THE MAXIMUM DEVIATION OF SAMPLE SPECTRAL DENSITIES

BY

MICHAEL B. WOODROOFE and JOHN W. VAN NESS

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Summary.

Due to the non-parametric nature of many time series problems, confidence bands for spectral densities are of paramount importance. The present paper provides some partial results towards obtaining such bands (and therefore goodness of fit tests). Specifically, it is shown that for linear processes

$$\frac{1}{\|W\|_2} \left( \frac{N}{2m_N \log m_N} \right)^{1/2} \max_{-\pi \leq \lambda \leq \pi} \left| \frac{f_N(\lambda) - f(\lambda)}{f(\lambda)} \right| \to 1$$

in probability, where $f_N(\lambda)$ is the usual windowed sample spectral density, $m_N W(m_N \lambda)$ is the (varying) window, and $m_N$ tends to $\infty$ with $N$.

Note that the difference between the maximum deviation and the deviation at a single $\lambda$ point (see Parzen (1957)) manifests itself in the factor $(\log m_N)^{-1/2}$. Thus in practice a confidence band for all $\lambda$ is $O((\log m_N)^{1/2})$ times that for a finite set.

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1. Introduction.

Let \( \{X_t\} \) be a real-valued, discrete parameter random process. We assume that \( \{X_t\} \) is linear and therefore has the following representation in terms of a pure white noise process, \( \{\xi_t\} \): 

\[
X_t = \sum_{k=-\infty}^{\infty} a_k \xi_{t-k}
\]

(1.1)

where \( a_k = O(|k|^{-1-\beta}) \) for some \( \beta > 0 \). Under these assumptions, \( \{X_t\} \) has a spectral density 

\[
f(\lambda) = \frac{1}{2\pi} \left| \sum_{v=-\infty}^{\infty} a_v e^{-iv\lambda} \right|^2
\]

(1.2)

so that 

\[
R(v) = E X_t X_{t+v} = \int_{-\pi}^{\pi} e^{-iv\lambda} f(\lambda) d\lambda.
\]

(1.3)

The most commonly used spectral density estimates, \( f_N(\lambda) \), are those obtained by weighting the periodogram, 

\[
I_N(\lambda) = \frac{1}{2\pi N} \left| \sum_{v=1}^{N} X_v e^{-iv\lambda} \right|^2,
\]

(1.4)

in the following manner (see Parzen (1957)): 

\[
f_N(\lambda) = m_N \int_{-\infty}^{\infty} W(m_N(\omega-\lambda)) \lambda_N(\lambda) d\omega
\]

(1.5)

where \( \{m_N\} \) is a sequence of positive integers increasing to \( \infty \) with \( N \) and \( W(\cdot) \) is a suitable positive, even, weight function. It is assumed that \( W(\cdot) \) has a Fourier representation, 

\[
W(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iv\lambda} W(v) dv
\]

(1.6)
\[ w(v) = \int_{-\infty}^{\infty} e^{-iv\lambda} W(\lambda) d\lambda \]

with \( \omega(0) = 1 \). Then (1.5) can be written

\[ f_N(\lambda) = \frac{1}{2\pi} \sum_{v=-N+1}^{N-1} e^{-iv\lambda} R_N(v) w(vm_N^{-1}) \]

where \( R_N(\cdot) \) is the covariance estimate,

\[ R_N(v) = \frac{1}{N} \sum_{t=1}^{N-v} X_t X_{t+v} = R_N(-v) \quad v \geq 0. \]

If \( w(\cdot) \) has compact support (i.e. for some \( V, w(v) = 0 \) if \( |v| > V \)), \( f_N(\cdot) \) is called a truncated spectral estimate.

With these assumptions and definitions, the main result is

**Theorem 1.1** Let

a) \( \{X_t\} \) have the representation (1.1) with \( \beta \geq 1/5 \),

b) \( E \xi_j = 0, E \xi_j^2 = 1, E \xi_j^8 < \infty \),

c) \( f(\cdot) \) be everywhere positive and satisfy a uniform Lipschitz condition,

d) \( f_N(\cdot) \) be a truncated estimate,

e) \( \int_{0}^{\infty} \lambda^{2+\gamma} W(\lambda) d\lambda < \infty, \quad \gamma > 0, \ W(\cdot) \in L_1(-\infty, \infty) \cap L_\infty(-\infty, \infty), \) and

f) \( m_N = o(N^{\alpha}), \quad \alpha < 2/5, \quad N = o\left(\frac{\log m_N}{N}\right); \)

then as \( N \to \infty \)

\[ \frac{1}{\sqrt{2m_N \log m_N}} \max_{-\pi \leq \lambda \leq \pi} \frac{|f_N(\lambda) - f(\lambda)|}{\|w\|_2 f(\lambda)} \to 1 \]

in probability, where \( \|\cdot\|_p \) denotes the \( L_p(-\infty, \infty) \) norm. The result remains true if absolute value signs replace the parentheses in (1.10).
The proof of the theorem is divided into two major parts. First, (1.10) is proved for the pure white noise process itself (Section 3). This is accomplished by reducing the \( (f_N(\lambda) - f(\lambda)) \) to a sum of independent identically distributed random variables plus a term which is negligible in the limit and applying a theorem on large deviations to the sum. The second part of the proof (Section 4) involves reducing the linear process case to the pure white noise case.

Section 2 discusses some background material.

2. **Background.**

There are some results related to this paper in the literature. Walker (1964) discusses similar problems for the periodogram (1.4) itself rather than for \( f_N(\cdot) \). Whittle (1958) has studied the maxima of trigonometric polynomials with random coefficients which, of course, is closely related. Grenander and Rosenblatt (1957, Chapter 6) give asymptotic confidence bands for spectral distribution functions and heuristically discuss confidence bands for spectral densities. See also Hannan (1960) Chapters 3 and 4.

