THE DIVISION OF SPACE BY HYPERPLANES WITH
APPLICATIONS TO GEOMETRICAL PROBABILITY

BY

THOMAS COVER and BRADLEY EFRON

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DEPARTMENT OF STATISTICS
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0. Summary 

This paper investigates invariant combinatorial properties of convex cones and their dual cones generated by collections of vectors in a Euclidean space. These properties, which include the number of non-degenerate cones, the number of k-faces of these cones, and the natural measures of the set of k-faces, do not depend on the configuration of the set of generating vectors, except for a weak non-degeneracy requirement. Applications to geometrical probability are given. 

1. Introduction. 

It is known that $N$ hyperplanes in general position in $E^d$ divide $E^d$ into 

$$(1.1) \quad C(N,d) = 2^{d-1} \sum_{i=0}^{d-1} \binom{N-1}{i}$$

regions. (A set of $N$ vectors in Euclidean d-space $E^d$ is said to be in general position if every $d$-element subset is linearly independent, and a set of $N$ hyperplanes through the origin of $E^d$ is said to be in general position if the corresponding set of normal vectors is in general position.) 

This result, first proved by Schlafli [1], in the 19$^{th}$ century, is an intrinsic property of collections of hyperplanes in the sense that
the number of nondegenerate cones formed is independent (subject to
general position) of the configuration of the normal vectors. Schlafli's
theorem, essentially combinatorial in nature has been restated to yield
useful results in many branches of mathematics, as the following examples
show:

a) Given \( x_1, x_2, \ldots, x_N \) in general position in \( \mathbb{R}^d \),
consider the set of simultaneous inequalities

\[
\operatorname{sgn}(x_i \cdot w) = \delta_i, \quad i = 1, 2, \ldots, N
\]

where \( \operatorname{sgn} \) is the signum function defined on the reals

\[
\operatorname{sgn}(y) = \begin{cases} 
1, & y > 0 \\
0, & y = 0 \\
-1, & y < 0
\end{cases}
\]

and each \( \delta_i = \pm 1 \). Then among the \( 2^N \) possible
assignments of the \( \delta_i \), exactly \( C(N,d) \) will admit
some solution vector \( w \). That is, \( C(N,d) \) of the
\( 2^N \) sets of inequalities will be consistent.

b) Of the \( 2^N \) partitions of the vectors \( x_1, x_2, \ldots, x_N \)
(d-dimensional and in general position) into two
subsets, exactly \( C(N,d) \) can be separated by a
hyperplane through the origin. (A dichotomy is
separable by a hyperplane if the two classes lie
entirely on opposite sides of the hyperplane.) This
formulation of 1) is relevant to the theory of linear
threshold devices, [2].
c) Let $N$ vectors be chosen independently according to a $d$-dimensional probability distribution which is symmetric about the origin ($\mu(A) = \mu(-A)$) for every measurable set $A$, where $-A = \{x: -x \in A\}$ and absolutely continuous with respect to Lebesgue measure on $E^d$. Then with probability $C(N,d)/2^N$ there will exist a half-space containing the set of $N$ vectors. This probabilistic formulation of 1) is due to Wendel [3].

The $C(N,d)$ regions generated by hyperplanes in general position through the origin of $d$-space are all proper, nondegenerate convex cones. It is the purpose of this paper to demonstrate other properties of these cones and their dual cones, which, like the number $C(N,d)$, depend on the orientation of the partitioning hyperplanes only through the condition of general position. Applications of invariant properties analogous to 1), 2), and 3) above will be obvious in most cases. When they are not, or when the result is deemed of independent interest, they will be stated explicitly.

2. Theorems and proofs.

Let $x_1, x_2, \ldots, x_N$ be a set of $N$ vectors in general position in Euclidean $d$-space, and let $H_1, H_2, \ldots, H_N$ be the $N$ corresponding hyperplanes through the origin:

* These conditions can be weakened. See Section 3.
(2.1) \[ H_i = \{ w : v \cdot x_i = 0 \} \quad i=1,2,...,N \ . \]

The \( N \) hyperplanes partition \( \mathbb{F}^d \) into \( C(N,d) \) proper, nondegenerate (of full dimension) cones \( \{ W_j : j=1,2,...,C(N,d) \} \), where \( C(N,d) \) is given by (1.1).

