ON THE MAXIMUM DEVIATION OF THE SAMPLE DENSITY

BY

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1. Introduction.

Consider a sequence $X_1, X_2, \ldots$ of independent, identically distributed (i.i.d.) random variables having a common density $f(\cdot)$. In some contexts one might wish to estimate $f(\cdot)$ from a finite segment, $X_1, \ldots, X_n$ say, of the sequence. This is commonly done with estimates of the form

$$f_n(x) = \tau \int_{-\infty}^{\infty} K(x-\omega) dF_n(\omega)$$

(1.1)

$$= \frac{\tau}{n} \sum_{i=1}^{n} K(x-X_i)$$

where $F_n(\cdot)$ is the sample distribution function of $X_1, \ldots, X_n$, $K(\cdot)$ is a kernel, and $\tau$ is a function of $n$ for which $\tau \to \infty$ and $\tau = o(n)$ as $n \to \infty$. By "kernel" we mean a positive bounded, measurable function $K(\cdot)$ for which

$$\int_{-\infty}^{\infty} |\omega| K(\omega) d\omega < \infty ,$$

(1.2)

$$\int_{-\infty}^{\infty} K(\omega) d\omega = 1 ,$$

and throughout this paper $K(\cdot)$ will denote a kernel.

Parzen (1962) has considered estimates of the form (1.1) in some detail. One of his results states that if $f(\cdot)$ is uniformly continuous, the Fourier Transform of $K(\cdot)$ is absolutely integrable, and $\tau^2 = o(n)$ as $n \to \infty$, then

$$\lim_{n \to \infty} \mathbb{E} \left[ \sup_{x} |f_n(x) - f(x)|^2 \right] = 0 .$$

(1.3)
More recently Nadaraya (1965) has shown with the same assumptions on $f(\cdot)$ that if $K(\cdot)$ is of bounded variation and $\sum_{n=1}^{\infty} \exp(-\gamma n \tau^2)$ is finite for every positive $\gamma$, then the $\sup$ in (1.3) tends to zero w. p. one. Nadaraya's result, incidentally, may be obtained as an easy consequence of a theorem of Chung (1949).

The present paper also considers the maximum deviation of $f_n(\cdot)$ from $f(\cdot)$. The main result states that if $f(\cdot)$ and $K(\cdot)$ both satisfy uniform Lipschitz conditions, $f(\cdot)$ is positive on $[a,b]$, and $\tau^\alpha = o(n)$ and $n = o(\tau^\beta)$ as $n \to \infty$ with $1 < \alpha < \beta \leq 3$, then as $n \to \infty$

$$\max_{a \leq x \leq b} \left( \frac{f_n(x) - f(x)}{\sqrt{f(x)}} \right)$$

is relatively stable (converges in probability to one when properly normalized). The other states that under the same conditions (1.4) tends to zero w. p. one as $n \to \infty$ and that if $\tau^2 (\log \tau)^\alpha = o(n)$ as $n \to \infty$ with $\alpha > 1$, then (1.4) is, in fact, $o(\tau^{-1/2})$ w. p. one as $n \to \infty$. The key to establishing these results is a theorem on the large deviations of sums of i.i.d. bivariate random variables which extends a theorem of Cramer (1938). In its simplest form it says that if $S_n = (S_{n1}, S_{n2})$ is the $n$-th partial sum, of i.i.d., standardized, bounded, bivariate random variables and if $\lambda_n^6 = o(n)$ as $n \to \infty$, then as $n \to \infty$

$$P[S_{ni} \geq \lambda_n, i=1,2] \sim (1-2\Phi(\lambda_n) + \Phi_r(\lambda_n, \lambda_n))$$

where $\Phi(\cdot)$ and $\Phi_r(\cdot, \cdot)$ are standardized normal univariate and bivariate d.f.'s respectively. For the applications intended, however, we need (1.5) to hold uniformly in a sense made precise in section 3. This fact complicates its proof.
The organization of the paper is briefly as follows: notation and some preliminaries are given in section 2; section 3 contains the theorem on large deviations; section 4 contains two lemmas which allow us to pick out a sequence of finite sets of points on which the max in (1.4) will essentially be obtained with probability approaching one; and the main results are given in section 5.

2. Preliminaries.

Throughout the remainder of the paper \( f(\cdot) \) will be assumed to be bounded on \(( -\infty, \infty )\) and positive and continuous on some neighborhood of \([a,b]\), which without loss of generality we take as \([-1,1]\). If we introduce the functions

\[
f(x; \tau) = \tau \int_{-\infty}^{\infty} K(\tau(x-\omega))f(\omega) \, d\omega,
\]

(2.1)

\[
\sigma^2(x; \tau) = \tau \int_{-\infty}^{\infty} K^2(\tau(x-\omega))f(\omega) \, d\omega - \frac{1}{\tau} f(x; \tau)^2,
\]

then the representation of \( f_n(x) \) as a sum in (1.1) shows that

\[
E[f_n(x)] = f(x; \tau),
\]

(2.2)

\[
\text{Var}(f_n(x)) = \frac{1}{n} \sigma^2(x; \tau).
\]

Lemma 2.1 (below) is some standard Fourier analysis which will be useful in what follows. In it we have denoted the \( L_p \) norm of \( K(\cdot) \) by \( \| K \|_p \).

**Lemma 2.1.** Let \( f(\cdot) \) be bounded on \(( -\infty, \infty )\) and continuous on \((a',b')\), and let \( K(\cdot) \) be a kernel. Then for \( 1 \leq p < \infty \)
\begin{equation}
(2.3) \quad \lim_{\tau \to \infty} \int_{-\infty}^{\infty} K^p(\tau(x-\omega))f(\omega)d\omega = f(x) \|K\|^p_p,
\end{equation}

uniformly on every closed subinterval of \((a', b')\). If, in addition, \(f(\cdot)\) satisfies a uniform Lipschitz condition of order \(\ell, 0 \leq \ell \leq 1\), on \((a', b')\) then as \(\tau \to \infty\) \(|f(x; \tau) - f(x)| = O(\tau^{-\ell})\) uniformly on every closed subinterval of \((a', b')\).