In the remainder of this section we state some results to which we will refer. The first concerns the large deviations of independent, identically distributed, bivariate random variables. A bivariate distribution function (b.d.f.) will be called standardized (st.) if the means and variances of the associated random variables are zero and one respectively. If \( F(\cdot) \) is a st. b.d.f. of compact support we let
\[ r(F) = \int_{-\infty}^{\infty} z_1 z_2 dF(z), \]
\[ \gamma(F) = \inf \left\{ \gamma : \int_{\gamma} dF(z) = 1 \right\}. \]

Also, we let \( F^{(n)}(\cdot) \) be the \( n \)-fold convolution of \( F(\cdot) \) with itself and denote by \( \Phi(\cdot) \) and \( \Phi_{\rho}(\cdot, \cdot) \) the st. normal d.f. and the st. normal b.d.f. with parameter \( \rho \) respectively. We may now state the following theorem which is proved by Woodrooffe (1966).

**Theorem 2.1.** Let \( \frac{1}{\gamma} \) be a family of st. b.d.f.'s of compact support for which \( |r(F)| \leq 1 - 28, 0 < \delta < 1/2 \), for all \( F(\cdot) \in \frac{1}{\gamma} \); let \( \{\gamma_n\}_{n=1}^{\infty} \) be a fixed sequence of real numbers for which \( \gamma_n \geq 1 \) for all \( n \) and \( \gamma_n^2 = o(n) \) as \( n \to \infty \); and let \( \frac{\gamma}{\gamma_n} \leq \frac{1}{\gamma_n} \) be the subfamily of \( \frac{1}{\gamma} \) consisting of those \( F(\cdot) \in \frac{1}{\gamma} \) for which \( \gamma(F) \leq \gamma_n \).

If \( \{z_n\}_{n=1}^{\infty} \) is another sequence of real numbers for which \( z_n \to \infty \) and \( z_n \gamma_n^2 = o(n) \) as \( n \to \infty \), then as \( n \to \infty \)

\[ \int_{z_n}^{\infty} \int_{z_n}^{\infty} dF^{(n)}(z \sqrt{n}) \sim \int_{z_n}^{\infty} \int_{z_n}^{\infty} d\Phi_{r(F)}(z) \]

uniformly for \( F(\cdot) \in \frac{1}{\gamma_n} \).

**Lemma 2.1.** Let \( p(\lambda) = \sum_{v=-k}^{k} \alpha_v \exp(iv\lambda) \) be a trigonometric polynomial. Then

\[ \max_{\lambda} |p'(\lambda)| \leq (2k+1) \max_{\lambda} |p(\lambda)| \]

where \( p'(\cdot) \) denotes the derivative of \( p(\cdot) \) with respect to \( \lambda \).
Lemma 2.1 is proved by Zygmund (1959) on p. 11; it will be used as follows. Denote by \( J_N(\cdot) \) and \( g_N(\cdot) \) respectively the periodogram and sample density in the white noise case -- i.e., \( J_N(\cdot) \) and \( g_N(\cdot) \) are defined by (1.4) and (1.5) with \( \xi_t \) replacing \( X_t \) for \( t=1,\ldots,N \).

Let \( r \) be a positive integer and let \( \lambda_{N,1},\ldots,\lambda_{N,N} \) be \( p = \frac{\pi}{2m_N} \) equally spaced points in \( [0,\pi] \). If \( w(\cdot) \) is of compact support, say \( w(u) = 0, \ |u| \geq 1 \), then \( g_N(\cdot) \) is a trigonometric polynomial involving no frequencies higher than the \( m_N \)-th. It follows from the Mean Value Theorem and Lemma 2.1 that

\[
(2.1) \quad \max_\lambda |2\pi g_N(\lambda)-1| \leq \max_i |2\pi g_N(\lambda_{N,i})-1| + \frac{\pi}{2m_N} (2m_N+1) \max_\lambda |2\pi g_N(\lambda)-1| ,
\]

and therefore

\[
(2.2) \quad \max_\lambda |2\pi g_N(\lambda)-1| \leq \max_i \frac{|2\pi g_N(\lambda_{N,i})-1|}{(1-3\pi^{-1})} .
\]

3. The White Noise Case.

Theorem 3.1. Let

a) \( \{\xi_t\} \) be a pure white noise process;

b) \( E[\xi_1] = 0, \ E[|\xi_1|^2] = 1, \ E[|\xi_1|^8] < \infty \);

c) \( g_N(\cdot) \) be a truncated estimate;

d) \( \int_0^\infty \lambda^\gamma w(\lambda) d\lambda < \infty , \ 0 < \gamma < 1, \) and \( w(\cdot) \in L_1(\infty,\infty) \cap L_\infty(-\infty,\infty) . \)

e) \( m_N \to \infty \) and \( m_N = o(N^\alpha) , \ \alpha < 2/5 \);
(3.1) \[ \sqrt{\frac{N}{2mN \log mN}} \max_{0 \leq \lambda \leq \pi} (2\pi g_N(\lambda)-1) \to 1 \]

in probability as \( N \to \infty \). (3.1) remains true if absolute value signs replace the parentheses.

