the flow.

\[ (\nabla \varphi_n)^2 \leq \omega + \lambda_1 + b + 1, \]

by the maximum principle, where \(b\) is a term representing the estimate analogous to \(\omega\) coming from the tail and tends to zero in the finite part of the plane.
The derivatives $\partial \psi / \partial z$ are thus uniformly bounded in the circle $|z - z_0| \leq 2 \rho$, and (11.3) follows. The results of Sections 7, 8 and 9 now apply to $L$ in the neighborhood of $z_0$, and we deduce that $L$ is, indeed, an analytic, constant pressure, free boundary for the axially symmetric flow $\psi$.

We have thus obtained an infinite cavity $W$ behind $C$ as the limit of the finite cavities $W_n$. It is worth while to obtain a crude estimate on the geometrical nature of the infinite monotonic free streamline $L$ (cf. [8]).

Let us suppose that $W$ contains a large circle with center on the $x$-axis, $|z - x_0| < R$, whose circumference is tangent to $L$ at $z = t$. Then if $\psi_0$ denotes the stream function of the flow past this circle, we get by (10.3)

$$\frac{1}{y^2} (\nabla \psi_0)^2 \geq \lambda$$

at $t$. Explicit calculation gives

$$\frac{1}{y^2} (\nabla \psi_0)^2 = \frac{2}{4} \frac{y^2}{|z - x_0|^2}$$

and hence if $t = t_1 + t_2$,

$$\frac{t_2^2}{R^2} \geq \frac{4\lambda}{9}, \quad \frac{(x_0 - t_1)^2}{R^2} \leq 1 - \frac{4\lambda}{9},$$

and

$$\mu R \leq R - (x_0 - t_1) \leq t_1,$$

with

$$\mu = 1 - \left(1 - \frac{4\lambda}{9}\right)^{1/2} > 0.$$

But $t_2 \leq R$, and therefore

$$t_2 \leq \frac{t_1}{\mu}.$$
This shows that \( L \) contains a sequence of points within the angle \((11.4)\) which tend to infinity. In particular, the cavitation region \( D \) has points with arbitrarily large, positive abscissas. These results indicate that the cavity, or wake, \( W \) in some sense follows the nose \( C \).

12. Summary and discussion of results.

The principal result of Sections 4 through 10 is the existence of a unique axially symmetric cavitation flow leaving a prescribed nose surface generated by rotation of a monotonic curve about the \( x \)-axis and returning to a tail surface which is the reflection of the nose in a plane perpendicular to the \( x \)-axis. Section 4 shows in addition that the free boundary is generated by a curve which ascends to a maximum and then descends. The proof is based on the minimum virtual mass problem \((3.7)\) and consists of five parts: demonstration of the existence of a minimum, application of symmetrization to get a smooth free boundary, interior variational analysis \((6.11)\) leading to the boundary condition \((7.9)\), proof of analyticity of the free boundary by means of the functional equations \((8.12)\), and uniqueness.

In Section 11, for the same nose it is shown that there is a cavitational flow with an infinite wake rather than a tail, which is of interest in view of the classical infinite wake theory of Helmholtz and Kirchhoff. This flow is obtained as a limiting case of the previous model.

The application of the integral equation \((8.12)\) presented in Sections 8 and 9 has significance in its own right, since the analysis there shows that any axially symmetric, constant pressure, free surface must be analytic. The method has quite general implications for overdetermined analytic boundary value problems in elliptic partial differential equations.
Many further flow problems and free boundary extremal problems for linear elliptic partial differential equations in two independent variables can be treated by the method and techniques developed in the present paper. We indicate here one of these applications.

One can carry through the existence proof for axially symmetric cavitation flow in the case where the flow region \( D \) is bounded outside by a cylindrical tunnel, or pipe, \( y = H \). The hypotheses on the object \( B \) and the cavity \( W \) are the same as before, except that in addition \( B + W \) must lie always within the cylinder \( 0 \leq y \leq H \). The line \( y = H \) is a streamline, and the virtual mass \( M \) is defined exactly as before, except that \( D \) is now a region contained in the cylinder of radius \( H \). The coefficient \( a \) no longer appears, but it suffices to replace (3.7) by the equally valid extremal problem

\[
(12.1) \quad M - (\lambda - 1) \nu = \text{minimum}.
\]

Symmetrization, interior variations, the functional equation (8.12), and the comparison method for uniqueness can be carried through essentially in the same way as before, and the differences encountered for the new model are merely formal [5]. The flow described here occurs in actual experimental set-ups.

From the point of view of the applied mathematician, it is important to discuss the possibility of calculating axially symmetric cavities by use of the present methods. It is suggestive that the minimum problem (3.7) and the uniqueness technique of Section 10 might allow one to approximate free streamline flows by a Ritz procedure.

Another, possibly more interesting, approach would start from the functional equations (8.12). Given any analytic curve \( L \), we can calculate explicitly analytic functions \( f(z) \) and \( g(z) \) with the values \( \overline{z} \) and \( \overline{z} \) on \( L \).
Formulas (8.12) can then be used to calculate the analytic function \(-\lambda^{1/2} F(\mathbf{z}) / 2\pi i\), which merely represents the axially symmetric stream function \(\psi^i\) along a characteristic plane in the 4-dimensional, complex domain. Thus, for complex values of \(x, y\) with \(z = x + iy, \overline{z}_0 = x - iy\), formulas (8.6) and (8.12) imply for \(z_0\) on \(L\)
\[
\psi(z, \overline{z}_0) = i \lambda^{1/2} \int_{z_0}^{z} A(\zeta, g(\zeta); z, \overline{z}_0) g'(\zeta)^{1/2} d\zeta.
\]
Since \(g(z_0) = \overline{z}_0\) on \(L\), we obtain further
\[
(12.2) \quad \psi(z, g(z_0)) = i \lambda^{1/2} \int_{z_0}^{z} A(\zeta, g(\zeta); z, g(z_0)) g'(\zeta)^{1/2} d\zeta.
\]
This is an analytic identity in \(z_0\) along \(L\), and hence by analytic continuation it holds for all values of \(z_0\) in a neighborhood of \(L\). Let \(w = g(z_0)\) in some region near \(L\) and let \(z_0 = G(w)\) be the inverse of that analytic function. By (12.2), we find
\[
\psi(z, w) = i \lambda^{1/2} \int_{G(w)}^{z} A(\zeta, g(\zeta); z, w) g'(\zeta)^{1/2} d\zeta.
\]
In particular, for \(w = \overline{z}\) we obtain in the real \((x, y)\)-plane the real stream function
\[
(12.3) \quad \psi(z, \overline{z}) = i \lambda^{1/2} \int_{G(z)}^{z} A(\zeta, g(\zeta); z, \overline{z}) g'(\zeta)^{1/2} d\zeta.
\]
The stream function (12.3) yields an explicit example of an axially symmetric flow for which the original analytic curve \(L\) is a free streamline. This indirect method yields examples of cavitating flow, but it has the disadvantage that the fixed boundaries and singularities cannot be prescribed and may not be physically significant.
In order to get useful examples, one might try to use for the analytic curve $L$ the exact free streamlines connected with the known two-dimensional cavitating flow patterns. In this case, the analytic functions $f$ and $g$ are particularly easily computed in terms of the complex potential of the plane flow.
The following list contains references to papers known to us on axially symmetric flow plus papers from which we have borrowed ideas; we do not pretend to cover the extensive bibliography of plane cavity flow.

REFERENCES


