APPROXIMATIONS TO THE DISTRIBUTION FUNCTION OF SUMS OF INDEPENDENT CHI RANDOM VARIABLES

BY

HERMAN RUBIN and JAMES ZIDEK

TECHNICAL REPORT NO. 106
AUGUST 3, 1965

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APPROXIMATIONS TO THE DISTRIBUTION FUNCTION OF SUMS
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This report is concerned with the problem of approximating the
distribution function, denoted by $F_n$, of

\begin{equation}
\sum_{i=1}^{n} |X_i|, \quad n = 1, 2, \ldots,
\end{equation}

where the $\{X_i\}_{i=1}^{\infty}$ are independent standard normal random variables.

In the first of its three sections, we present three well-known approximations adapted to the problem at hand. These are the Edgeworth (cf., Cramér [1], page 228), the Cramér [2], and a saddlepoint approximation.

In addition, a second very effective saddlepoint approximation is derived which, as far as is known by the authors, has not appeared previously in the literature. The second section is devoted to a discussion of the problem of calculating the moment generating function, say $M$, of $F_1$ for complex values of its argument. This is of concern, for example, in the problem of numerically inverting $M^n$ in order to obtain the values of $F_n$ corresponding to prescribed values of its argument. Finally, in section three, we tabulate calculations which provide a comparison of the approximations given in section one for the cases $n = 10$ and $40$. To facilitate this comparison, the corresponding values (to eight decimal places) of $F_{10}$ and $F_{40}$ are given.

1. **Four Approximation to $F_n$.**

Let $f$ denote the probability density function for the distribution of $|X_1|$. Then

\begin{equation}
\text{1}
\end{equation}
\[ f(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2}, \quad x \geq 0 \]
\[ = 0 \quad , \quad x < 0 . \]

Also, if \( M(z) = E(e^{iX_1}) \), it is easily shown that

\[ M(z) = 2e^{z^2/2} \phi(z) , \]

where \( z \) is complex and

\[ \phi(z) = \frac{1}{2\pi} \int_{-\infty}^{z} e^{-v^2/2} dv . \]

Let \( \alpha_r \ (r = 1, 2, \ldots) \) and \( \sigma \) denote the \( r^{th} \) cumulant and standard deviation, respectively, of \( |X_1| \). Then

\[ \alpha_1 = \mu_1 \approx 0.79788 45608 03 \]
\[ \alpha_2 = 1 - \mu_1^2 \approx 0.36338 02276 32 \]
\[ \alpha_3 = \mu_1 (\mu_1^2 - \alpha_2) \approx 0.21801 36141 45 \]
\[ \alpha_4 = 2\mu_1^2 (2 - 3\mu_1^2) \approx 0.11477 0682054 \]
\[ \alpha_5 = \mu_1 (3 - 20\mu_1^2 + 24\mu_1^4) \approx 0.00443 76884 6262 \]
\[ \sigma \approx 0.60281 02749 89 . \]

Now, using the Edgeworth expansion of \( F_n \) we obtain, as \( n \to \infty \),
1 - F_n(x) \sim 1 - \Phi(w_n) + \frac{1}{\sqrt{n}} \cdot \frac{\lambda_n^2}{3!} \cdot \phi(3)(w_n)

(6)
\[
\frac{1}{n} \left[ \frac{\lambda_n^4}{4!} \cdot \phi(4)(w_n) + \frac{10}{6!} \cdot \lambda_n^2 \cdot \phi(6)(w_n) \right]
\]
\[
+ \left( \frac{1}{\sqrt{n}} \right)^3 \left[ \frac{\lambda_n^5}{5!} \cdot \phi(5)(w_n) + \frac{35}{7!} \cdot \lambda_n^3 \lambda'_4 \phi(7)(w_n) + \frac{280}{9!} \cdot \lambda_n^3 \cdot \phi(9)(w_n) \right]
\]

where
\[
w_n = (x - n \mu_1)/(\sqrt{n} \sigma)
\]

(7)
\[
\lambda_n = \alpha_n/\sigma^n
\]

(n = 1,2,...) and \( \phi(r) \) denotes the r\textsuperscript{th} derivative of \( \Phi(r = 0,1,2,...) \).