**Proof.** Since \( m = m_N \) satisfies assumption e) iff \( cm \) does for every constant \( c \), we may assume that \( w(u) = 0 \) for \( |u| \geq 1 \). Thus from (1.8) and (1.9) we find that

(3.2) \[ \frac{N}{\sqrt{m}} (2\pi g_N(\lambda)-1) = Z_N(\lambda) + r_N(\lambda) + r_N \]

where for \( 0 \leq \lambda \leq \pi \), \( 1 \leq t \leq N \), and \( N = 1, 2, \ldots \)

(3.3a) \[ Z_N(\lambda) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} Z_{N,t}(\lambda) , \]

(3.3b) \[ Z_{N,t}(\lambda) = \frac{2}{\sqrt{m}} \sum_{v=1}^{m-1} \xi_t \xi_{t+v} w(\sqrt{m}) \cos v\lambda , \]

\[ r_N(\lambda) = \frac{2}{\sqrt{Nm}} \sum_{t=N-m}^{m} \sum_{v=N-t+1}^{m} \xi_t \xi_{t+v} w(\sqrt{m}) \cos v\lambda , \]

\[ r_N = \frac{1}{\sqrt{Nm}} \sum_{t=1}^{N} (\xi_t^2 - 1) . \]

Since for \( N = 1, 2, \ldots \)

\[ E(\max_{\lambda} |r_N(\lambda)|) \leq \frac{2}{\sqrt{Nm}} \sum_{v=1}^{m} E\left| \sum_{t=N-m}^{N} \xi_t \xi_{t+v} \right| , \]

(3.4) \[ E\left| \sum_{t=N-v}^{N} \xi_t \xi_{t+v} \right|^2 \leq v \leq m , \quad v = 1, \ldots, N , \quad \text{and} \]

\[ E|r_N|^2 \leq m^{-1} , \]
we have \( \max_{\lambda} |r_N(\lambda) + r_{-N}(\lambda)| \to 0 \) in probability as \( N \to \infty \), and it suffices to consider the stochastic processes \( Z_N(\lambda), 0 \leq \lambda \leq 1, \ N = 1,2,\ldots \)
defined by (3.3a) and (3.3b).

**Lemma 3.1.** Under the hypotheses of Theorem 3.1, the random variables
\[ Z_{N,1}(\lambda), \ldots, Z_{N,N}(\lambda) \]
have zero means and variances
\[ \sigma_N^2(\lambda) = \frac{m}{m-1} \sum_{v=1}^{m-1} w(vm)^{-1} \cos(v\lambda)^2 \]
for \( 0 \leq \lambda \leq \pi \) and \( N = 1,2,\ldots \). If \( t_1 < t_2 < t_3 < t_4 \) and \( 0 \leq \lambda_i \leq \pi, \ i = 1,\ldots,4 \), then
\[ (3.5a) \quad E\left[ \prod_{i=1}^{4} Z_{N,t_i}(\lambda_i) \right] = 0 = E\left[ \prod_{i=1}^{4} Z_{N,t_i}(\lambda_i) \right] . \]

Moreover, there exists a constant \( A \) for which
\[ (3.5b) \quad |E\left[ \prod_{i=1}^{4} Z_{N,t_i}(\lambda_i) \right]| \leq \begin{cases} A & \text{if } t_1 = t_2 \text{ and } t_3 = t_4 \\ A m^{-1} & \text{if } t_1 = t_2 \neq t_3 \neq t_4 \end{cases} \]
for \( 0 \leq \lambda_i \leq \pi, \ i = 1,\ldots,4 \) and \( N = 1,2,\ldots \).

**Proof.** The first assertion is obvious. The second, (3.5a), follows from the fact that if \( t_1 < t_i \) for \( i \neq 1 \), then \( E[\prod_{i=1}^{4} \xi_{t_i}^{v_1}] \)
\( = E[\xi_{t_1}] \cdot E[\prod_{i=1, i \neq 1}^{4} \xi_{t_i}^{v_1}] \)
in each of the multiple sums which compose its left and right-hand sides. (3.5b) involves a rather tedious consideration of cases the details of which will be omitted.

**Lemma 3.2.** Under the hypotheses of Theorem 3.1

1) \( \sigma_N^2(\lambda) \) is bounded uniformly in \( N = 1,2,\ldots \) and \( 0 \leq \lambda \leq \pi \)
and \( \sigma_N^2(\lambda) \to ||W||_2^2 \) uniformly in \( m^{-1}(\log m) \leq \lambda \leq \pi \) as \( N \to \infty \).
\( \text{Proof.} \) The boundedness of \( \hat{\sigma}_N^2(\lambda) \) is obvious from that of \( w(\cdot) \). To establish the uniform convergence we note that from well-known trigonometric identities we have

\[
(3.6a) \quad |\hat{\sigma}_N^2(\lambda) - \|w\|^2_2| \leq 2m^{-1} \sum_{v=1}^{m-1} w(vm^{-1})^2 \|w\|^2_2 + 2m^{-1} \sum_{v=1}^{m-1} w(vm^{-1})^2 \cos 2v\lambda.
\]

The first term on the right is independent of \( \lambda \) and tends to zero as \( N \to \infty \) since the Fourier Transform is an isometry of \( L_2(-\infty, \infty) \). Moreover, if we let \( \ast \) denote convolution in \( L_1(-\infty, \infty) \) and use the fact that

\[
w(vm^{-1})^2 = \int_{-\infty}^{\infty} e^{ivy} \hat{w}(m\hat{w})(dy),
\]

we find that the second sum in (3.6a) does not exceed

\[
\int_{-\infty}^{\infty} \left| \frac{\sin(m - \frac{1}{2})(y+2\lambda)}{\sin \frac{1}{2}(y+2\lambda)} \right| \hat{w}(m\hat{w})(dy) + 2m^{-1}
\]

\[
= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} 2(k+1)\pi \left| \frac{\sin(m - \frac{1}{2})(y+2\lambda)}{\sin \frac{1}{2}(y+2\lambda)} \right| \hat{w}(m\hat{w})(dy) + 2m^{-1}
\]

\[
= \int_{-\pi}^{\pi} \left| \frac{\sin(m - \frac{1}{2})(y+2\lambda)}{\sin \frac{1}{2}(y+2\lambda)} \right| W_N(y)dy + 2m^{-1}
\]