\textbf{Proof}. Let \(f_p(x; \tau)\) denote the integral whose limit is taken in (2.3), and let \([a, b] \subset (a', b')\). Then clearly for all \(x \in [a, b]\)

\[ |f_p(x; \tau) - f(x)| \|K\|^p_p \leq \int_{|\omega| \leq \delta \tau} |f(x) - f(x - \omega \tau^{-1})|K^p(\omega)d\omega + 2B_p|\omega| \int_{|\omega| \geq \delta \tau} K^p(\omega)d\omega \]

where \(B_p\) is an upper bound for \(f(\cdot)\). The first assertion follows by taking \(\delta\) so small that \(|f(x) - f(x + y)|\) is uniformly less than some preassigned \(\varepsilon\) for \(x \in [a, b]\) and \(|y| \leq \delta\); the second from taking any \(\delta\) for which \(|f(x) - f(x + y)| \leq B|y|\) uniformly in \(x \in [a, b]\) and \(|y| \leq \delta\).

In view of Lemma 2.1 and the assumption on \(f(\cdot)\) there exist an integer \(n_1\) and positive real numbers \(\eta, \eta'\) for which

\begin{equation}
(2.4) \quad \sigma(x, \tau) \geq \eta
\end{equation}

for all \(x \in [-1 - \eta', 1 + \eta']\) and all \(n \geq n_1\). To avoid needless repetition we shall always assume \(n \geq n_1\) in the following.

At this point it is convenient to introduce the stochastic processes \(Y_n(y), |y| \leq 1 + \eta', n \geq n_1\) defined by

\[...

\]
\[ Y_n(y) = \sqrt{n} \frac{f_n(y) - f(y; \tau)}{\sigma(y; \tau)} \]

(2.5)

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{ni}(y) \]

where, of course,

(2.6) \[ Z_{ni}(y) = \sqrt{\tau} \frac{K(\tau(y-x_i)) - \tau^{-1}f(y; \tau)}{\sigma(y; \tau)}. \]

The covariance function of the \( Y_n(y) \) process is

\[ r_n(x, y) = \frac{\tau \int_{-\infty}^{\infty} K(\tau(x-w))K(\tau(y-w))f(w)dw - \frac{1}{\tau} f(x, \tau)f(y; \tau)}{\sigma(x; \tau) \sigma(y; \tau)}, \]

and we note in passing that if \( y-x \geq 2\tau^{-1} \log \tau \), then it follows from (1.2) that as \( \tau \to \infty \)

\[
\left| \tau \int_{-\infty}^{\infty} K(\tau(x-w))K(\tau(y-w))f(w)dw \right|
\]

\[ \leq B_f B_k \left\{ \int_{-\infty}^{-\log \tau} K(\mu) d\mu + \int_{\log \tau}^{\infty} K(\mu) d\mu \right\} = o(\log \tau^{-1}) \]

where \( B_f \) and \( B_k \) are upper bounds for \( f(\cdot) \) and \( K(\cdot) \) respectively.

Hence from Lemma 2.1 we have

(2.7) \[ r_n(x, y) = o(\log \tau^{-1}), \text{ as } n \to \infty, \]

uniformly in \( |x-y| \geq 2\tau^{-1} \log \tau \), \( |x| \leq 1 \) and \( |y| \leq 1 \).
3. **Large Deviations.**

In this section we extend a theorem of Cramer (1938) on the large deviations of sums of i.i.d. random variables to the bivariate case.

We need the following notation and terminology. Points in Euclidian two-space \( \mathbb{R}_2 \) will be column vectors, and \( \| . \| \) will denote their Hilbert Space norm — e.g., \( z = (z_1, z_2)' \in \mathbb{R}_2 \) where ' denotes transpose, and \( \| z \|^2 = z'z \). A bivariate distribution function (b.d.f.) will be called standardized (st.) if the means and variances of the associated random variables are 0 and 1 respectively. If \( F(\cdot) \) is a st.b.d.f. we let

\[
r(F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_1 z_2 \, dF(z),
\]

\[
\gamma(F) = \inf \{ \gamma \leq \infty : \int_{-\gamma}^{\gamma} \int_{-\gamma}^{\gamma} dF(z) = 1 \}.
\]

Finally we let \( \Phi(\cdot) \) and \( \Phi_\rho(\cdot, \cdot) \) denote the st. normal d.f. and st. normal b.d.f. with parameter \( \rho \) respectively.

**Theorem 3.1.** Let \( \mathcal{F} \) be a family of st.b.d.f.'s for which

\[
\sup \{ |r(F)| : F(\cdot) \in \mathcal{F} \} \leq 1-2\delta \text{ with } 0 < \delta < \frac{1}{2}; \text{ let } \{ \alpha_n \}_{n=1}^{\infty} \text{ be a fixed sequence of real numbers for which } \alpha_n \geq 1, \ n = 1, 2, \ldots, \text{ and } \alpha_n^2 = o(n) \text{ as } n \to \infty; \text{ and let } \mathcal{F}_n \subseteq \mathcal{F} \text{ be the sub-family of } \mathcal{F} \text{ consisting of those } F(\cdot) \in \mathcal{F} \text{ for which } \gamma(F) \leq \alpha_n. \text{ If } \{ \lambda_n \}_{n=1}^{\infty} \text{ is another sequence of real numbers for which } \lambda_n \to \infty \text{ and } \lambda_n^2 = o(n) \text{ as } n \to \infty, \text{ then as } n \to \infty
\]

\[
[1-F(n)(\lambda_n \sqrt{n}, \infty)] - F(n)(\infty, \lambda_n \sqrt{n}) + F(n)(\lambda_n \sqrt{n}, \lambda_n \sqrt{n})]
\]

(3.1)

\[
\sim (1-2\Phi(\lambda_n) + \Phi(r(F)(\lambda_n, \lambda_n))
\]

uniformly for \( F(\cdot) \in \mathcal{F}_n \), where \( (n) \) denotes n-fold convolution.
Proof. Let \( \mathcal{A} \) and \( \{ \alpha_n \}_{n=1}^{\infty} \) be as in the statement of the theorem. Then for each \( F(\cdot) \in \mathcal{A}_n \) the moment generating function

\[
m(t) = m(t; F) = \int_{-\infty}^{\infty} \exp(t z) dF(z)
\]

exists (finitely) for all \( t \) such that \( 0 \leq t_1, t_2 \), and we may define a new b.d.f. \( G(\cdot) \) by

\[
G(z) = G(z; t, F) = \frac{1}{m(t)} \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} \exp(t \omega) dF(\omega).
\]

(Throughout this section we will denote dependence on \( t \) and \( F \) only when a symbol is first introduced and will suppress it thereafter.)