From equations (5) and (6),

1 - F_n(x) \sim 1 - \Phi + \frac{1}{\sqrt{n}} \cdot (.16587 86244 .05) \cdot \phi(3)

\[- \frac{1}{n} \left[ (.03621 57209 836)\phi(4) + (.01375 78590 173)\phi(6) \right]
\]

(8)
\[- \left( \frac{1}{\sqrt{n}} \right)^3 \left[ (.00046 45926 47496)\phi(5) - (.00600 74139 7860)\phi(7)
\right.
\]
\[\left. - (.00076 07115 7618)\phi(9) \right] \]

where, for convenience, the argument, \( w_n \), has been suppressed on the right of this asymptotic equality.
The Cramér approximation gives

\[
1 - F_n(x) \sim \begin{cases} 
(1 - \Phi(w_n)) \exp \left[ -\frac{w_n^2}{2} \frac{\lambda(w_n)}{\sqrt{n}} \right] & x \geq n\mu_1 \\
\phi(w_n) \exp \left[ -\frac{w_n^3}{2} \frac{\lambda(w_n)}{\sqrt{n}} \right] & x < n\mu_1
\end{cases}
\]

where, if the series

\[
\sigma z = \sum_{r=2}^{\infty} \frac{\alpha_r}{(r-1)!} h^r
\]

is inverted to obtain \( h \) as a power series, \( h(z) \), in \( z \),

\[
\lambda(z) = \frac{\sqrt{n}}{z^3} \left( \frac{z^2}{2} - \sum_{r=2}^{\infty} \frac{(r-1)\alpha_r}{r!} - h(z) \right)
\]

\[
= \left( \frac{\lambda_3}{6} + \left( \frac{\lambda_4}{24} - \frac{\lambda_3^2}{8} \right) z + \left( \frac{\lambda_5}{120} - \frac{\lambda_3 \lambda_4}{12} + \frac{\lambda_3^3}{18} \right) z^2 \right) + \ldots
\]

\[
\approx 0.16587 \ 86244 \ 05 - 0.08760 \ 50101 \ 719 \ z
\]

\[
+ 0.05161 \ 09001 \ 629 \ z^2 + \ldots
\]

We now derive two saddlepoint approximations to \( F_n \) using the

Gurland [3] inversion formula which asserts

\[
1 - F_n(x) = \frac{1}{2} + \lim_{T \to \infty} \lim_{\eta \to 0^+} \left\{ \int_{-T}^{T} + \int_{-T}^{-\eta} \right\} \frac{e^{-i x t}}{2 \pi i t} M_n(it) \ dt,
\]

where \( i = \sqrt{-1} \). Equation (12) is equivalent to
\[ 1 - F_n(x) = \frac{1}{2}(1 - \text{sign}(c)) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[-(c+iu)x + n \log M(c+iu)\right] \frac{du}{c+iu} \]

for every real number \( c \), and this is the form in which we shall make use of it. Since,

\[ \frac{d}{dh} M(n) = \sqrt{\frac{2}{\pi}} \left[ 1 + \frac{n\Phi(h) - \Phi(h)}{\Phi(h)} \right], \]

where \( \Phi(h) = \Phi(1)(h) \), the critical point, say \( c_n = (n=1,2,...) \) of the integrand is the solution of the equation

\[ \Phi(h)/\Phi(h) + h = x/n. \]

Equation (14) can be solved numerically using Newton's method with initial approximation \( c_{n0} = x/n \), and \( k \) \text{th} iterate

\[ c_{nk} = c_{n,k-1} - \frac{[g(c_{n,k-1}) + c_{n,k-1} - x/n]}{[1-c_{n,k-1}^2 - g(c_{n,k-1}) - g^2(c_{n,k-1})]}, \]