where \( W_N(y) = \sum_{k=-\infty}^{\infty} \hat{w}(m\hat{w}(y+2k\pi)) \) with the sum converging in \( L_1(-\pi, \pi) \). (The propriety of interchanging summation and integration follows from the Monotone Convergence Theorem.)
Since
\[ \int_{-\pi}^{\pi} W_N(\lambda) d\lambda = \int_{-\infty}^{\infty} W*W(my) dy \leq \|W\|_1^2 m^{-1}, \]
and
\[ \int_{-\epsilon}^{\epsilon} W_N(y) dy \geq \int_{-\epsilon m}^{\epsilon m} W*W(y) dy \]
for all \( \epsilon > 0 \), we find that if \( m^{-1} \log m \leq \lambda \leq \pi \), then
\[
\left| y \right| \leq m^{-1} \log m \quad \left| \frac{\sin(m - \frac{1}{2})(y+2\lambda)}{\sin \frac{1}{2}(y+2\lambda)} \right| W_N(y) dy
\]
\[ \leq \frac{1}{\sin \frac{1}{2}(m^{-1} \log m)} \int_{-\pi}^{\pi} W_N(y) dy \leq \frac{\pi}{\log m} \]
and
\[
\int_{m^{-1} \log m < \left| y \right| \leq \pi} \left| \frac{\sin(m - \frac{1}{2})(y+2\lambda)}{\sin \frac{1}{2}(y+2\lambda)} \right| W_N(y) dy
\]
\[ \leq m \int_{m^{-1} \log m < \left| y \right| \leq \pi} W_N(y) dy \leq 2 \int_{\log m}^{\infty} W*W(y) dy \]
so that i) follows by the triangle inequality. The proof of ii) is similar.

The random variables \( Z_{N,1}(\lambda), \ldots, Z_{N,N}(\lambda), \ 0 \leq \lambda \leq \pi, \) \( N = 1, 2, \ldots \) have the desirable property of \( m \)-dependence, which we will now exploit. Let \( k = k_N = \lceil m(\log m)^2 \rceil \) where \( \lceil \cdot \rceil \) denotes the greatest integer function. We may then write

\[
N = nk + r
\]

where \( 0 \leq r < k \). For \( i = 1, \ldots, n, \ N = 1, 2, \ldots \) and \( 0 \leq \lambda \leq \pi \), define

\[
U_{N,i}(\lambda) = \frac{1}{\sqrt{k}} \left( Z_{N,(i-1)k+1}(\lambda) + \cdots + Z_{N,ik-m}(\lambda) \right)
\]

\[
V_{N,1}(\lambda) = \frac{1}{\sqrt{m}} \left( Z_{N,ik-m+1}(\lambda) + \cdots + Z_{N,ik}(\lambda) \right)
\]

\[
V_{N,0}(\lambda) = Z_{N,nk+1}(\lambda) + \cdots + Z_{N,nk}(\lambda)
\]

Then clearly

\[
(3.7) \quad Z_N(\lambda) = \frac{nk}{\sqrt{N}} (U_N(\lambda) + \sqrt{m} V_N(\lambda)) + \frac{1}{\sqrt{N}} V_{N,0}(\lambda)
\]

for \( 0 \leq \lambda \leq \pi \) and \( N = 1, 2, \ldots \) where

\[
U_N(\lambda) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_{N,i}(\lambda), \ 0 \leq \lambda \leq \pi, \ N = 1, 2, \ldots
\]

\[
V_N(\lambda) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_{N,i}(\lambda), \ 0 \leq \lambda \leq \pi, \ N = 1, 2, \ldots
\]
Moreover, \( U_N(\lambda) \) and \( V_N(\lambda) \) are for each fixed \( \lambda \) and \( N \) sufficiently large sums of independent, identically distributed random variables. We wish to demonstrate that as \( N \to \infty \)

\[
(3.8a) \quad \max_{\lambda} |V_{N,\lambda}(\lambda)| = o_p(1)
\]

\[
(3.8b) \quad \max_{i} |V_N(\lambda_N, i)| = o_p(\sqrt{kN^{-1}})
\]

where \( \{\lambda_N, 1, \ldots, \lambda_N, p\} \) is a suitable sequence of partitions of \( [0, \pi] \). (We remind the reader that \( a_N = o_p(b_N) \) as \( N \to \infty \) iff \( a_N b_N^{-1} \to 0 \) in probability as \( N \to \infty \).) (3.8a) follows easily from

\[
E[\max_{\lambda} |V_{N,\lambda}(\lambda)|] \leq \frac{2}{\sqrt{Nm}} \sum_{v=1}^{m^{-1}} E\left| \sum_{t=kn+1}^{N} \xi_t \xi_{t+v} \right|
\]

\[
E\left| \sum_{t=kn+1}^{N} \xi_t \xi_{t+v} \right|^2 \leq N-nk = r < k
\]

and assumption e).

(3.8b) requires more care. For \( i = 1, \ldots, n \), \( N = 1, 2, \ldots \), and \( 0 \leq \lambda \leq \pi \) let

\[
U_{N, i}(\lambda)^{'} = \begin{cases} 
U_{N, i}(\lambda) : \text{if } |U_{N, i}(\lambda)| \leq N^{0.3} \\
0 : \text{if } |U_{N, i}(\lambda)| > N^{0.3}
\end{cases}
\]

\[
V_{N, i}(\lambda)^{'} = \begin{cases} 
V_{N, i}(\lambda) : \text{if } |V_{N, i}(\lambda)| \leq N^{0.3} \\
0 : \text{if } |V_{N, i}(\lambda)| > N^{0.3}
\end{cases}
\]
\[ U_{N,i}(\lambda)'' = \frac{U_{N,i}(\lambda)' - E(U_{N,i}(\lambda)')}{\sqrt{\text{Var}(U_{N,i}(\lambda)'')}} \]