The interior of each cone \( W_j \) is the set of all solution vectors \( w \) to a certain set of simultaneous linear inequalities,

(2.2) \[ \text{sgn}(x_i \cdot w) = \delta_i \quad i=1,2,...,N, \]

where each \( \delta_i = \pm 1 \). (Of the \( 2^N \) possible vectors of \( \pm 1 \)'s, \( \delta = (\delta_1, \delta_2, ..., \delta_N) \), exactly \( C(N,d) \) yield consistent inequalities and hence non-empty solution cones.)

The boundary of the \( d \)-dimensional solution cone \( W_j \) is the union of a finite number of \( (d-1) \)-dimensional cones, which will be referred to as the \( (d-1) \)-faces of \( W_j \). The boundaries of the \( (d-1) \)-faces are in turn composed of \( (d-2) \)-dimensional cones, the \( (d-2) \)-faces of \( W_j \). In general, the \( k \)-faces of \( W_j \) will be proper cones contained in a \( k \)-dimensional but not \( (k-1) \)-dimensional subspace of \( \mathbb{F}^d \). The \( 1 \)-faces are the extreme rays of \( W_j \), while the origin is the only \( 0 \)-face.

In the following \( k \) will always satisfy \( 1 \leq k \leq d-1 \).

The interior (relative to the smallest subspace containing it) of each \( k \)-face of \( W_j \) is the totality of solutions to some set of simultaneous relations.
\[(2.3) \quad \text{sgn}(x_i \cdot w) = \delta_i^* \quad i=1,2,...,N,\]

where \( I \) is a subset of size \( d-k \) of the integers \( \{1,2,...,N\} \) and

\[(2.4) \quad \delta_i^* = 0 \quad i \in I, \]

\[(2.4) \quad \delta_i^* = \delta_i \quad i \notin I. \]

**Theorem 1.** (Counting the k-faces of the solution cones).

Let \( R_k(W_j) \) be the number of k-faces of the cone \( W_j, j=1,2,...,C(N,d) \).

Then

\[(2.5) \quad \sum_{j=1}^{C(N,d)} R_k(W_j) = 2^{d-k} \binom{N}{d-k} C(N-d+k,k), \]

Proof. Let \( H = \bigcap_{i=1}^{d-k} H_i \) be the k-dimensional linear subspace orthogonal to the vectors \( x_1, x_2, ..., x_{d-k} \). The remaining \( N-d+k \) hyperplanes \( H_{d-k+1}, H_{d-k+2}, ..., H_N \) partition \( H \) into \( C(N-d+k,k) \) convex cones \( \{V_f\} \). (This is easily verified by noting that the projections of \( x_{d-k+1}, x_{d-k+2}, ..., x_N \) into \( H \) are in general position in that space, and that the intersection of \( H_i \) with \( H \), for \( d-k < i \leq N \), is the \( (k-1) \)-dimensional subspace of \( H \) orthogonal to the projection of \( x_i \). Hence the result (1.1) applies.

The interiors (in \( H \)) of each of the cones \( V_f \) can be characterized as the set of solution vectors to the simultaneous relations

\[(2.6) \quad \text{sgn}(x_i \cdot w) = \delta_i^* \quad i=1,2,...,N, \]

where

\[\delta_i^* = 0 \quad i=1,2,...,d-k, \]

and

\[\delta_i^* = 1 \quad i=d-k+1,...,N. \]
Let $\delta = (\delta_1, \ldots, \delta_N)$ be a vector of $1$'s such that $\delta_i = \delta_1^* \text{ for } i \geq d-k+1$. It follows by continuity that every such $\delta$ represents (as in (2.2) and (2.3)) a non-empty solution cone having $V_{\delta}$ as a $k$-boundary, and that these $2^{d-k}$ solution cones are the only ones having this property.