We then have the relation

\[
(3.2) \quad F(z) = m(t) \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} \exp(-t \omega) dG(\omega).
\]

The means, variances, and covariances of \( G(\cdot) \) are

\[
\theta_i = \theta_i(t, F) = \int_{-\infty}^{\infty} z_i dG(z) = \left. \frac{\partial}{\partial s_i} \log m(s) \right|_{s=t},
\]

\[
(3.3) \quad \sigma_{ij} = \sigma_{ij}(t, F) = \int_{-\infty}^{\infty} z_i z_j dG(z) - \theta_i \theta_j = \left. \frac{\partial^2}{\partial s_i \partial s_j} \log m(s) \right|_{s=t},
\]

where \( i, j = 1, 2. \) If now we let \( G_n(z) = G^{(n)}(\sqrt{n}_1 z_1 \sqrt{n}_1 + n \theta_1, \sqrt{n}_2 z_2 + n \theta_2) \) and \( F_n(z) = F^{(n)}(\sqrt{n}_1 z_1, \sqrt{n}_2 z_2) \), then we have the following identity:

\[
(3.4) \quad F_n(z) = \exp(-n \theta' t m(t)) \int_{-\infty}^{\sqrt{\sigma}_{11}} \int_{-\infty}^{\sqrt{\sigma}_{22}} \exp(-\omega' t \sqrt{n}) dG_n(\omega)
\]

7
where \( \theta = (\theta_1, \theta_2)' \) and \( t^* = (t_1 \sqrt{\sigma_{11}}, t_2 \sqrt{\sigma_{22}})' \). In particular (3.4) implies

\[
1 - F_n(\theta_1 \sqrt{n}, \omega) - F_n(\omega, \theta_2 \sqrt{n}) + F_n(\theta_1 \sqrt{n}, \theta_2 \sqrt{n})
\]

\( (3.4') \)

\[
= \exp(-\theta'^tn)n \int_0^\infty \int_0^\infty \exp(-z't^* \sqrt{n})dG_n(z) ,
\]

which will be used as follows. First we will solve the equation

\( (3.5) \)

\[ \theta \sqrt{n} = (\lambda_n, \lambda_n)' , \]

obtaining a point \( t = t(n, F) \) for which the left-hand sides of (3.1) and (3.4') are equal. Then an application of the Central Limit Theorem to \( G_n(\cdot) \) and some manipulations will show that the ratio of their right-hand sides tends to one uniformly for \( F \in \mathcal{F}_n \) as \( n \to \infty \).

Call the right hand side of (3.4) \( H(z) \); to establish (3.4) it suffices to show that \( F_n(\cdot) \) and \( H(\cdot) \) have the same characteristic functions.

A change of variables and the definition of \( G_n(\cdot) \) show that

\[
H(z) = m(t)n \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} \exp(-ut' \sqrt{n})dG(n)(\mu \sqrt{n}) .
\]

Therefore the characteristic function of \( H(\cdot) \) at an arbitrary \( s \) is

\[
= m(t)n \int_{-\infty}^{\infty} \exp([is-t \sqrt{n}]'z)dG(n)(z \sqrt{n})
\]

\[
= \left\{ m(t) \int_{-\infty}^{\infty} \exp(is'z)exp(-t'z \sqrt{n})dG(z \sqrt{n}) \right\}^n
\]

\[
= \left\{ \int_{-\infty}^{\infty} \exp(is'z)dF(z \sqrt{n}) \right\}^n
\]

which is also the characteristic function of \( F_n(\cdot) \) at \( s \). This establishes the identity.
Now we have to establish some uniform approximations. To this end we introduce
\[ \Delta(t; F) = \max_{i+j=3} \left. \frac{\partial^3}{\partial s_i \partial s_j \partial s_2} \log m(s; F) \right|_{s=t} \]
and prove

**Lemma 3.1.** There is a constant \( \Delta_0 \) (independent of \( n \)) for which
\[ \Delta(t; F) \leq \Delta_0 \alpha_n \]
uniformly in \( 0 \leq t_1, t_2 \leq \alpha_n^{-1} \), \( F(\cdot) \in \mathcal{F}_n \) and \( n = 1, 2, \ldots \).

**Proof.** Repeated use of the chain rule and Hölder's inequality yields
\[ \Delta(t; F) \leq 5 \int_{-\infty}^{\infty} (|z_1|^3 + |z_2|^3) \exp(t'z) dF(z) \]
for each \( F(\cdot) \in \mathcal{F}_n \). If now \( 0 \leq t_1, t_2 \leq \alpha_n^{-1} \) and \( F(\cdot) \in \mathcal{F}_n \), then
\[ |z_i|^3 \leq \alpha_n |z_i|^2, \quad i = 1, 2 \]
and \( \exp(t'z) \leq \exp(2) \) over the effective range of integration. Therefore the integral on the right does not exceed
\[ \alpha_n e^2 \int_{-\infty}^{\infty} (|z_1|^2 + |z_2|^2) dF(z) \leq 2e^2 \alpha_n \]
and we may take \( \Delta_0 = 10e^2 \).

