DISTRIBUTION OF A SUM OF WAITING TIMES
IN COUPON COLLECTION

BY
GERALD CHASE and HERMAN RUBIN

TECHNICAL REPORT NO. 109
AUGUST 12, 1965

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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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The distribution of balls into urns, or equivalent problems, has been treated by many authors. The many variations of the problem, known as occupancy problems, may be characterized by two assumptions:

(1) the distinguishability of the balls, and

(2) the manner in which the distribution of the balls is to be carried out. The classical variations of the problem may be found in Fanzen [7] or Feller [3]. One well known variation is often referred to as the coupon collector's problem, which is illustrated by the following non-urn example.

Suppose a soap manufacturer randomly encloses a coupon bearing one of the integers 1 to M in each package of soap. If n packages of soap are purchased, the probability that exactly m of the M integers will not be obtained is given by

\[
\binom{M}{m} \frac{\Delta^{M-m}(O^n)}{M^m}
\]

where

\[
\Delta^r(O^n) = \sum_{K=0}^{r} (-1)^K \binom{r}{K}(r-K)^n \quad r = 0,1,\ldots,n
\]

\[
n = 1,2,\ldots
\]
The function $\Delta^r(0^n)$ is tabulated in [4], Table XXII, for $n = 2(1)25$ and $r = 2(1)n$.

The general problem, which allows the probabilities of the coupons to be unequal was treated by H. von Schelling in 1936 [8].

One important aspect of the coupon collectors problem is the waiting time (number of trials) until $r$ of the $M$ numbers have been collected. Let $S^r_M$ denote the waiting time (a random variable) until this event occurs.

We may characterize the distribution as follows: let $X^r_{MK}$ have a geometric distribution with $P^r_{MK} = \frac{M-K}{M}$, $K = 0, 1, \ldots, M-1$.

\begin{equation}
Pr[X^r_{MK} = \ell] = (1-P^r_{MK})^{\ell-1}P^r_{MK}, \quad \ell = 1, 2, \ldots \\
K = 0, 1, \ldots, M-1
\end{equation}

with the $X^r_{MK}$'s independent.

Define

\begin{equation}
Y^r_M = \sum_{K=0}^{r-1} X^r_{MK},
\end{equation}

Then $S^r_M$ is distributed the same as $Y^r_M$.

The distribution of $Y^10_M$ has been tabulated by R. E. Greenwood [5]. He proposes its use as a test for random digits by using a goodness of fit test. That is, the number of random digits is counted until all ten digits have been observed. This number is recorded and the process is repeated, beginning with the next random digit, until the table of digits has been exhausted. The resulting sample is tested for
goodness of fit with the tabled distribution. A similar test was used by Kendall and Smith [6] to test a table of numbers for randomness.

This note is concerned with the approximation of the c.d.f. of sums of independent random variables distributed as $Y^M_M$. (We collect a complete set of coupons, then another set, and so on until we have a specified number $(N)$ of complete sets. We then look at the total number of coupons we have.)

The moment generating function for $X_{MK}$ is given by

$$G_{X_{MK}}(t) = \frac{P_{MK}e^t}{1 - (1-P_{MK})e^t}$$

Using independence and (4) we then get

$$G_{Y^M_M} = \prod_{K=0}^{M-1} \frac{P_{MK}e^t}{1 - (1-P_{MK})e^t}$$

$$= \frac{\Gamma(M+1) \Gamma(M[e^{-t}-1] + 1)}{\Gamma(Me^{-t} + 1)}$$

Let $Y_{MI}, i = 1, \ldots, N$, be i.i.d. as $Y^M_M$. We are interested in the distribution of

$$S_{M,N} = \sum_{i=1}^{N} Y_{MI} .$$

The investigation was done for $M = 30$, and the number of convolutions $(N)$ equal to 8, 16, and 32.
Define $x_{M,N,\alpha}$ such that

$$P_r[S_{M,N} \leq x_{M,N,\alpha}] \leq \alpha < P_r[S_{M,N} \leq x_{M,N,\alpha} + 1].$$ \hspace{1cm} (8)

The following points were used in the estimation procedure:

$$x_{30,N,\alpha} = 0.999, 0.990, 0.900, 0.100, 0.010, 0.001$$

$$N = 8, 16, 32$$ \hspace{1cm} (9)

The exact distribution of $Y_{30}$ was generated and convolved on the Stanford University IBM 7090 digital computer. A check was made on the generation routine by using $M = 10$ and the table of Greenwood's [5]. The convolution routine was checked with the Poisson distribution.