Finally, any of the $\binom{N}{d-k}$ subsets of the size $d-k$ from $x_1, \ldots, x_N$ may be used in place of $x_1, x_2, \ldots, x_{d-k}$ in the discussion above, yielding a total of $\binom{N}{d-k} 2^{d-k} C(N+d-k, k)$ $k$-boundaries for the solution cones.

To each choice of $d-k$ vectors $x_{i_1}, x_{i_2}, \ldots, x_{i_{d-k}}$ from the set $\{x_1, x_2, \ldots, x_N\}$, there corresponds a $k$-dimensional orthogonal subspace, $L_k(i)$. These subspaces are distinct because of the condition of general position. The proof of Theorem 1 provides some obvious but useful additional information on the $k$-faces of the solution cones, which is summarized in Theorem 2.

**Theorem 2.** Each $k$-face of a solution cone $W_j$ is contained in exactly one $L_k(i)$, and the union of the $k$-faces of all the $W_j$ is the set formed by the union of the $\binom{N}{d-k}$ subspaces $L_k(i)$. Each $k$-face bounds exactly $2^{d-k}$ solution cones.

Given any convex cone $W$, the **dual cone** $W^*$ is defined to be the set of vectors within a right angle of every vector in $W$, $W^* = \{w^*: w^* \cdot w \geq 0 \text{ for all } w \in W\}$. In particular, if $W$ is the solution cone corresponding to the set of linear inequalities

$$\text{sgn}(x_i \cdot w) = \delta_i \quad \delta_i = \pm 1 \quad i = 1, 2, \ldots, N,$$

then it is known that the dual cone $W^*$ is given by
(2.8) \[ W^* = \{ w^* : w^* = \sum_{i=1}^{N} \alpha_i s_i x_i, \alpha_i \geq 0 \quad i=1,2,\ldots,N \} \]

That is, \( W^* \) is the proper convex cone spanned by the vectors 
\( s_1 x_1, s_2 x_2, \ldots, s_N x_N \) (The \( 2^N - C(N,d) \) sets of assignments of the \( s_i \)'s which lead to an inconsistent set of inequalities (2.7), generate improper cones in (2.8).)

As has been shown above, a k-face of the solution cone \( W_j \) is orthogonal to exactly \( d-k \) of the vectors \( x_i \), say \( x_{i_1}, x_{i_2}, \ldots, x_{i_{d-k}} \). In the \( (d-k) \)-dimensional subspace generated by these vectors

\[
(2.9) \quad L^*_{d-k} = \{ x : x = \sum_{m=1}^{d-k} c_i x_i \} ,
\]

there is one \( d-k \) face to the dual cone \( W^*_j \), which is the convex cone generated by the vectors \( s_1 x_{i_1}, s_2 x_{i_2}, \ldots, s_{d-k} x_{i_{d-k}} \). Thus there is a one-to-one correspondence between the k-faces of a solution cone \( W_j \) and the \( d-k \) faces of its dual cone \( W^*_j \). Immediately, from Theorem 1, we obtain

**Theorem 3.** (Counting the k-faces of the dual cones).

Let \( R_k(W^*_j) \) be the number of k-faces of the dual cone \( W^*_j \). Then

\[
(2.10) \quad \sum_{j=1}^{C(N,d)} R_k(W^*_j) = 2^k \binom{N}{k} C(N-k,d-k), \quad k=1,2,\ldots,d-1.
\]

A statement corresponding to Theorem 2 can also be made for the dual cones. Let \( \{ L^*_k(1), L^*_k(2), \ldots, L^*_k(N) \} \) represent the class of k-dimensional linear subspaces of \( \mathbb{R}^d \) generated by the \( \binom{N}{k} \) possible k-element subsets of the N vectors \( x_1, x_2, \ldots, x_N \).
Theorem 4. Each $k$ face of a dual cone $W^*_j$ is contained in exactly one $L^*_k(i)$, and the union of the $k$-faces of all the $W^*_j$ is the set formed by the union of the $\binom{N}{k}$ subspaces $L^*_k(i)$. Each $k$-face bounds exactly $C(N-k,d-k)$ dual cones. The $(k-l)$-dimensional interiors of the $k$-faces overlap only if the two $k$-faces are identical.