**Corollary 3.1.** Let \( \rho = \rho(t, F) = \frac{\sigma_{12}}{\sqrt{\sigma_{11} \sigma_{22}}} \) and let \( \mathcal{F}_0 \) be a correlation matrix with off diagonal element \( \rho \). There exist constants \( \Delta_i, \quad i = 1, \ldots, 4 \) for which
i) \[ | \rho - r(F) | \leq \Delta_1 \alpha_n \| t \| \]

ii) \[ \| \theta - \Psi_0 t^* \| \leq \Delta_2 \alpha_n \| t \|^2 \]

iii) \[ \| \psi^{-1}_0 \theta - t \| \leq \Delta_3 \alpha_n \| t \|^2 \]

iv) \[ | \log m(t) - \theta' t + \frac{1}{2} \theta' \psi^{-1}_0 \theta | \leq \Delta_4 \| t \|^3 \alpha_n \]

uniformly in \( 0 \leq t_1, t_2 \leq 8/20 \Delta_n \alpha_n F(\cdot) \epsilon \mathcal{F}_n \) and \( n \).

**Proof.** A routine consequence of Taylor's Theorem, (3.3), and Lemma 3.1.

Before proceeding to the completion of the proof of Theorem 3.1, we state a mapping theorem which may not be generally known. It has also been used by Matthes (1963) in connection with the Sequential Probability Ratio Test.

**Lemma 3.2.** (Rado and Reichelderfer). Let \( T \subseteq \mathbb{R}^2 \) be a topologically closed, simply connected Jordan region with boundary \( \partial T \). If \( f(\cdot) \) is a complex valued function which is continuous on \( T \) and does not vanish on \( T \), then \( \text{Var}_{\partial T}(\text{arg} f(\cdot)) = 0 \).


Lemma 3.2 will be used to establish the existence of an integer \( n_\alpha \) for which (3.5) may be solved for every \( P(\cdot) \epsilon \mathcal{F}_n \) provided only that \( n \geq n_\alpha \). Our \( n_\alpha \) is chosen so that

\[
\frac{\lambda_n \alpha_n}{\sqrt{n}} \leq \frac{(1-\delta)^3}{20 \Delta_0}
\]

for all \( n \geq n_\alpha \). If we then let \( h_n = \lambda_n / (1-\delta) \sqrt{n} \), it is clear that \( T_n \), the convex hull of the points \( (0,0)', (h_n,0)', (\delta^{-1} h_n, \delta^{-1} h_n)' \), and
(0, h_n)' will be contained in the square S_n consisting of those t \in \mathbb{R}_2 with 0 \leq t_1,t_2 \leq 8/20 \Delta \alpha_n. Thus Lemma 3.1 applies inside T_n and in conjunction with Taylor's Theorem and (3.5), it yields the following inequalities:

\[ t_1(1-s^2)-t_2(1-s^2) \leq \theta_1(t;F) \leq t_1(1+s^2)+t_2(1-s^2) \]
\[ -t_1(1-s^2)+t_2(1-s^2) \leq \theta_2(t;F) \leq t_1(1-s^2)+t_2(1+s^2) \]

(3.6)

for all \( t \in T_n \), \( F \in \mathcal{F}_n \) and \( n \geq n_0 \).

Now suppose that for some \( n \geq n_0 \) and some \( F_0(\cdot) \in \mathcal{F}_n \) (3.5) failed to have a solution inside \( T_n \). Then the function \( g(\cdot) \) defined by

\[ g(t) = (\theta_1 - \frac{\lambda}{\sqrt{n}}) + i(\theta_2 - \frac{\lambda}{\sqrt{n}}) \]

would satisfy the hypotheses of Lemma 3.2, and since (3.6) implies

\[ \text{Var}_{T_n}(\arg g(\cdot)) = 2\pi, \] we would have a contradiction. Thus there can exist no such \( n \) and \( F_0(\cdot) \) which is what we wished to demonstrate.

For the remainder of the proof \( n \) will be \( \geq n_0 \), \( F(\cdot) \) will denote an arbitrary element of \( \mathcal{F}_n \), \( t = t(n,F) \) will denote a solution of (3.5), and the functions \( \theta_i \), \( \sigma_{ij} \), and \( \rho \) are evaluated at \( (t(n,F),F) \). Since we have actually shown that the equations (3.5) may be solved inside \( T_n \subset S_n \), we have

\[ t_1 \leq \frac{\lambda_n}{(1-\delta)\sqrt{n}} \leq \frac{\delta}{20 \Delta \alpha_n} \]  \quad i=1,2

(3.7)

uniformly. This fact will be used below.
Returning now to (3.4') we may, according to a theorem of Bergström (1944), write

\[ G_n(z) = \Phi_\rho(z) + R_n(z) \]

where

\[ R_n(z) \leq \frac{C}{1-\rho^2} \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} \left\{ \frac{|z_1-\theta_1|^3}{\sigma_1^3} + \frac{|z_2-\theta_2|^3}{\sigma_2^3} \right\} dG(z) \leq \frac{C'\alpha_n}{\sqrt{n}} \]

where \( C \) and \( C' \) are independent of \( F(\cdot) \epsilon \mathcal{F}_n \) and \( n \geq n_0 \). Here we have used Corollary 3.1 and (3.7) and have bounded the integral as we did in the proof of Lemma 3.1. Corresponding to (3.8), of course, there is an analogous decomposition of (3.4') the components of which we now consider separately. Integration by parts shows easily that

\[ \left| \int_0^\infty \int_0^\infty \exp(-z't^*\sqrt{n})dR_n(z) \right| \leq \frac{2C'\alpha_n}{\sqrt{n}}. \]

The integral with respect to \( \Phi_\rho(\cdot,\cdot) \) presents more difficulty. We note first that

\[ \int_0^\infty \int_0^\infty \exp(-z'\mathcal{Z}_n^{-1}\theta^*\sqrt{n})d\Phi_\rho(z) \]

\[ = \exp\left(\frac{\lambda_n}{1+\rho}\right) \int_\lambda_n^\infty \int_\lambda_n^\infty d\Phi_\rho(z) \]

\[ \approx \frac{(1+\rho)^{3/2}}{2\pi\sqrt{1-\rho}} \cdot \frac{1}{\lambda_n^2} \]