An Edgeworth series (Cramer [1], page 228) was used as follows:

$$\phi(Z) = \int_{-\infty}^{Z} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$ \hspace{1cm} (10)

and suppose we wish to estimate

$$P_r[S_{M,N} \leq x] = P_r\left[\frac{S_{M,N} - E[S_{M,N}]}{(\text{Var}[S_{M,N}])^{\frac{1}{2}}} \leq Z'_{M,N}(x)\right]$$ \hspace{1cm} (11)

where

$$Z'_{M,N}(x) = \frac{x - E[S_{M,N}]}{(\text{Var}[S_{M,N}])^{\frac{1}{2}}}$$ \hspace{1cm} (12)
Using a continuity correction, let

\begin{equation}
\frac{Z^{(x)}}{\chi_{M,N}} = Z_{M,N}^{(x + \frac{1}{2})}
\end{equation}

Define \( \lambda_{M,U} \) to be the \( U \)th cumulant of \( \frac{Y_{M}^{M} - \mathbb{E}[Y_{M}^{M}]}{(\text{Var}[Y_{M}^{M}])^{\frac{1}{2}}} \).

The Edgeworth series approximation to the order of \((N^{-\frac{1}{2}})^{3}\) is given by:

\begin{equation}
\hat{F}_{M,N}(x) = \phi(Z_{M,N}(x)) - \frac{\lambda_{M,3}}{3!} \frac{1}{N^{2}} \phi(3)(Z_{M,N}(x))
\end{equation}

\begin{equation}
\quad + \frac{\lambda_{M,4}}{4!} \frac{1}{N} \phi(4)(Z_{M,N}(x)) + \frac{10}{6!} \frac{(\lambda_{M,3})^{2}}{N} \phi(6)(Z_{M,N}(x))
\end{equation}

\begin{equation}
\quad - \frac{\lambda_{M,5}}{5!} \frac{1}{N^{3/2}} \phi(5)(Z_{M,N}(x)) - \frac{35(\lambda_{M,4})(\lambda_{M,3})}{7!} \frac{1}{N^{3/2}} \phi(7)(Z_{M,N}(x)) +
\end{equation}

\begin{equation}
\quad - \frac{280}{9!} \frac{(\lambda_{M,3})^{3}}{N^{3/2}} \phi(9)(Z_{M,N}(x))
\end{equation}

where

\begin{equation}
\phi^{(K)}(Z) = \frac{d^{K}}{dZ^{K}} \phi(Z)
\end{equation}

The cumulants \( \lambda_{M,U} \) may be obtained from \( G_{M}^{M} \) in (6). Letting

\begin{equation}
\chi_{M,U} = \frac{d}{dt} \log[G_{M}^{M}(t)] \bigg|_{t=0}
\end{equation}
we then have

\begin{equation}
\chi_{M,U} = \frac{\chi_{M,U}}{\chi_{M,2}^{U/2}} \quad U = 2, 3, \ldots.
\end{equation}

If \( \psi(X) \) denotes the Psi function, i.e.,

\begin{equation}
\psi(X) = \frac{d}{dX} \log \Gamma(X),
\end{equation}

the cumulants may be expressed as follows:

\begin{align*}
\chi_{M,2} &= M[\psi(1) - \psi(1+M)] + M^2[\psi'(1) - \psi'(1+M)] \\
\chi_{M,3} &= -M[\psi(1) - \psi(1+M)] - 3M^2[\psi'(1) - \psi'(1+M)] \\
&\quad - M^3[\psi''(1) - \psi''(1+M)] \\
\chi_{M,4} &= M[\psi(1) - \psi(1+M)] + 7M^2[\psi'(1) - \psi'(1+M)] \\
&\quad + 6M^3[\psi''(1) - \psi''(1+M)] + M^4[\psi'''(1) - \psi'''(1+M)] \\
\chi_{M,5} &= -M[\psi(1) - \psi(1+M)] - 15M^2[\psi'(1) - \psi'(1+M)] \\
&\quad - 25M^3[\psi''(1) - \psi''(1+M)] - 10M^4[\psi'''(1) - \psi'''(1+M)] \\
&\quad - M^5[\psi''''(1) - \psi''''(1+M)].
\end{align*}
The mean of the waiting time is given by