Proof. Each of the $2^k$ cones generated by the vectors $(\delta_1 x_1, \delta_2 x_2, \ldots, \delta_k x_k)$, $\delta_i = \pm 1$, is a proper cone, and these cones partition the linear space $L^*_k$ generated by $x_1, x_2, \ldots, x_k$. (The cones overlap only on their boundaries, not on their interiors.)

Let $V^*_k$ be the cone generated by $x_1, x_2, \ldots, x_k$. $V^*_k$ will be a $k$-face of the convex cone $W^*$ generated by $x_1, x_2, \ldots, x_k$, $\delta_{k+1} x_{k+1}, \delta_{k+2} x_{k+2}, \ldots, \delta_N x_N$ if and only if the projections of the vectors $\delta_i x_i$, $i = k+1, k+2, \ldots, N$, into $L_{d-k}$, the orthocomplement of $L^*_k$ generate a proper convex cone in that space. (Since, as mentioned previously, $V^*_k$ is a $k$-face of $W^*$ if and only if it corresponds to a $(d-k)$-face $V_{d-k}$ of its dual cone $W$. If so, any vector $w$ within $V_{d-k}$ will lie in $L_{d-k}$ and will satisfy $\text{sgn}(x'_i \cdot w) = \delta_i$, where $x'_i$ is the projection of $x_i$ into $L_{d-k}$, $i = k+1, \ldots, N$. Conversely, the existence of such a $w$ easily implies $W^*$ is proper and $V^*_k$ is on its boundary.) By Schl"{a}fli's theorem exactly $C(N-k,d-k)$ assignments of the signs $\delta_i$, $i = k+1, \ldots, N$, will have this property. (Note that the projected vectors will be in general position in the $(d-k)$-dimensional space $L_{d-k}$.)

Thus the $k$-faces of dual cones partition the subspace $L^*_k$ into $2^k$ cones, and each $k$-face bounds $C(N-k,d-k)$ different dual cones. Repeating this argument for the $\binom{N}{k}$ possible selections of $k$ vectors from $x_1, x_2, \ldots, x_N$ completes the proof.
A separate argument is required to establish the next theorem, application of which will yield the expected volume of the cone spanned by a random collection of vectors. The statement is a logical generalization of a theorem proved by Samelson, Thrall, and Wesler [5] concerning the partitioning of $\mathbb{E}^d$ by cones. Let $W^*(x_1, x_2, \ldots, x_N)$ denote the convex cone spanned by $x_1, x_2, \ldots, x_N$, and consider the $2^N$ cones $W^*(\delta_1 x_1, \delta_2 x_2, \ldots, \delta_N x_N)$ where $\delta_i = \pm 1$, $i = 1, 2, \ldots, N$. Then Samelson et al show that, for $N = d$, this collection of $2^d$ cones partitions $\mathbb{E}^d$. We shall now show that for $N > d$ the cones $W^*(\delta_1 x_1, \ldots, \delta_N x_N)$ partition $\mathbb{E}^d$ many times over in a systematic manner.

**Theorem 5.** Let $x_1, x_2, \ldots, x_N$ lie in general position in $\mathbb{E}^d$. If $v$ is a point in $\mathbb{E}^d$ such that $x_1, x_2, \ldots, x_N$ and $v$ jointly lie in general position, then $v$ is a member of precisely $\binom{N-1}{d-1}$ proper convex cones of the form $W^*(\delta_1 x_1, \delta_2 x_2, \ldots, \delta_N x_N)$, $\delta_i = \pm 1$, $i = 1, 2, \ldots, N$.