\( (k = 1,2,...) \), where \( g(x) = \Phi(x)/\Phi(x) \). In Table 1, we give the values of \( c_n \) which were obtained by this method in the case \( n = 20 \) for various values of \( x \). For simplicity we shall hereafter write \( c = c_n \).
TABLE I

SOLUTIONS OF EQUATION (14) OBTAINED USING NEWTON'S METHOD

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</tr>
<tr>
<td>26</td>
<td>1.01960</td>
</tr>
</tbody>
</table>

Now, for values of \( |u| \) in some neighbourhood of the origin

\[
-(c + iu)x + n \log M(c + iu) = (-cx + n \log M(c)) - nu^2 \psi^{(2)}(c)/2 + n \sum_{r=0}^{\infty} \frac{\psi^{(r)}(c)}{r!} (iu)^r ,
\]

where \( \psi(z) = \log M(z) \). Let

\[
\left\{
\begin{align*}
\sigma^* &= [\psi^{(2)}(c)]^{1/2} \\
b_r &= \psi^{(r)}(c) i^r / (r! \cdot \sigma^r) \\
a_r &= (-i)^r / (c \sigma^r)
\end{align*}
\]

(18) \( K(c,n,x) = \frac{1}{\sqrt{2\pi}} \exp[-cx + n \psi(c)] , \)

(19) \( I(x,n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-(c + iu) + n\psi(c + iu)] \frac{du}{c+iu} , \)

\((r = 1,2,\ldots; n = 1,2,\ldots) . \)
With this notation,

\[
\left(1 + \frac{iy}{\sqrt{n} \sigma^*}\right)^{-1} \exp \left[ n \sum_{r=2}^{\infty} b_r \left( \frac{y}{\sqrt{n}} \right)^r \right]
\]

(20)

\[= \sum_{m=0}^{\infty} d_m(y) \left( \frac{1}{\sqrt{n}} \right)^m,
\]

where

\[d_0(y) = 1\]

(21)

\[d_2(y) = a_2 y^2 + \left( b_4 + a_1 b_3 \right) y^4 + \frac{1}{2} b_3^2 y^6\]

\[d_4(y) = a_4 y^4 + \left( b_6 + a_1 b_5 + a_2 b_4 + a_3 b_3 \right) y^6\]

\[+ \left( \frac{1}{2} b_4^2 + b_3 b_5 + a_1 b_3 b_4 + \frac{1}{2} a_2 b_2^2 \right) y^8\]

\[+ \left( \frac{1}{2} b_5 b_3^2 + \frac{1}{6} a_1 b_3^3 \right) y^{10} + \frac{1}{24} b_3^4 y^{12},\]

and in general \(d_{2k-1}(y)\) is an odd polynomial in \(y\), \(d_{2k}(y)\) an even polynomial in \(y\), \(k = 1, 2, \ldots\).

According to an argument given by Daniels [4],

(22)

\[I(x, n) \sim K(c, n, x) / (c \sqrt{n} \sigma^*) \sum_{m=0}^{\infty} d_{2m} \left( \frac{1}{\sqrt{n}} \right)^{2m},\]

where
\[ d_0 = 1 \]

\[ d_2 = a_2 + 3(b_4 + a_1 b_3) + 15 b_2^2 / 2 \]

\[ d_4 = 3a_4 + 15(b_6 + a_1 b_5 + a_2 b_4 + a_3 b_3) \]

\[ + 105(\frac{1}{2} b_4^2 + b_3 b_5 + a_1 b_3 b_4 + \frac{1}{2} a_2 b_3^2) \]

\[ + 945(\frac{1}{2} b_4 b_5^2 + \frac{1}{5} a_1 b_5^2) + 1035 b_3^4 / 24 \]