\begin{align*}
(20) \quad E[S_{M,N}] &= N \chi_{M,1} \\
&= -NM[\psi(1) - \psi(1+M)] .
\end{align*}

The \( \psi \) function and its derivatives (the polygamma functions) are
tabled up to \( M = 99 \) in [2]. For higher values of \( M \) the following approximations derived from Stirling's formula may be used:

\begin{align*}
\psi(1+M) &\approx \log M + \frac{1}{2M} - \frac{1}{12M^2} + \frac{1}{120M^4} \\
\psi^{(1)}(1+M) &\approx \frac{1}{M} - \frac{1}{2M^2} + \frac{1}{6M^3} - \frac{1}{30M^5} \\
\psi^{(2)}(1+M) &\approx -\frac{1}{M^2} + \frac{1}{3M^3} - \frac{1}{2M^4} + \frac{1}{6M^6} \\
\psi^{(3)}(1+M) &\approx \frac{2}{M^3} - \frac{3}{M^4} + \frac{2}{5M^5} - \frac{1}{7M^7} \\
\psi^{(4)}(1+M) &\approx \frac{6}{M^4} + \frac{12}{M^5} - \frac{10}{M^6} + \frac{7}{8M^8} \\
\end{align*}

Due to rounding errors, the tabled distributions were in error by
the following amounts:
\[ F_{30,1}(\alpha) = 0.999997 \]
\[ F_{30,8}(\alpha) = 0.999986 \]
\[ F_{30,16}(\alpha) = 0.999975 \]
\[ F_{30,16}(\alpha) = 0.999952 \]

The errors were taken into account in the upper tails, but ignored in the lower tails.

The unit normal c.d.f. and its derivatives were taken from [9]. The tables were in steps of 0.001 in the range of interest, and linear interpolation was used.

The results are given on the following tables for four approximations. The Edgeworth series

(1) to the order of \( n^0 \) (normal approximation),
(2) to the order of \( n^{-1/2} \),
(3) to the order of \( n^{-1} \), and
(4) to the order of \( n^{-3/2} \).
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<th>$F_{30,8}(x_{30,8},\alpha+1)$</th>
<th>$\hat{F}<em>{30,8}(x</em>{30,8},\alpha)$</th>
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Summary: The Edgeworth series to powers of \((N^{-1/2})^3\) appears to give a good approximation for \(P_{S_{M,N}} \leq x\), when \(x\) is an integer.

To get this approximation, compute

\[
Z_{M,N}(x) = \frac{x - \text{NE}[Y_M^M] + 1/2}{N^{1/2}(X_{M,2})^{1/2}}, \quad \text{where}
\]

\(X_{M,2}\) is given in (19) and \(\text{NE}[Y_M^M]\) is given in (20). Compute \(\hat{P}_{S_{M,N}}(x)\) as in (13).
REFERENCES


**Distribution of a Sum of Waiting Times in Coupon Collection**

**Report Date**: August 12, 1965

**Contract or Grant No.**: Nbr 255(52)

**Project No.**: NR 342-022

**Other Report No.(s)**: Technical Report No. 109

**Availability/Limitation Notices**: Releaseable without limitations on dissemination

**Abstract**

The problem of approximating the distribution of the sum of $n$ times to collect all of $M$ coupons is considered. The Edgeworth expansion is obtained to terms of order $n^{-3/2}$. Tables are given for $M = 30$, $n = 8$, $15$, and $32$, of the accuracy of the Edgeworth expansion for the cumulative distribution function approximately (because of discreteness) .001, .01, .1, .9, .99, and .999.
Waiting time distribution
Coupon collector's problem

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