ASYMPTOTIC EXPANSIONS AND LARGE DEVIATIONS IN MANY DIMENSIONS

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1. Introduction.

The present paper will concern itself with triangular arrays (T.A.'s)

\[(x^{(n, k)}_{k=1, \ldots, k_n}, \; n=1, 2, \ldots)\]

of p-dimensional random vectors which are independent and identically
distributed (i.i.d.) by rows in the sense that

\[x^{(n, 1)}, \ldots, x^{(n, k_n)}\]

are i.i.d. for each \(n\). If, in addition,

\[E[x^{(n, 1)}] = 0, \; n=1, 2, \ldots\]

\[(1.2) \quad \mathcal{Y}_n' = E[x^{(n, 1)}x^{(n, 1)'}] \rightarrow \mathcal{T}_0 \text{ (pos. def.), as } n \rightarrow \infty,\]

\[(1.3) \quad \int \cdots \int_{\|x\| \geq A} \|x\|^2 \Pr(\|x^{(n, 1)}\| \leq x) \rightarrow 0, \; \text{as} \; A \rightarrow \infty,\]

uniformly in \(n\), then it is easy to show that

\[k_n^{-1/2} s(n) = k_n^{-1/2} \sum_{k=1}^{k_n} x^{(n, k)}\]

is asymptotically normal with mean vector zero and covariance matrix \(\mathcal{T}_0\).

One of our results gives an asymptotic expansion of

\[(1.4) \quad \mathcal{F}[k_n^{-1/2}s(n) \leq x, i=1, \ldots, p] \sim \Phi(x; \mathcal{Y}_n),\]

where \(\Phi(\cdot; \mathcal{Y}_n)\) denotes the normal distribution function (d.f.) with
mean vector zero and covariance matrix \( \eta_n \). Indeed, it is shown in section 3 that under the appropriate regularity conditions (1.4) may be written in the form

\[
(1.5) \\
\phi(x; \eta_n) \sum_{k=1}^{s-3} Q_{n,k}(x_n) - \frac{k}{2} + R_n
\]

where \( \phi(\cdot; \eta_n) \) is the density of \( \Phi(\cdot; \eta_n) \), the \( Q_{n,k}(\cdot) \)'s are polynomials depending on the moments of \( \chi_{(n,1)} \), and \( R_n \) is of smaller order than the last term in the sum. In section 4 we consider the problem of estimating

\[
R(n) = P(k_n^{-1/2} \delta_1(n) \geq \lambda_n x_i, i=1,\ldots,p)
\]

where \( x_i > 0, i=1,\ldots,p \) and \( \lambda_n \to \infty \) as \( n \to \infty \), by a function \( q(\cdot) \) for which \( p(n) \sim q(n) \) as \( n \to \infty \). Section 5 consists of some remarks on possible extensions of our results, and section 2 presents some preliminaries.

Our development of (1.5), which resembles that of Cramér (1937) and Esseen (1944) in the one-dimensional case, may be termed the characteristic function approach. Another approach is due to Bergström, who has shown (1945) that (1.4) is bounded in absolute value by \( \max_{i,j} |a_{ij} \| \beta_{i,j} / k_n^{1/2} \| \) where \( C_p \) depends only on \( p \), \( (a_{i,j}) = \chi_n^{-1} \), and \( \beta_n \) is the sum of the absolute third moments of the components of \( \chi_{(n,1)} \). He has also given an expansion of (1.4) from which (1.5) may be deduced with a better estimate of the error term \( R_n \) than we give (Bergström (1951)). His results, however, are valid only for sums of sequences of i.i.d. random vectors and the conditions under which his results hold are somewhat restrictive and hard to check. The results of section 4 also generalize a theorem of Cramér (1938); a special case has appeared previously (Woodroofe (1966)).
2. **Preliminaries.**

In this section we will extend the following theorem to the multi-dimensional random variables. The theorem is due originally to Cramér (1937) and may also be found in Kolmogorov and Gnedenko (1954).