(23)

and, in general,

\[ d_{2m} = \int_{-\infty}^{\infty} \varphi(y) d_{2m}(y) \, dy \]

(24)

\[ m = 0, 1, 2, \ldots \]. After simplification, equations (23) become, with \[ \psi(r)(c) = U_r, \ r = 1, 2, \ldots, \]

\[ d_0 = 1 \]

\[ d_2 = -1/(\sigma^2 c^2) + \left[ \frac{1}{8} U_4 - \frac{1}{2c} U_3 \right] / \sigma^2 \]

\[ - 5 U_5^2 / (24 \sigma^3) \]

(25)

\[ d_4 = 3/(\sigma^2 c^4) \]

\[ + \frac{5}{8 \sigma^3} \left( - \frac{1}{16} U_6 + \frac{1}{5c} U_5 - \frac{1}{2c} U_4 + \frac{4}{3c} U_3 \right) \]

\[ + \frac{105}{24 \sigma^4} \left( \frac{1}{16} U_4^2 + \frac{1}{10} U_3 U_5 - \frac{1}{2c} U_3 U_4 + \frac{1}{c^3} U_3^2 \right) \]

\[ + \frac{35}{16 \sigma^3} \left( - \frac{1}{4} U_4 U_3^2 + \frac{1}{5c} U_3^2 \right) + \frac{385}{48 \sigma^6} U_3^4 \]
Observe that equation (14) implies

\[(26) \quad K(c, n, x) = \left( e^{-xc} / \sqrt{2\pi} \right) \left[ \frac{2}{\pi} \left( \frac{x}{n} - \frac{c}{2} \right)^{-1} \right]^n \]

and (after considerable simplification) on letting \( u = x/n \),

\[U_1 = u\]

\[U_2 = cu + 1 - u^2 = 2u^2\]

\[U_3 = uc^2 + c(1 - 3u^2) - u(1 - 2u^2)\]

\[U_4 = uc^3 + c^2(1 - 7u^2) - uc(5 - 12u^2) + u^2(4 - 6u^2)\]

\[U_5 = uc^4 + c^3(1 - 15u^2) - uc^2(16 - 50u^2)\]

\[\quad - c(3 - 35u^2 + 60u^4) + u(3 - 20u^2 + 24u^4)\]

\[U_6 = uc^5 + c^4(1 - 31u^2) - uc^3(42 - 180u^2)\]

\[\quad - c^2(13 - 191u^2 + 390u^4) + cu(41 - 270u^2 + 360u^4)\]

\[\quad - u^2(28 - 120u^2 + 120u^4)\]

Thus,

\[(28) \quad 1 - F_n(x) \sim \frac{1}{2} \left( 1 - \text{sign} (c_n) \right)\]

\[\quad + \left[ e^{-xc_n} / (\sqrt{2\pi} c \sqrt{n} u_2) \right] \left[ \frac{2}{\pi} \left( \frac{x}{n} - c_n \right)^{-1} \right]^n \left[ 1 + \frac{d_2}{n} + \frac{d_4}{n^2} \right],\]

where \( c_n (n = 1, 2, \ldots) \) is obtained from equation (14), \( d_2 \) and \( d_4 \) from equations (25) with the aid of equations (27).
Let us return now to equation (19) and by a somewhat more delicate argument obtain a second saddlepoint approximation to $1 - F_n$. From this equation together with equations (16), (17), (18) and the argument of Daniels which was previously referred to, we have on letting

$$\phi = \rho_n = c_n \sqrt{n} U_2, \quad b_r = b_r' / i^r \quad (r = 0, 1, 2, \ldots) ,$$

(29) \quad $1 - F_n(x) \sim \frac{1}{2} \left(1 - \text{sign}(c)\right) + K(c, n, x) \int_{-\infty}^{\infty} \frac{\phi(u)}{\rho + iu} \exp\left[\sum_{r=3}^{\infty} b_r' \frac{(iu)^r}{(\sqrt{n})^{r-2}}\right] du$ .