12
uniformly in $F(\cdot) \in \mathcal{F}_n$ as $n \to \infty$. Here we have used (3.5) and changed variables in the first step; in the second we have used Lemma 2.2 of Berman (1962) which gives the asymptotic equivalence as uniform in $|\rho| \leq 1-\delta$. Since also by a change of variables, the convexity of $\exp(\cdot)$, and Corollary 3.1

$$
\sqrt{\delta} \cdot \lambda_n^2 \int_0^\infty \int_0^\infty \left| \exp(-z't^* / \lambda_n^2) - \exp(-z' \varphi_0^{-1} \theta \sqrt{n}) \right| d\varphi_0(z) 
\leq \int_0^\infty \int_0^\infty \left| \exp(-z't^* \sqrt{n}) - \exp(-z' \varphi_0^{-1} \theta \sqrt{n}) \right| dz 
\leq \int_0^\infty \int_0^\infty \left( \exp(\Delta \|z\|t^2 \sqrt{n}) - 1 \right) \exp(-\frac{1}{2} z_1^2) dz_1 dz_2,
$$

which is independent of $F(\cdot) \in \mathcal{F}_n$ and tends to zero as $n \to \infty$ by (3.7) and Lebesgue's Dominated Convergence Theorem, we have

$$(3.10) \int_0^\infty \int_0^\infty \exp(-z't^* \sqrt{n}) \varphi_0(z) \sim \int_0^\infty \int_0^\infty \exp(-z' \varphi_0^{-1} \theta \sqrt{n}) d\varphi_0(z)$$

uniformly for $F(\cdot) \in \mathcal{F}_n$ as $n \to \infty$.

Combining (3.4), (3.5), (3.9), and (3.10), we find

$$1 - F_n(\lambda_n, \omega_n) - F_n(\omega_n, \lambda_n) + F_n(\lambda_n, \lambda_n)$$

$$\sim \exp(-n\theta't)m(t)^n \int_0^\infty \int_0^\infty \exp(-z' \varphi_0^{-1} \theta \sqrt{n}) d\varphi_0(z) \left[ 1 + O\left(\frac{\lambda_n^2}{\sqrt{n}}\right) \right]$$

$$= \exp\{n(\log m(t) - \theta' t + \frac{1}{2} \theta' \varphi_0^{-1} \theta)(1 - 2\theta(\lambda_n') + \phi(\lambda_n', \lambda_n'))[1 + O\left(\frac{\lambda_n^2}{\sqrt{n}}\right)]\}$$

13
uniformly for $F(\cdot) \in F_n$ as $n \to \infty$; and since Corollary 3.1, (3.7), and the Lemma of Berman give

$$n |\log m(t) - \theta' t + \frac{1}{2} \theta' Z_0^{-1} \theta| \to 0$$

$$\left(1 - 2 \phi'(\lambda_n) + \phi'_{\theta_n}(\lambda_n, \lambda_n') \right) \sim \left(1 - 2 \phi'(\lambda_n) + \phi_{\theta_n}(\lambda_n, \lambda_n') \right)$$

uniformly for $F(\cdot) \in F_n$. Theorem 3.1 is established.

**Corollary 3.2.** Let $\lambda = \lambda_n \to \infty$ with $n$ and let $\lambda^6 \tau = o(n)$ as $n \to \infty$. Then as $n \to \infty$

$$P(Y_n(y) \geq \lambda) \sim (1 - \phi(\lambda))$$

uniformly in $|y| \leq 1$.

**Proof.** Let $\phi$ be the family of b.d.f.'s of the form

$$F_n = P[Z_n(y) \leq z_n]P[Z_n(y) \leq z_n]$$

for $|y| \leq 1$ and $n \geq n_1$. $\alpha_n$ may then be taken as $(B_r + B_K) \sqrt{\tau / \eta}$

where $B_r$ and $B_K$ are upper bounds for $f(\cdot)$ and $K(\cdot)$ respectively, and $\delta$ may be taken as $1/4$. The conclusion of the corollary is then simply the square root of that of the theorem.

**Corollary 3.3.** Let $\lambda = \lambda_n \to \infty$ with $n$ and let $\lambda^6 \tau = o(n)$ as $n \to \infty$. Then as $n \to \infty$

$$P(Y_n(x) \geq \lambda, Y_n(y) \geq \lambda) \sim (1 - 2 \phi(\lambda) + \phi_{\theta_n}(\lambda, \lambda)),$$

where $r = r_n(x, y)$, uniformly in $|x|$, $|y| \leq 1$ and $|x - y| \geq 2 \tau^{-1} \log \tau$.

**Proof.** Essentially the same as above. By (2.7) $\delta$ may again be taken as $1/4$. 

14
4. Some Lemmas.

For \( p = 0, \ldots, [2\tau] \) and \( n = 1, 2, \ldots \) let \( y_{n,p} = -1 + p \tau^{-1} \) and

\[
Y_{n,p}(y) = \frac{\sigma(y, p, p^{-1}, \tau)}{\sigma(y, p, \tau)} Y_n(y, p^{-1}) , \quad 0 \leq y \leq 1.
\]

Lemma 4.1. If \( K(\cdot) \) satisfies a uniform Lipschitz condition of order one on \(( -\infty, \infty)\), then there is a constant \( C_1 \) for which

\[
\mathbb{E}[\exp\left(\frac{Y_{n,0}(x) - Y_{n,0}(y)}{\sqrt{|x-y|}}\right)] \leq C_1,
\]

uniformly in \( 0 \leq x, y \leq 1 \), \( p = 0, \ldots, [2\tau] \), and \( n \) sufficiently large.