**Proof.** Let $W(x_1, x_2, \ldots, x_N)$ be defined as following 2.1. Let $v$ partition the set $S$ of cones $W(\delta_1 x_1, \ldots, \delta_N x_N)$, the dual cones to the cones $W^*(\delta_1 x_1, \ldots, \delta_N x_N)$, $\delta_i = \pm 1$, $i = 1, 2, \ldots, N$, into three sets defined by

$$S^+ = \{ W \in S : v \cdot w > 0 , \text{ all } w \in W \},$$

(2.11) $$S^0 = \{ W \in S : v \cdot w = 0 , \text{ some } w \in W \},$$

$$S^- = \{ W \in S : v \cdot w < 0 , \text{ all } w \in W \}.$$  

There are $C(N,d)$ non-empty cones in $S^+$ and there are $C(N,d-1)$ non-empty cones in $S^0$ by Schlaflili's theorem applied to the projections of the vectors $X_i$ into the space orthogonal to $v$. Since $S^-$ is the...
set of reflected cones of $S^+$, the number of elements in $S^+$ and $S^-$ is equal, and thus the number of elements in $S^+$ is

\[(2.12) \quad \frac{1}{2} (C(N,d) - C(N,d-1)) = \binom{N-1}{d-1}.
\]

Finally, by the duality of $W$ and $W^*$, $v \in W^* (\delta_1 x_1, \delta_2 x_2, \ldots, \delta_N x_N)$ if and only if $W(\delta_1 x_1, \delta_2 x_2, \ldots, \delta_N x_N)$ is in $S^+$.

3. Applications to geometrical probability.

Let $X_1, X_2, \ldots, X_N$ be $N$ random points in $E^d$ having a joint distribution invariant under reflections through the origin - that is, for any $N$ sets $A_1, A_2, \ldots, A_N$ in $E^d$, the probability $P(\delta_1 x_1 \in A_1, \delta_2 x_2 \in A_2, \ldots, \delta_N x_N \in A_N)$ has the same value for all $2^N$ choices of $\delta_i = \pm 1$. (Actually, as will be clear, it is sufficient for the symmetry condition to hold for all cones $A_1, A_2, \ldots, A_N$ in $E^d$.)

Furthermore, let us suppose that with probability one the set of points is in general position. (This is satisfied in the important case where the $X_i$ are selected independently according to a distribution absolutely continuous with respect to the natural Lebesgue measure.)

Wendel utilizes Schl"afli's theorem in the following manner to establish result (c) of the introduction. Given that $X_1 = \delta_1 x_1, X_2 = \delta_2 x_2, \ldots, X_N = \delta_N x_N$ for some fixed set of points $x_1, x_2, \ldots, x_N',$ the symmetry condition implies that all $2^N$ choices of $\delta_i = \pm 1$ are equally likely; and by Schl"afli's theorem, for exactly $C(N,d)$ of these choices the vectors $\delta_1 x_1, \delta_2 x_2, \ldots, \delta_N x_N$ will generate
a proper convex cone. The probability that $X_1, X_2, \ldots, X_N$ all lie in some half-space of $E^d$, or that the unit vectors along the $X_i$ all lie in some one hemisphere of the unit $d$-sphere, is therefore $\binom{N}{d}/2^N$.

This same argument yields probabilistic statements of Theorems 1 and 3:

**Theorem 1'.** Let $W$ be the random polyhedral convex cone resulting from the intersection of $N$ random half-spaces in $E^d$ with positive normal vectors $X_1, X_2, \ldots, X_N$ having a joint distribution as described above. Then the expected number of $k$-faces $R_k(W)$ of $W$, conditioned on $W = \emptyset$, is given by

$$
(3.1) \quad E(R_k(W)) = \binom{N}{d-k} \binom{N}{d-k} \frac{C(N-d+k)}{C(N,d)}
$$

and

$$
(3.2) \quad \lim_{N \to \infty} E(R_k(W)) = 2^{d-k} \binom{d-1}{d-k}.
$$

**Theorem 3'.** Let $W^*$ be the random polyhedral convex cone spanned by the collection of random vectors $X_1, X_2, \ldots, X_N$. Then the expected number of $k$-faces of $W^*$, conditioned on $W^*$ being a proper cone, is given by

$$
(3.3) \quad E(R_k(W^*)) = \binom{N}{k} \frac{C(N-k,d-k)}{C(N,d)},
$$

and

$$
(3.4) \quad \lim_{N \to \infty} E(R_k(W^*)) = 2^{d-k} \binom{d-1}{k}.
$$

(Note: by Wendel's result, $P(W \neq \emptyset) = P(W^* \text{ proper}) = \frac{C(N,d)}{2^N}$.)