Now

(30) \quad $\exp\left[\sum_{r=3}^{\infty} b_r' \frac{(iu)^r}{(\sqrt{n})^{r-2}}\right] = \sum_{k=0}^{\infty} g_k(iu) \left(\frac{1}{\sqrt{n}}\right)^k ,$

where

$$g_0(y) = 1$$

$$g_1(y) = b_3' y^3$$

(31) \quad $$g_2(y) = b_4' y^4 + \frac{1}{2} b_3^2 y^6$$

$$g_3(y) = b_5^2 y^5 + b_3 b_4 y^7 + \frac{1}{6} b_3^3 y^9$$

$$g_4(y) = b_6^2 y^6 + \left(\frac{1}{2} b_4'^2 + b_3 b_5'\right) y^8 + \frac{1}{2} b_3^2 b_4' y^{10} + \frac{1}{24} b_3^4 y^{12} .$$

Define $Q_k(\phi) \quad (k = 0, 1, \ldots)$ by

(32) \quad $Q_k(\phi) = \int_{-\infty}^{\infty} \frac{\phi(u)}{\rho + iu} (iu)^k du$ .

Then

10
\[ Q_0(\rho) = \frac{1}{\phi(\rho)} \left[ \frac{1}{2} (1 + \text{sign}(c)) - \Phi(\rho) \right] \]

and the \( \{q_k\} \) satisfy the recurrence formulae,

\[
Q_{2k-1}(\rho) = (-1)^{k-1} \frac{(2k-3)(2k-5)}{3!} \ldots \frac{1}{\rho} Q_{2k-2}(\rho) \tag{34}
\]

\[
Q_{2k}(\rho) = -\rho Q_{2k-1}(\rho), \quad (k = 1, 2, \ldots) .
\]

Thus,

\[
1 - F_n(x) \approx \frac{1}{2} (1 - \text{sign}(c))
\]

\[
+ \left( e^{-x^2/2} \right) \left[ \frac{1}{\sqrt{n}} \left( \frac{x}{n} - c \right)^{-1} \right]^n \sum_{k=0}^{\frac{4}{n}} \tilde{g}_k(\rho) \left( \frac{1}{\sqrt{n}} \right)^k ,
\]

where

\[
g_0(\rho) = Q_0(\rho)
\]

\[
g_1(\rho) = \frac{1}{6} U_2 \left( \sqrt{U_2} \right)^{-3} \cdot Q_3(\rho)
\]

\[
g_2(\rho) = \frac{1}{24} U_4 U_2^{-5} Q_4(\rho) + \frac{1}{12} U_3^2 U_2^{-3} Q_6(\rho)
\]

\[
g_3(\rho) = \frac{1}{120} U_6 U_2^{-5} Q_5(\rho) + \frac{1}{144} U_5 U_4 \left( \sqrt{U_1} \right)^{-7} Q_7(\rho)
\]

\[
+ \frac{1}{1296} U_3^2 \left( \sqrt{U_2} \right)^{-3} Q_9(\rho)
\]

\[
g_4(\rho) = \frac{1}{720} U_6 U_2^{-3} Q_6(\rho) + \left( \frac{1}{1152} U_4^2 + \frac{1}{720} U_3 U_5 \right) U_2^{-4} Q_8(\rho)
\]

\[
+ \frac{1}{1728} U_3^2 U_4 U_2^{-5} Q_{10}(\rho) + \frac{1}{31104} U_3^4 U_2^{-6} Q_{12}(\rho)
\]

where the \( \{q_1\}_{l=3}^{12} \) are readily obtained using recurrence relations (34), \( Q_0 \) is obtained from equation (33) and the \( \{U_{l,1=1}^5\} \) are given by equation (27).
2. The Calculation of \( M(z) \).