Proof. For \( x, y, n \) and \( p \) fixed, let

\[
U_{ni} = \frac{K(x + \tau(y, p, p^{-1}, \tau)) - K(y + \tau(y, p, p^{-1}, \tau))}{\sigma(y, p, \tau)}
\]

so that \( U_{n1}, \ldots, U_{nn} \) are independent and identically distributed and

\[
(4.1) \quad Y_{n,p}(x) - Y_{n,p}(y) = \sqrt{\tau} \sum_{i=1}^{n} (Y_{ni} - \mathbb{E}(U_{ni})).
\]

If \( M(t) \) denotes the moment generating function of \( (U_{n1} - \mathbb{E}(U_{n1})) \), then by (4.1) and the Mean Value Theorem

\[
\mathbb{E}[\exp\left(\frac{Y_{n,0}(x) - Y_{n,0}(y)}{\sqrt{|x-y|}}\right)] = [M(|x-y|^{-1/2} \sqrt{\tau})]^n \leq (1 + \frac{1}{2} \frac{1}{|x-y|} \frac{\tau}{n} M'(t_o))^n
\]

where \( 0 < t_o < |x-y|^{-1/2} \sqrt{\tau} \leq |x-y|^{-1/2} \). Now the uniform Lipschitz condition which \( K(\cdot) \) satisfies and (2.4) imply

\[
|U_{n1} - \mathbb{E}(U_{n1})| \leq C_1 |x-y|.
\]
w.p. one where \( C'_1 \) is independent of \( x, y, n, \) and \( p \). Moreover

\[
\text{Var}(U_{nl}) \leq \frac{1}{\sigma^2(y_{n, p}, \tau)} \int_{-\infty}^{\infty} |K(x+\tau(y_{n, p}, w))-K(y+\tau(y_{n, p}, w))|^2 f(w) dw
\]

\[
\leq \frac{C'_1 B_p |x-y|}{\tau \eta^2} \int_{-\infty}^{\infty} |K(x-w)-K(y-w)| dw.
\]

It follows that

\[
M''(t) \leq \exp\left[tC'_1 |x-y|\right] \text{Var}(U_{nl}) \leq \frac{C''_1}{\tau} |x-y|
\]

for \( 0 < t < |x-y|^{-1/2} \), where \( C''_1 \) is independent of \( x, y, n, \) and \( p \).

Therefore

\[
E\left[\exp\left[-\frac{Y_{n, p}(x)-Y_{n, p}(y)}{\sqrt{|x-y|}}\right]\right] \leq (1 + \frac{C'_1}{2n})^{n} \leq e^{\frac{1}{2} C'_1} = C_1,
\]

which is also independent of \( x, y, n, \) and \( p \).

**Corollary 4.1.** If \( K(\cdot) \) satisfies a uniform Lipschitz condition of order one on \((-\infty, \infty)\), then

\[
P\left(\left|Y_{n, p}(x)-Y_{n, p}(y)\right| \geq \varepsilon\right) \leq C_1 e^{-\frac{\varepsilon}{\sqrt{|x-y|}}} , \text{ for all } \varepsilon > 0 .
\]

**Corollary 4.2.** If \( K(\cdot) \) satisfies a uniform Lipschitz condition of order one on \((-\infty, \infty)\), then there is a constant \( C_2 \) for which

\[
E\left|Y_{n, p}(x)-Y_{n, p}(y)\right|^k \leq C_2|x-y|^2
\]

uniformly in \( 0 \leq x, y \leq 1, \) \( p = 0, \ldots, [2\tau] \), and \( n \) sufficiently large.
To state the next lemma the following notation will be useful.

For \( g(\cdot) \in C[0,1] \) a continuous function on \([0,1] \), denote by
\[ \pi_k(g)(\cdot) \]
the broken line one with vertices at \((0,g(0)), (1\cdot 2^{-k}, g(1\cdot 2^{-k})), \ldots, (l, g(l))\); that is
\[ \pi_k(g)(x) = g(1\cdot 2^{-k}) + 2^k [g((i+1)\cdot 2^{-k}) - g(i\cdot 2^{-k})](x - i\cdot 2^{-k}) \]
for \( 1\cdot 2^{-k} \leq x \leq (i+1)\cdot 2^{-k}, i = 0, \ldots, 2^k - 1 \). Let \( Q_{n,p}(\cdot) \) denote the distribution induced in \( C[0,1] \) by the process \( Y_{n,p}(y), 0 \leq y \leq 1 \), (the existence of \( Q_{n,p}(\cdot) \) is guaranteed by Corollary 4.2 and the discussion in Prokhorov (1956) on pp. 178-179), and let \( \|\cdot\|_s \) denote the sup norm in \( C[0,1] \).

**Lemma 4.2.** If \( K(\cdot) \) satisfies a uniform Lipschitz condition of order one on \( (-\infty, \infty) \), then there is a constant \( C_3 \) for which
\[ Q_{n,p} \left( \{ g: \|g - \pi_k(g)\|_s \geq \lambda \cdot 2^{-\frac{1}{4}k} \} \right) \leq C_3 \exp \left( -\lambda \cdot 2^{\frac{1}{4}k} \right) \]
uniformly in \( p = 0, \ldots, [2^k] \) and \( n \) sufficiently large, if \( \lambda \geq 1 \).

**Proof.** Clearly,
\[ Q_{n,p} \left( \{ g: \|\pi_k(g) - \pi_{k-1}(g)\|_s \geq \lambda \cdot 2^{-\frac{1}{4}k+1} \} \right) \]
\[ \leq Q_{n,p} \left( \{ g: \max_{i=0, \ldots, 2^k-1} |g(i\cdot 2^{-k}) - g((i+1)\cdot 2^{-k})| \geq \lambda \cdot 2^{-\frac{1}{4}k} \} \right) \]
\[ \leq \sum_{i=0}^{2^k-1} P \left( |Y_{n,p}(i\cdot 2^{-k}) - Y_{n,p}((i+1)\cdot 2^{-k})| \geq \lambda \cdot 2^{-\frac{1}{4}k} \right) \]
\[ \leq 2^{k-1} C_1 \exp \left( -\lambda \cdot 2^{\frac{1}{4}k} \right) \]
by Corollary 4.1. By the triangle inequality we find

$$Q_{n,p}(g;\|\pi_k(g) - \pi_{k+r}(g)\|_s \geq \frac{-1/4}{k} \frac{1/4}{r}) \geq 4\lambda \frac{2}{1-2^{-1/4}}$$

(4.2)

$$\leq C_1 \sum_{j=k+1}^{k+r+1} 2^j \exp(-\lambda 2^j/4).$$

Now when \( r \to \infty \) \( \|\pi_r(g)\|_s \) converges in distribution to \( \|g\|_s \) as is shown in Prokhorov (1956) on pp. 178-179. Therefore, letting \( r \to \infty \) in (4.2) yields the conclusion of the lemma.