\[ V_{N,i}(\lambda)'' = \frac{V_{N,i}(\lambda)' - E(V_{N,i}(\lambda)')}{\sqrt{\text{Var}(V_{N,i}(\lambda)'')}} \]

and let \( U_N(\lambda)'', V_N(\lambda)'', U_N(\lambda)'', V_N(\lambda)'' \) be \( n^{-1/2} \) times their respective sums. (For example, \( U_N(\lambda)' \) is defined exactly as was \( U_N(\lambda) \) with \( U_{N,i}(\lambda)' \) replacing \( U_{N,i}(\lambda) \) for \( i=1,\ldots,n \).) Then it follows easily from Lemma 3.1 that for \( 0 \leq \lambda \leq \pi \) and \( N \) sufficiently large

\[ E(U_{N,i}(\lambda)) = 0 = E(V_{N,i}(\lambda)) , \]

\[ \text{Var}(V_{N,i}(\lambda)) = \sigma_N^2(\lambda) = (1 - mk^{-1})^{-1} \text{Var}(U_{N,i}(\lambda)) , \]

\[ E|U_{N,1}(\lambda)|^4 \leq A k^{-1} , \]

\[ E|V_{N,1}(\lambda)|^4 \leq A , \]

where \( A \) is independent of \( N \) and \( \lambda \). Moreover, (3.9) implies (via Chebyshev-like inequalities) that

\[ P(U_{N,1}(\lambda) \neq U_{N,1}(\lambda)') \leq A k^{-1} N^{-1.2} , \quad 0 \leq \lambda \leq \pi , \]

\[ P(V_{N,1}(\lambda) \neq V_{N,1}(\lambda)') \leq A N^{-1.2} , \quad 0 \leq \lambda \leq \pi , \]

\[ |E(U_{N,i}(\lambda)')| \leq A k^{-1} N^{-0.9} , \quad |E(V_{N,1}(\lambda)'')| \leq A N^{-0.9} , \quad 0 \leq \lambda \leq \pi , \]

\[ |\text{cov}(U_{N,1}(\lambda), U_{N,1}(\lambda)) - \text{cov}(U_{N,1}(\lambda)', U_{N,1}(\lambda)')| \leq A k^{-1} N^{-0.6} \]
(3.12b) \[ |\text{Var}(U_{N,1}(\lambda)) - \sigma_N^2(\lambda)| \leq 2A k m^{-1} N^{-0.6}, \quad 0 \leq \lambda \leq \pi, \]

(3.12c) \[ |\text{Var}(V_{N,1}(\lambda)) - \sigma_N^2(\lambda)| \leq 2A N^{-0.6}, \quad 0 \leq \lambda \leq \pi. \]

We now turn to the proof of (3.8b). For each \( N \) let \( \lambda_{N,1}, \ldots, \lambda_{N,p} \) be \( p = m \log m \) equally spaced points in \([0, \pi]\). Since for large \( N \)

\[
 P(V_{N,1}(\lambda_{N,j}) \neq V_{N,1}(\lambda_{N,j}'), \text{ for some } j),
\]

\[
 \leq \sum_{j=1}^{p} \sum_{i=1}^{n} P(V_{N,1}(\lambda_{N,j}) \neq V_{N,1}(\lambda_{N,j}'), \leq (A m \log m) N^{-1.2} \leq A N^{-0.2}
\]

which is \( o(1) \) as \( N \to \infty \), (3.8b) is equivalent to

(3.13) \[ \max_j |V_{N}(\lambda_{N,j})'| = o_p(\sqrt{km^{-1}}) \]

as \( N \to \infty \). Since also by (3.11), (3.12), and Lemma 3.1

\[ \frac{E(V_{N,1}(\lambda)' \sqrt{n}}{\text{Var}(V_{N,1}(\lambda)' \sqrt{} \to 0 \text{ uniformly in } \lambda \text{ as } N \to \infty}, \]

\[ \text{Var}(V_{N,1}(\lambda)' \leq A_1 \text{ uniformly in } N \text{ and } \lambda}, \]

(3.13) would follow from

(3.14) \[ \max_j |V_{N}(\lambda_{N,j})''| = o_p(\sqrt{km^{-1}}) \text{ as } N \to \infty. \]

If we now let \( \mathcal{F} \) be the family of b.d.f.'s of the form

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\[ P(v_1, v_2) = P(V_{N,1}(\lambda) \leq v_1) P(V_{N,1}(\lambda)'' \leq v_2), \quad -\infty < v_1, v_2 < \infty, \]

for \( 0 \leq \lambda \leq \pi \) and \( N = 1, 2, \ldots \), an application of Theorem 2.1 yields

\[ P(V_N(\lambda)'' \geq 4 \sqrt{2 \log m}) \sim (1 - \phi(4 \sqrt{2 \log m})) \]

uniformly in \( 0 \leq \lambda \leq \pi \) as \( N \to \infty \). Therefore, if \( \epsilon > 0 \) is given, we have

\[ P(\max_j |V_N(\lambda_{N,j})''| \geq \epsilon \sqrt{km^{-1}}) \]

\[ \leq 2n \log m(1 - \phi(4 \sqrt{2 \log m})) \leq m^{-1.5} \log m \]

for \( N \) sufficiently large. Since \( \epsilon > 0 \) was arbitrary (3.8b) is established.