We consider now, the problem of computing \( M(z) \) as defined in equation (3). This computation is necessary to the numerical inversion procedure by means of which one obtains, using equation (13), exact values of \( P_n \) for specified values of its argument. The difficulty here is that excessive and uncontrollable round off errors occur, even for moderate values of \(|z|\) when \( \Phi(z) \) is computed using its Taylor expansion. Some examples illustrating this remark are presented below in Table 2.

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<tr>
<th>( z )</th>
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<th>( \text{Im}(z) )</th>
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<th>( \text{Re}(M(z)) )</th>
<th>( \text{Im}(M(z)) )</th>
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<table>
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<th>( z )</th>
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<th>( M(z) )</th>
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</table>

The solution to this problem which we propose here involves the complex forms of Shenton's [5] continued fraction for small values of \(|z|\) and Laplace's continued fraction (cf., Kendall and Stuart [6], p. 138) otherwise. Since
(37) \[ M(z) = 2e^{z^2/2} - M(-z), \]

there is no loss of generality in considering the case \( R(z) \geq 0 \), alone. The notation

\[
\frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ldots}}} = \frac{a_1}{b_1} + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ldots}}
\]

will be adopted here.

Using the Shenton fraction we obtain

(39) \[ M(z) = e^{z^2/2} + \sqrt{\frac{2}{\pi}} \left( \frac{1}{1 - \frac{z^2}{3+}} + \frac{z^2}{5+} - \frac{2z^2}{7+} \ldots \right), \quad \Re(z) > 0 \]

and using that of Laplace

(40) \[ M(z) = 2e^{z^2/2} + \sqrt{\frac{2}{\pi}} \left( \frac{1}{\frac{1}{t^+}} + \frac{1}{\frac{1}{t^+}} \frac{2}{\frac{3}{t^+}} \ldots \right), \quad \Re(z) > 0 . \]

Equation (39) may be rewritten with \( t = 1/z^2 \) as

(41) \[ M(z) = e^{z^2/2} + \sqrt{\frac{2}{\pi}} \left( \frac{z}{1 - \frac{1}{t^+}} + \frac{1/3}{1 - \frac{2/(3.5)}{t^+}} \frac{3/(5.7)}{t^+} \ldots \right), \quad \Re(z) > 0 . \]

The \( 2n \text{th} \) approximant to the continued fraction in this expression is

(42) \[ \frac{z}{1 - \frac{1}{t^+}} \ldots \left(\frac{2n-1}{\left[(4n-3)(4n-1)\right]}\right) \quad (n = 1, 2, \ldots) \]

while the remainder satisfies
\[
\begin{align*}
(43) \quad \frac{2n/[(4n-1)(4n+1)]}{1-} & \quad \frac{(2n+1)/[(4n+1)(4n+3)]}{t+} \\
& \sim \frac{1/8n}{1-} \quad \frac{1/8n}{t+} \quad \frac{1/8n}{1-} \\
& \text{as } n \to \infty. \text{ The continued fraction appearing on the right of (43)}
\end{align*}
\]

represents the function \( u(t) \) which satisfies the equation

\[
(44) \quad u = a_n/[1 - \frac{a_n}{t+u}] ,
\]

that is,

\[
(45) \quad u(t) = (a_n - \frac{t}{2}) + \sqrt{(\frac{t}{2})^2 + a_n^2} ,
\]

where \( a_n = 1/8n \).

Let

\[
R_n(t) = \text{Re}\left[\left(\frac{t}{2}\right)^2 + a_n^2\right] \\
I_n(t) = \text{Im}\left[\left(\frac{t}{2}\right)^2 + a_n^2\right] \\
R_n'(t) = \left[\frac{1}{2} \left(R_n(t) + \left[I_n^2(t) + R_n^2(t)\right]\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} ,
\]

\( (n = 1,2,\ldots) \).