5. Relative Stability.

**Theorem 5.1.** Let \( f(\cdot) \) be a bounded density which satisfies a uniform Lipschitz condition of order \( \ell \), \( 0 < \ell \leq 1 \), and is positive on some neighborhood of \([a,b]\); let \( K(\cdot) \) be a kernel which satisfies a uniform Lipschitz condition of order one on \((-\infty, \infty)\); and let \( \tau^\alpha = o(n) \) for some \( \alpha > 1 \) and \( n = o(\tau^\beta) \) with \( \beta \leq 1 + 2\ell \), as \( n \to \infty \).

Then

$$\lim_{n \to \infty} \max_{a \leq x \leq b} \sqrt{\frac{n}{2\pi \log \tau}} \left( \frac{f_n(x) - f(x)}{\|K\|_2 \sqrt{f(x)}} \right) = 1$$

and (5.1) remains true if absolute value signs replace the parentheses.

**Proof.** By translating the origin and changing scale, we may without loss of generality assume that \( a = -1 \) and \( b = +1 \). In view of Lemma 2.1, then, it will be sufficient to show that if \( f(\cdot) \) is continuous on a neighborhood of \([-1,1]\) and \( \tau^\alpha = o(n) \) as \( n \to \infty \) with \( \alpha > 1 \), then for every \( \epsilon > 0 \).
\[(5.2) \quad \lim P\left\{ \max_{|y| \leq 1} Y_n(y) \leq (1-\epsilon) \sqrt{2 \log \tau} \right\} = 0 \]

\[(5.2') \quad \lim P\left\{ \max_P \max_{0 \leq y \leq 1} |Y_{n_P}(y)| \geq (1+\epsilon) \sqrt{2 \log \tau} \right\} = 0. \]

The argument given to establish (5.2) is essentially that given by Cramér (1962) for stationary normal processes. We define for \(|y| \leq 1\) and \(n = 1, 2, \ldots\)

\[
W_n(y) = \begin{cases} 
0: & \text{if } Y_n(y) \leq \lambda_n \\
1: & \text{if } Y_n(y) > \lambda_n 
\end{cases}
\]

\[
W_n = \int_{-1}^{+1} W_n(y) \, dy
\]

where \(\left(\lambda_n\right)_{n=1}^{\infty}\) is a sequence of constants for which \(\frac{\lambda_n}{\tau} = o(n)\) as \(n \to \infty\). Then clearly for \(n = 1, 2, \ldots\)

\[(5.3) \quad P\left\{ \max_{|y| \leq 1} Y_n(y) \leq \lambda_n \right\} = P[W_n = 0] \leq \text{Var}(W_n) / (E[W_n])^2. \]

To estimate the right hand side of (5.3) we note that since each of the processes \(Y_n(y)\) is continuous w.p. one, each is measurable (Loève 1963, Sect. 35); therefore so is each of the \(W_n(y)\) processes. Since each of the latter processes is also bounded, Fubini's Theorem applies, and we may write
(5.4) \[ E[\mathcal{W}_n] = \int_{-1}^{1} P(Y_n(y) > \lambda_n) dy = 2(1 - \Phi(\lambda_n))[1 + o(1)] \]

where the second equality follows from Corollary 3.2. The numerator in (5.3) presents a bit more difficulty, and the following identity will be useful:

(5.5) \[ \int_{\lambda_n}^{\infty} \int_{\lambda_n}^{\infty} d\Phi(z) = (1 - \Phi(\lambda))^2 + \frac{1}{2\pi} \int_0^{|\rho|} \frac{-\lambda^2}{1 + \rho} \frac{e^{-\rho z}}{\sqrt{1 - \rho^2}} d\rho \]

identically in \( \lambda > 0 \) and \( |\rho| < 1 \). (5.5) may be verified by differentiation. Now by Corollaries 3.2 and 3.3 and Fubini's Theorem \( E[\mathcal{W}_n]^2 \) does not exceed

\[
\iint_{|x-y| \geq 2\tau^{-1}\log \tau} P(Y_n(x) \geq \lambda_n, Y_n(y) \geq \lambda_n) dx dy + \iint_{|x-y| \leq 2\tau^{-1}\log \tau} P(Y_n(y) \geq \lambda_n) dy
\]

(5.6) \[
\leq \iint_{|x-y| \geq 2\log \tau} dx dy \left\{ \int_{\lambda_n}^{\infty} \int_{\lambda_n}^{\infty} d\Phi(z) \right\} (1 + o(1)) + \frac{\log \tau}{\tau} (1 - \Phi(\lambda_n))[1 + o(1)]
\]

where \( r = r_n(x,y) \) and \( o(1) \) is uniform in \( x \) and \( y \). If we replace the term in braces by the right-hand side of (5.5) and subtract the square of the right-hand side of (5.4) from the last line in (5.6), we find that \( \text{Var}(\mathcal{W}_n) \) does not exceed
\[4(1 - \Phi(\lambda_n))^2 \times o(1) + \frac{h \log \tau}{\tau} (1 - \Phi(\lambda_n)) [1 + o(1)]
\]

(5.7)

\[
\int_{|x-y| \geq \frac{2\log \tau}{\tau}} \left\{ \frac{1}{2\pi} \int_0^{|r|} \frac{\exp\left(-\frac{\lambda_n^2}{1 + w}\right)}{\sqrt{1 - w^2}} \, dw \right\} \, dx \, dy.
\]

Moreover, the last integral is majorized by

(5.8) \[o([\log \tau]^{-1}) \exp(-\lambda_n^2) \exp(\lambda_n^2 \cdot o([\log \tau]^{-1})) [1 + o(1)]\]

in view of (2.7).