We may now write the left-hand side of (3.1) in a manageable form. Indeed from (3.2), (3.4), (3.7), and (3.8) we may write

\[ \sqrt{\frac{N}{m}} (2\pi g(\lambda) - 1) = U_N(\lambda) + r_N(\lambda)'' \]

where \( \max_i |r_N(\lambda_{N,i})''| \to 0 \) in probability as \( N \to \infty \). Moreover, by essentially the same arguments used to reduce (3.8b) to (3.14) we may write

\[ U_N(\lambda) = U_N(\lambda)''_n(\lambda) + r_N(\lambda)'' \]

where \( \max_i |r_N(\lambda_{N,i})''_n| \to 0 \) in probability as \( N \to \infty \). (See (3.10a), (3.11), and (3.12)). Therefore, in view of Lemma 2.1 and Slutsky’s Theorem, (3.1) would follow from
(3.15a) \[ \lim_{N \to \infty} P(\max_{j} \left| U_{N}(\lambda_{N,j})'' \right| \sigma_{N}(\lambda_{N,j}) \geq (1+\epsilon)\|w\|_{2} \sqrt{2 \log m}) = 0 \]

(3.15b) \[ \lim_{N \to \infty} P(\max_{j} U_{N}(\lambda_{N,j})'' \sigma_{N}(\lambda_{N,j}) \leq (1-\epsilon)\|w\|_{2} \sqrt{2 \log m}) = 0 \]

for every \( \epsilon > 0 \).

To establish (3.15a) let \( S \) be the set of \( j \) for which \( 1 \leq j \leq p = [m \log m] \) and \( \lambda_{N,j} \geq m^{-1} \log m \). Then if \( \epsilon = 2\epsilon' > 0 \) is given, we find from Theorem 2.1 and Lemma 3.2 that for \( N \) sufficiently large

\[
P(\max_{j \in S} \left| U_{N}(\lambda_{N,j})'' \right| \sigma_{N}(\lambda_{N,j}) \geq (1+\epsilon)\|w\|_{2} \sqrt{2 \log m})
\]

\[
\leq \sum_{j \in S} P\left( \left| U_{N}(\lambda_{N,j})'' \right| \geq (1+\epsilon') \sqrt{2 \log m} \right)
\]

\[
\leq 4m \log m \left( 1 - \Phi((1+\epsilon') \sqrt{2 \log m}) \right) = o(1)
\]

as \( N \to \infty \), and

\[
P(\max_{j \notin S} \left| U_{N}(\lambda_{N,j})'' \right| \sigma_{N}(\lambda_{N,j}) \geq (1+\epsilon)\|w\|_{2} \sqrt{2 \log m})
\]

\[
\leq \sum_{j \notin S} P\left( \left| U_{N}(\lambda_{N,j})'' \right| \geq \epsilon \sqrt{2 \log m} \right)
\]

\[
\leq 2 \log m \left( 1 - \Phi(\epsilon \sqrt{2 \log m}) \right) = o(1)
\]

as \( N \to \infty \) where \( \epsilon^2 > 0 \) is a lower bound for \( \|w\|_{2}^{2}/\sigma_{N}^{2}(\lambda) \). This establishes (3.15a).
The proof of (3.15b) is a bit more involved. We begin by defining for $N = 1, 2, \ldots$

$$Y_N(\lambda) = \begin{cases} 0 & \text{if } U_N(\lambda)'' \leq \mu_N \\ 1 & \text{if } U_N(\lambda)'' > \mu_N \end{cases}$$

$$Y_N = \sum_{j \in S} Y_N(\lambda_N, j)$$

where $\mu_N \to \infty$ and $\mu_N^6 (\log m)^{2 - \alpha} = o(N^{2 - \alpha})$ as $N \to \infty$. Then clearly

(3.16) \quad P(\max_{j \in S} U_N(\lambda_N, j)'' \leq \mu_N) = P(Y_N = 0) \leq \frac{\text{Var}(Y_N)}{(E[Y_N])^2}

by Chebyshev's Inequality.

To estimate the right-hand side of (3.16) we first notice that by Theorem 2.1

(3.17) \quad E[Y_N] = \sum_{j \in S} P(U_N(\lambda_N, j)'' \geq \mu_N) \geq c(S)(1 - \Phi(\mu_N))(1 + o(1))

as $N \to \infty$, where $c(S)$ denotes the cardinality of $S$. To bound the numerator in (3.16) we will use the following identity, which is most easily verified by differentiation:

(3.18) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\Phi_p(y, z) = (1 - \Phi(\mu))^2 + \frac{1}{2\pi} \int_0^\infty \frac{-\mu^2}{(1 + w)^{1/2}} \, dw
identically in $\mu > 0$ and $|\rho| < 1$. Now $E|Y_N|^2$ clearly does not exceed

$$\sum_{i,j} P(U_N(\lambda_{N,i}''') > \mu_N, U_N(\lambda_{N,j}'') > \mu_N)$$

$$+ \sum_{i,j} P(U_N(\lambda_{N,j}'') > \mu_N)$$

where $\sum'$ and $\sum''$ denote respectively summation over $i,j \in S$

such that $|\lambda_{N,i} - \lambda_{N,j}| \geq m^{-1}(\log m)^{2/\gamma}$ and $i,j \in S$ such that

$|\lambda_{N,i} - \lambda_{N,j}| < m^{-1}(\log m)^{2/\gamma}$. Since by (3.12) and Lemma 3.2,

$$(3.19) \quad r_{ij} = \frac{\text{Cov}(U_N(\lambda_{N,i}''), U_N(\lambda_{N,j}''))}{\sqrt{\text{Var}(U_N(\lambda_{N,i}'')) \text{Var}(U_N(\lambda_{N,j}''))}} = O(\text{[log m]}^{-2})$$

uniformly in $|\lambda_{N,i} - \lambda_{N,j}| \geq m^{-1}(\log m)^{2/\gamma}$, Theorem 2.1 applies, and

we have

$$\sum_{i,j} P(U_N(\lambda_{N,i}''') > \mu_N, U_N(\lambda_{N,j}'') > \mu_N)$$

$$\leq \sum_{i,j} (1+o(1)) \int_{\mu_N}^{\infty} \int_{\mu_N}^{\infty} d\Phi_{r_{ij}}(y,z)$$