Then (cf. Ahlfors [7], p. 3),

\[
[R_n^2(t) + I_n^2(t)]^{\frac{1}{2}} = \pm [R_n'(t) + \frac{i}{2} I_n(t)/R_n'(t)]
\]

if \( R_n'(t) \neq 0 \) and 0 otherwise. The square root in this equation has
branch points at \( \pm 2ai \), and the function obtained by choosing either sign
is a branch of the square root. Rather than fix the sign, we take
to obtain a continuous approximation to the continued fraction.

Using equation (45) we obtain the following asymptotic approximation to the continued fraction of equation (41):

\[
\frac{2}{1} - \frac{1}{z^{t+1}} + \frac{2}{z^2} - \cdots (2n-1)/[(4n-1)(t^2/2 + a_n + \sqrt{(t^2/2 + a_n^2)}], \quad \text{Re}(z) > 0,
\]

\[(n = 1, 2, \ldots)\]

which, as it happens, gives satisfactory results in the case \( \text{Re}(z) = 0 \).

By a similar argument we obtain the following approximation to the continued fraction of equation (40):

\[
\frac{1}{z+} + \frac{1}{z+} + \frac{2}{z^2} + \cdots \frac{2(n-1)}{z + \sqrt{z^2 + 4n}}, \quad \text{Re}(z) > 0,
\]

\[(n = 2, 3, \ldots)\]

We shall make, in turn, a modification of this last approximation.

Observe that if

\[
H_n(z) = \int_0^{\infty} \frac{t^{n-1}}{(n-1)!} e^{-\frac{1}{2}(t+z)^2} dt, \quad n = 1, 2, \ldots, \quad \text{Re}(z) > 0
\]

\[(50)\]

\[
H_0(z) = \sqrt{\pi} e^{-z^2/2},
\]

then
\[(51) \quad H_n(z) = (n+1) H_{n+2}(z) + zH_{n+1}(z) \quad (n = 0, 1, 2, \ldots) .\]

The substitution, \( q_n = H_{n-1}/H_n \), yields

\[(52) \quad q_n(z) = z + n/q_{n+1}(z) , \quad n = 1, 2, \ldots .\]

Thus

\[\begin{align*}
q_1(z) &= \varphi(z)/(1 - \varphi(z)) \\
&= z + 1/q_2(z) \\
&= z + \frac{1}{z^+} \frac{2}{z^+} \ldots \frac{(n-1)}{q_n(z)} , \quad (n = 2, 3, \ldots) .
\end{align*}\]

In (49) replace \( q_n \) by \( q'_n \), where the \( \{q'_n\}_{n=2}^\infty \) are chosen so that the approximation \( \frac{1}{2}(z + \sqrt{z^2 + q'_n}) \) to \( q_n(z) \) is exact at \( z = 0 \), that is,

\[(54) \quad q'_n = 8 \frac{\Gamma^2(n + \frac{1}{2})}{\Gamma^2(n + \frac{3}{2})} , \quad n = 2, 3, \ldots ,\]

where \( \Gamma \) denotes the Gamma-function.

Let \( R_n(z) \) and \( I_n(z) \) denote \( \text{Re}(z^2 + q'_n) \) and \( \text{Im}(z^2 + q'_n) \), respectively. Then, if

\[(55) \quad R_{kn}(z) = \left[ \frac{k}{2}((-1)^{k-1} R_n(z) + [R_n^2(z) + I_n^2(z)]^{1/2} \right]^{1/2} , \]

\( k = 1, 2; \quad n = 2, 3, \ldots , \) we take, as a result of the same considerations which gave rise to equation (47),