11
Let \( \mu \) be any probability measure absolutely continuous with respect to natural Lebesgue measure.

**Theorem 2'.** The expected \( \mu \)-measure of a non-empty random \( W \) described in Theorem 1' is \( 1/C(N,d) \). The expected \( \mu \)-measure of a proper random dual cone \( W^* \) described in Theorem 3' is \( \binom{N-1}{d-1}/C(N,d) \).

**Proof.** Given that \( X_1 = \delta_1 x_1, \ldots, X_N = \delta_N x_N \), the \( C(N,d) \) non-empty cones \( W_j \) generated (as in Theorem 1') by different choices of the \( \delta_i = \pm 1 \) partition \( E^d \) disjointly, ignoring their boundaries, which have \( \mu \)-measure \( 0 \). Therefore, \( \sum_{j=1}^{C(N,d)} \mu(W_j) = 1 \), and \( EW = \frac{1}{C(N,d)} \) by the symmetry condition. From Theorem 5, the \( C(N,d) \) proper dual cones \( W_j^* \) cover almost every point in \( E^d \) exactly \( \binom{N-1}{d-1} \) times. Therefore, \( \sum_{j=1}^{C(N,d)} \mu(W_j^*) = \binom{N-1}{d-1} \), and the second half of the theorem follows by symmetry.

4. **Remarks.**

The total number of non-empty cones \( W \), proper dual cones \( W^* \), and \( k \)-faces of these cones have been determined and shown to be independent, up to general position, of the configuration of \( x_1, x_2, \ldots, x_N \). The extreme vectors of \( x_1, x_2, \ldots, x_N \) are the \( 1 \)-faces of \( W^* \), the expected number of which is given in Eq. (3.3) and Eq. (3.4).

Thus

\[
(4.1) \quad ER_{1}(W^*) = \frac{2NC(N-1,d-1)}{C(N,d)}
\]

and

\[
(4.2) \quad \lim_{N \to \infty} ER_{1}(W^*) = 2(d-1).
\]
As a special case, suppose $N$ points are chosen at random on the surface of the unit sphere in $E^3$. Then, given that they all lie in the same one hemisphere, the expected number of extreme points of their convex hull (taken with great circles on the surface of the sphere) does not grow without bound as $N$ increases, but rather approaches the limit 4. This is perhaps surprising, particularly since the number of vertices can in no case be less than 3! For a comparison with the case of random points in the plane, where the expected number of extreme points goes to infinity, see [6]. On the other hand, the great circles having the $N$ chosen points as poles partition the surface of the sphere into regions having an expected number of sides 4 as $N$ goes to infinity. This agrees with the known result for regions formed by random lines in the plane [7, p. 57].

Closer inspection of (4.1) reveals that the expected number of extreme vectors of a random proper cone generated by $N$ random vectors in $E^d$ monotonically increases to $2(d-1)$ as $N$ increases to infinity. We also remark that (3.2) expressing the asymptotic expected number of k-faces of $W$ corresponds to the number of (k-1)-faces of a (d-1)-cube [7].

The $\binom{N-1}{d-1}$-fold overlapping of the cones spanned by the $\delta_iX_i$'s reduces, in the case $N = d$, to a corroboration of the result in [4] that the cones generated by $d$ points in $E^d$ partition $E^d$. Samelson et al extend their proof to cones generated by sets of pairs of vectors in $E^d$. 

13
(4.3) \[ \left\{ \begin{array}{c} \alpha_1 \\ \beta_1 \\ \vdots \\ \alpha_d \\ \beta_d \end{array} \right\}, \quad \alpha_1, \beta_1 \in \mathbb{R}^d, \quad i=1,2,\ldots,d \]

where each \((\alpha_i, \beta_i)\) pair is separated by each of the \(2^{d-1}\) hyperplanes generated by a set of representatives of the remaining pairs. Clearly each of the proofs in the present paper can be generalized in the same manner, the theorems holding true if \((X_{i'}, X_i)\) pairs are replaced by \((\alpha_i, \beta_i)\) pairs subject to the above conditions.
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