$$\leq \sum_{i,j} (1+o(1)) \int_{\mu_N}^{\infty} \int_{\mu_N}^{\infty} d\Phi_r(y,z)$$

and

$$\sum_{i,j} P(U_N(\lambda_{N,i}'') > \mu_N)$$

$$(3.21) \quad \leq 2c(S)(\log m)^{2/\gamma} (1-\Phi(\mu_N))(1+o(1)).$$
If we now apply the identity (3.18) to the right-hand side of (3.20), add the right-hand side of (3.21), and subtract the square of the right-hand side of (3.17), we find that \( \text{Var}(Y_N) \) does not exceed

\[
(3.22) \quad c(s)^2(1-\Phi(\mu_N))^2 o(1) + (1+o(1)) \sum_{i,j} \frac{1}{2\pi} \int_0^{x_{ij}} e^{\frac{-\mu_N}{1+w}} \frac{dw}{\sqrt{1+w^2}} + 2c(s)(\log m)^{2/\gamma} (1-\Phi(\mu_N))(1+o(1)).
\]

Now let \( \varepsilon = 2\varepsilon' > 0 \) be given and let \( \mu_N = (1-\varepsilon') \sqrt{2\log m} \); then for \( N \) sufficiently large (3.15b) does not exceed the left-hand side of (3.16) which by (3.17), (3.19), and (3.22) is majorized by

\[
(3.23) \quad \frac{(1-\Phi(\mu_N))^2}{c(s)^2} \sum_{i,j} \frac{1}{2\pi} \int_0^{x_{ij}} e^{\frac{-\mu_N}{1+w}} \frac{dw}{\sqrt{1+w^2}} + 2 \frac{(1-\Phi(\mu_N))^{-1}}{c(s)} (\log m)^{2/\gamma} (1+o(1)) + o(1);
\]

and since \( x_{ij} = O[(\log m)^{-2}] \) uniformly in the first sum, (3.23) is less than or equal to

\[
\frac{m^2(1-\varepsilon')^2}{2\pi} (\log m) [\log m]^{-2} \exp(-\mu_N^2) (1+o(1)) + 2 \sqrt{2\pi} \frac{\sqrt{2\log m} m(1-\varepsilon')^2}{\pi m (\log m)} (\log m)^{2/\gamma} + o(1)
\]

which is obviously \( o(1) \) as \( N \to \infty \).
4. Reduction to White Noise.

The second half of the proof of Theorem 1 will be accomplished by proving three lemmas.

Lemma 4.1. Under the conditions of Theorem 1

\[ \sqrt{\frac{N}{m_N \log m_N}} \max_{\lambda} \left| \frac{f(\lambda) - E f_N(\lambda)}{f(\lambda)} \right| = o\left( \sqrt{\frac{N}{m_N^3 \log m_N}} \right). \]

Proof. The term inside the absolute value signs is for some constant A

\[ \leq A |f(\lambda) - m \int_{-\infty}^{\infty} W(m(\lambda-\omega)) f(\omega) d\omega| + \]

(4.1)

\[ A|m \int_{-\infty}^{\infty} W(m(\lambda-\omega))(f(\omega) - E f_N(\omega)) d\omega|. \]

By assumption c), the first term of (4.1) is

\[ \leq A \int_{-\infty}^{\infty} |f(\lambda - \omega m^{-1}) - f(\lambda)| W(\omega) d\omega \]

\[ \leq A_1 \int_{-\infty}^{\infty} |\omega m^{-1}| W(\omega) d\omega \leq \frac{A_2}{m}. \]

The second term of (4.1) is by a well-known result (see e.g. Rosenblatt (1962) p. 171)

\[ \leq A_3 \frac{\log N}{N} m \int_{-\infty}^{\infty} W(m(\lambda-\omega)) d\omega \leq \frac{A_3 \log N}{N}. \]
Lemma 4.2: Under the conditions of Theorem 1,

\[
\sqrt{\frac{m_N}{m_N \log m_N}} \max_{\lambda} \frac{m_N}{f(\lambda)} \int_{-\infty}^{\infty} W(m_N(\lambda-\omega)) \left[ (I_N(\omega)-E_N(\omega)) \right. \\
- \left. 2\pi f(\omega) (J_N(\omega)-EJ_N(\omega)) \right] d\omega \\
= o_p \left( \frac{1}{m_N} (\log m_N)^{-1} N^{-\beta} \right).
\]

Proof.

\[
m \int_{-\infty}^{\infty} W(m(\lambda-\omega)) \left[ (I_N(\omega)-E_N(\omega)) \right. \\
- \left. 2\pi f(\omega) (J_N(\omega)-EJ_N(\omega)) \right] d\omega \\
= m \int_{-\infty}^{\infty} W(m(\lambda-\omega)) \frac{1}{2\pi N} \sum_{r,s=-\infty}^{\infty} a_r a_s \left[ \sum_{v_1,v_2=1}^{N} e^{-i(v_1-v_2)\omega} (\xi_{v_1-r+s} - R_{v_1-v_2+s}) \right] \\
- e^{-i(r-s)\omega} \sum_{v_1,v_2=1}^{N} e^{-i(v_1-v_2)\omega} (\xi_{v_1-v_2} - R_{v_1-v_2+s}) \right] d\omega \\
= \frac{1}{2\pi N} \sum_{r,s=-\infty}^{\infty} a_r a_s d_{rs}(\lambda)
\]

where

\[
d_{rs}(\lambda) = \left( \sum_{v_1,v_2=1}^{N} - \sum_{v_1=r+1}^{r+N} \sum_{v_2=s+1}^{s+N} \right) \frac{v_1-v_2}{m} e^{-i(v_1-v_2)\lambda} (\xi_{v_1-r+s} - R_{v_1-v_2+s}).
\]