16
\[ [R_n(z) + iI_n(z)]^{1/2} = \begin{cases} 
\text{sign}[I_n(z)][I_n(z)/\{2R_{2n}(z)\}] + iR_{2n}(z), \\
R_n(z) < 0, I_n(z) \neq 0, R_{2n}(z) \neq 0, \\
I_n(z)/\{2R_{2n}(z)\} + iR_{2n}(z), \\
R_n(z) < 0, I_n(z) = 0, R_{2n}(z) \neq 0, \\
R_{1n}(z) + iI_n(z)/\{2R_{1n}(z)\}, R_n(z) > 0, R_{1n}(z) \neq 0, \\
0, R_{2n}(z) = 0 \text{ or } R_{1n}(z) = 0 . 
\end{cases} \]

In summary, we have obtained the following approximations to \( M(z) \),

\[ M(z) \sim e^{z^2/2} + \sqrt{\frac{2}{\pi}} \left( \frac{z}{1+} \frac{1}{3}^{+} \frac{2}{5}^{+} \ldots \right) \]

\[ \frac{(2n-1)}{(4n-1)} \left( \frac{1}{2n} + \frac{1}{6n} + \sqrt{\frac{2}{2}} \frac{1}{4n} \right) \], \( \text{Re}(z) \geq 0 \)

and

\[ M(z) \sim 2e^{z^2/2} + \sqrt{\frac{2}{\pi}} \left( \frac{1}{z^{+}} \frac{1}{2}^{+} \frac{2}{3}^{+} \ldots \right) \]

\[ 2(n-1)/[z + \sqrt{z^2 + 8r^2 (n+1)/r^2 (n^{1/2})}], \text{Re}(z) > 0 \]

where the determination of the square roots involved is as given in equations (47) and (56).

These approximations were computed and compared on a grid for \( z \) comprised of 231 points spread over the region.
(59) \[ D = \{ z : -5 \leq \text{Re}(z) \leq 5, \quad 0 \leq \text{Im}(z) \leq 5 \} \, . \]

On the basis of this, Table 3 has been completed and lists for all sub-regions of \( D \), a suitable approximation to \( M \). The approximations given in equations (57) and (58) are denoted by \( S_r \) and \( C_r \) \((r=1,2,\ldots)\), respectively, where in each case, \( r \) designates a value of \( n \) sufficiently large as to give an accuracy of at least 10 significant figures for that subregion.
TABLE 3
APPROXIMATIONS TO M(z), z \in D WHICH ARE ACCURATE TO AT LEAST TEN SIGNIFICANT PLACES

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<th>I(z)</th>
<th>APPROXIMATION</th>
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Table 4, below, summarizes illustrative calculations which were carried out using the four approximations to \( 1 - F_n(x) \) discussed in Section 1 for the cases \( n = 10 \) and \( 40 \), for various values of \( x \). Also given are the corresponding exact values of \( 1 - F_n \) which serve as a basis for the comparisons of the four alternatives.

An examination of this table reveals that the saddlepoint approximation given in (35) yields substantially better results in all cases considered than those provided by any of the other three methods.

**TABLE 4**

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*Whenever several entries are tabulated for a given approximation and value of the argument, they represent the sequence of approximations obtained by taking for the \( k^{th} \) \((1 \leq k \leq 5)\), the sum of the first \( k \) terms of the original approximation as given in Section 1.*
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Rubin, Herman and Zidek, James, V.

August 3, 1965

Technical Report No. 106

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This report is concerned with the problem of approximating the distribution function, \( F_n \), of \( |X_1| + \cdots + |X_n| \), \( n = 1, 2, \ldots \), where the \( \{X_i\} \) are independent standard normal random variables. In Section One, three well known approximations, the Edgeworth, the Cramér, and a Saddlepoint approximation are described. Another saddlepoint approximation is derived. The second section is devoted to a discussion of the problem of calculating the moment generating function of \( F_{1/2} \) for complex values of its argument. Finally, the four approximations to \( F_n \) presented in Section One are compared for several cases in Section 3 and it is seen that the second saddlepoint approximation yields better results in each case.
Sums of Independent Chi Variables
Sums of Random Variables
Approximations to Distribution Functions

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