Now let \( \epsilon \) be given, 0 < 2\epsilon < 1. Then we may set \( \lambda_n = (1 - \epsilon) \sqrt{2 \log \tau} \), and the condition \( \lambda_n^2 = o(n) \) as \( n \to \infty \) will be satisfied. Thus from (5.3), (5.4), (5.7), and (5.8) we have, using the well-known expansion of \( (1 - \Phi(\cdot)) \),

\[
P\left\{ \max_{|y| \leq 1} Y_n(y) \leq (1 - \epsilon) \sqrt{2 \log \tau} \right\} \leq \text{Var}(W_n)/(E[W_n])^2
\]

\[
\leq \pi (1 - \epsilon)^2 \log \tau \left[ 2 \frac{\log \tau}{\pi} \times \left( \frac{1}{\tau^2 - \epsilon} \right) + \frac{o([\log \tau]^{-1})}{\tau (1 - \epsilon)^2} \right] (1 + o(1)) + o(1)
\]

which is \( o(1) \) as \( n \to \infty \).

To establish (5.2') let \( \epsilon = 3\epsilon' > 0 \) be given and let

\[Y_n = \max_p \max_{0 \leq y \leq 1} |Y_{n,p}(y)|.\]

Then from Corollary 3.2, Lemma 4.2, and Lemma 2.1, we find that \( P(\|Y_n\| \geq (1 + \epsilon) \sqrt{2 \log \tau}) \) is
\[
\leq \sum_{p=0}^{2^k} \sum_{i=1}^{2^k} \Pr[|Y_{n,p}(y)| \geq (1+2\epsilon') \sqrt{2\log \tau}]
\]
(5.9)

\[
+ \sum_{p=0}^{2^k} q_{n,p} \left( (g; \|g-K_n g\|_E \geq \epsilon' \sqrt{2\log \tau}) \right)
\]

\[\leq \tau 2^{k+2} \left( 1-\delta((1+\epsilon') \sqrt{2\log \tau})[1+o(1)] + 2\tau C_2 \exp(-2^{\frac{1}{4}} k-5 \epsilon') \sqrt{2\log \tau} \right)\]

\[\leq 64 \frac{(\log \tau)^{16}}{\sqrt{2\pi}} \times \frac{1}{\tau^{2\epsilon'+\epsilon'^2}} [1+o(1)] + \frac{2C_3}{\tau^{2\beta}},\]

for n sufficiently large, if \(2^k \sim [2\log \tau]^{\frac{1}{4}}\) as \(n \to \infty\). Since the last line in (5.9) is clearly \(o(1)\) as \(n \to \infty\), (5.2') is established.

Often a theorem asserting convergence in probability may be converted into a theorem asserting convergence w.p. one by a simple application of the Borel-Cantelli Lemma. This is true of Theorem 5.1 as we shall demonstrate.

**Theorem 5.2.** Let \(f(\cdot)\) be bounded on \((-\infty, \infty)\) and positive and continuous on some neighborhood of \([a,b]\); let \(K(\cdot)\) be a kernel which satisfies a uniform Lipschitz condition of order one on \((-\infty, \infty)\); and let \(\tau^\alpha = o(n)\) for some \(\alpha > 1\), and \(n = o(\tau^\beta)\) for some \(\beta\) as \(n \to \infty\). Then w.p. one

\[
(5.10) \quad \max_{a \leq x \leq b} |f_n(x)-f(x)| = o(1), \text{ as } n \to \infty.
\]
If, in addition \( f(\cdot) \) satisfies a uniform Lipschitz condition of order \( \ell, \frac{1}{2} < \ell \leq 1 \), on a neighborhood of \([a, b]\) and \( \tau^2(\log \tau)^\alpha = o(n) \), \( \alpha > 1 \), then \( o(1) \) may be replaced by \( o(\tau^{-1/2}) \) in (5.10).

**Proof.** By Lemma 2.1 it suffices to show that if \( f(\cdot) \) is bounded on \((-\infty, \infty)\) and positive and continuous on a neighborhood of \([a, b]\) then w.p. one

\[
(5.11) \quad \max_{a \leq x \leq b} |f_n(x) - f(x; \tau)| = \begin{cases} 
  o(1): & \text{if } \tau^\alpha = o(n) \\
  o(\tau^{-1/2}): & \text{if } \tau^2(\log \tau)^\alpha = o(n) 
\end{cases}
\]

as \( n \to \infty \); and, as above, there is no loss of generality in assuming that \([a, b] = [-1, 1]\). Under these assumptions (5.9) is true, and if we take \( \epsilon' = \beta \), we may recognize its right-hand side as the n-th term of a summable series. Thus by the Borel-Cantelli Lemma

\[
P(Y_n \geq (1 + 3\beta) \sqrt{2\log \tau}, "i.o." ) = 0 ,
\]

where "i.o." abbreviates "infinitely often". It follows easily from Lemma 2.1 that if \( \alpha > 1 \)

\[
(5.12) \quad \lim_{n \to \infty} \max_{a \leq x \leq b} \frac{f_n(x) - f(x; \tau)}{\|k\|^2 \sqrt{f(x)}} = 0 \quad \text{w.p. one}
\]

(5.11) now follows from the fact that \( \sqrt{n/\tau(\log \tau)^\alpha} \) is \( \geq 1 \) or \( \tau^{1/2} \) for large \( n \) accordingly as \( \tau^\alpha \) or \( \tau^2(\log \tau)^\alpha = o(n) \) as \( n \to \infty \).
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**13. ABSTRACT**

The maximum relative deviation of a sample density from the true one is shown to be relatively stable as the sample size tends to infinity provided both the sample and true densities are sufficiently regular. Conditions are given under which the sample density converges to the true density w.p. one uniformly on compact sets and uniformly on compact sets at a specified rate. The key to establishing this result is a theorem on the large deviations of independent, identically distributed bivariate random variables.
### Sample Density

Large Deviations

<table>
<thead>
<tr>
<th>KEY WORDS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample Density</td>
</tr>
<tr>
<td>Large Deviations</td>
</tr>
</tbody>
</table>

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