Let \( C_{r,s,N} \) denote the set of lattice points in the two sums not common to both sums, then

\[
d_{rs}(\lambda) = \sum_{C_{r,s,N}} \frac{v_1-v_2}{m} e^{-i(v_1-v_2)\lambda} e^{-i(v_1-v_2)\lambda} (\xi_{v_1-r+s} - R_{v_1-v_2+s}).
\]

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Let \( v = v_1 - v_2 \) and \( u = v_2 \) then
\[
\delta_{rs}(\lambda) = \sum_{v = \min(-N+1, -N+r+1-s)}^{\max(N-1, N+r-s-1)} e^{-iv\lambda} w(vm^{-1}) .
\]
\[
\sum_{D_{r,s,v}} (\xi_{v+u-r^2u-s} - R^2_\xi(v-r+s))
\]
where \( D_{r,s,v} \) is the integers in the projection onto the \( v_1 \) axis of that part of the line \( v_1 - v_2 = v \) which intersects \( C_{r,s,N} \).

Now
\[
E \max_{\lambda} \delta_{rs}(\lambda) \leq 2 \max_{\lambda} \sum_{v = \min(-N+1, -N+r+1-s)}^{\max(N-1, N+r-s-1)} |w(vm^{-1})E| \sum_{D_{r,s,v}} |\xi_{v+u-r^2u-s} - R^2_\xi(v-r+s)| ,
\]
and due to the independence of the \( \xi_j \),
\[
E \left| \sum_{D_{r,s,v}} \xi_{v+u-r^2u-s} - R^2_\xi(v-r+s) \right|^2 \leq \begin{cases} 2N & |r| \text{ or } |s| > N \\ |r|+|s| & \text{otherwise} \end{cases}
\]
if \( v \neq r-s \). If \( v = r-s \) then a constant term appears involving \( E\xi_j \). Therefore
\[
E(N\log m)^{-1/2} \max_{\lambda} \sum_{r,s = -\infty}^{\infty} a_{r,s} \delta_{rs}
\]
\[
\leq A_1 m^{1/2} (N\log m)^{-1/2} \left\{ \sum_{s = -\infty}^{\infty} \sum_{|r| \leq N} |a_{r,s}| (|r|+|s|)^{1/2} \\
+ \sum_{s = -\infty}^{\infty} \sum_{|r| > N} |a_{r,s}| N^{1/2} \right\}
\]
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\[ \leq A_2 m^{1/2} (N \log m)^{-1/2} \left[ \sum_{r \leq N} r^{1/2} a_r + N^{1/2} \sum_{|r| > N} a_r \right] \]
\[ \leq A_2 m^{1/2} (\log m)^{-1} N^{-\beta} . \]

**Lemma 4.3.** Under the conditions of Theorem 1,
\[ \sqrt{\frac{N}{m \log m}} \max_{\lambda} \left| \sum_{r \leq N} \frac{m_r}{f(\lambda)} \int_{-\infty}^\infty W(m(\lambda-\omega))(f(\lambda)-f(\omega))(J_N(\omega) - \frac{1}{2\pi}) d\omega \right| \]
\[ = O_P \left( \frac{N}{m^{3/2} \log m} \right) . \]

**Proof.** \[ |m \int_{-\infty}^\infty W(m(\lambda-\omega))(f(\lambda)-f(\omega))J_N(\omega) d\omega| \leq \]
\[ \leq \int_{-\infty}^\infty |f(\lambda)-f(\lambda-\omega m^{-1})| W(\omega) J_N(\lambda-\omega m^{-1}) d\omega \]
\[ \leq m^{-1} \int_{-\infty}^\infty |\omega| W(\omega) J_N(\lambda-\omega m^{-1}) d\omega . \]

This is dominated by
\[ (4.2) \quad m^{-1} \int_{-\infty}^\infty (1+\omega^2) W(\omega) J_N(\lambda-\omega m^{-1}) d\omega . \]

By the conditions of \( W(\cdot) \), the kernel \((1+\omega^2) W(\omega)\) satisfies the conditions of Theorem 3.1. Thus, interpolating the expected value of \((4.2)\),
\[ \sqrt{\frac{N}{m \log m}} \max_{\lambda} |m \int_{-\infty}^\infty W(m(\lambda-\omega))(f(\lambda)-f(\omega))J_N(\omega) d\omega| \]
\[ = O_P \left( \frac{N}{m^{3/2} \log m} \right) . \]

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since

\[
\sqrt{\frac{N}{m \log m}} \max_\lambda \frac{m}{2\pi} \int_{-\infty}^{\infty} W(m(\lambda-\omega))(f(\lambda)-f(\omega))d\omega
\]

= \text{O}_p \left( \frac{\sqrt{N}}{\sqrt{m \log m}} \right)

by the first part of the proof of Lemma 4.1.

Combining these three lemmas we see that

\[
\max_\lambda \sqrt{\frac{N}{m \log m}} \left| \frac{f_N(\lambda)-f(\lambda)}{f(\lambda)} - (2\pi g_N(\lambda)-1) \right| = \text{O}_p(1).
\]

This result and Theorem 5.1 prove Theorem 1.1.

5. Conclusion.

The hypotheses of Theorem 1.1 were chosen to make the proof as simple as possible without sacrificing a great degree of generality. Assumption d) could be weakened to read

\[d') \quad |w(v)| \leq Ke^{-\eta|v|} \text{ for some } K, \eta > 0.\]

Assumption e) could be stated in terms of how smoothly \( w(v) \) tends to 1 at the origin. Other assumptions could be juggled - strengthen one and weaken another - without affecting Theorem 1.1.
REFERENCES


THE MAXIMUM DEVIATION OF SAMPLE SPECTRAL DENSITIES

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