**Theorem A.** Let \( Y \) be real valued random variable with zero mean, variance \( \sigma^2 \), finite absolute \( s \)-th moment \( \beta_s (s \geq 4) \), and characteristic function (c.f.) \( \phi(\cdot) \). Let \( \rho_s = \beta_s^{1/s}/\sigma \). There exist polynomials \( P_1(\cdot), \ldots, P_{s-3}(\cdot) \) and a constant \( C(s) \), depending only on \( s \), for which

\[
\left| \phi(wn^{-1/2})^n - e^{-\frac{\sigma^2 w^2}{2} \left\{ 1 + \sum_{k=1}^{s-3} P_k(iw)n^{-\frac{k}{2}} \right\}} \right|
\]

\begin{equation}
(2.1)
\end{equation}

\[
\leq \frac{C(s)}{(\sqrt{n})^{s-3}} \rho_s^{3(s-2)} (|\sigma w|^s + |\sigma w|^{3(s-2)}) - \frac{1}{4} \frac{\sigma^2 w^2}{2^n}
\]

for all \( w \) with \( |\sigma w| \leq \sqrt{n}/8s \rho_s^3 \) and all \( n=1,2,\ldots \). The polynomials are defined by

\[
k!P_k(iw) = \frac{d^k}{dz^k} \left\{ \sum_{j=0}^{s-3} \frac{V^j}{j!} \right\} |z=0
\]

\begin{equation}
(2.2)
\end{equation}

\[
V = V(w,z) = \frac{\sigma^2 w^2}{2} + \frac{1}{z^2} \log \phi(wz), |\sigma w| \leq 1/8s \rho_s^{3/2}
\]

\( (2.2) \) is easily deduced from the equation atop p. 207 in Kolmogorov and Gnedenko's book.

In the multi-dimensional case we have a random vector \( X = (X_1, \ldots, X_p)' \) where ' denotes transpose. We will assume
\[ E(X) = 0 \]

(2.3)

\[ E(XX') = \mathcal{Z}(\text{pos. def.}) \]

(2.4)

\[ \beta_{s,i} = E|X_i|_s^s < \infty, \ i=1, \ldots, p, \ s \geq \frac{1}{4}. \]

For \( i=1, \ldots, p \), let \( \rho_{s,i} = \beta_{s,i}^{1/2}/\lambda^{1/2} \), where \( \lambda \) and \( \mu \) are respectively the minimum and maximum eigenvalues of \( \mathcal{Z} \), and let \( \rho_s = \sum_{i=1}^{p} \rho_{s,i} \).

Lemma 2.1. Let \( X \) be a \( p \)-dimensional random vector satisfying (2.3) and (2.4); let \( \psi(\cdot) \) denote its c.f.. There exist polynomials \( P_1(\cdot), \ldots, P_{s-3}(\cdot) \) and a constant \( C_1(s) \), depending only on \( s \), for which

\[
|\psi(tn^{-1/2}\mathbb{1}) - \exp(-\frac{1}{2} t' \mathbb{1} t)\{1 + \sum_{k=1}^{s-3} P_k(it)n^{-k/2}\}| \leq \frac{C_1(s)}{(\sqrt{n})^{s-2}} \rho_s^{3(s-2)} (\|ut\|_s^s + \|ut\|_s^s)^{3(s-2)} \exp(-\frac{1}{4} t' \mathbb{1} t) \]

(2.5)

for all \( t \in \mathbb{R}_p \) with \( \|ut\| \leq \sqrt{n}/8 \rho_s^3 \) and \( n = 1, 2, \ldots \). The polynomials are defined by

\[
k! P_k(it) = \frac{d^k}{dz^k}\left\{\sum_{j=0}^{s-3} U^{j/3} j!\right\}|_{z=0}
\]

(2.6)

\[ U = U(t,z) = \frac{1}{2} t' \mathbb{1} t + \frac{1}{z^2} \log \psi(tz), \|ut\| \leq 1/8 \rho_s^3 z^{1/2}. \]

Proof. Let \( 0 \neq t \in \mathbb{R}_p \) with \( \|ut\| = \mu(t't)^{1/2} \leq \sqrt{n}/8 \rho_s^3 \) be otherwise arbitrary; let \( v = t/\|t\| \). If we let \( \phi(\cdot;v) \) denote the c.f. of the (real) random variable \( Y = v'X \), then we obviously have
(2.7) \[ \phi(\|t\|; \nu) = \psi(t). \]

Moreover, \( Y \) satisfies the hypotheses of Theorem A with \( \sigma^2 = \sigma^2(\nu) = \nu' \Sigma \nu \) so that (2.1) holds. The Lemma is thus an obvious consequence of

\[ \rho_s(\nu) = \sqrt{\frac{E|\nu'X|^2}{\nu' \Sigma \nu}} \leq \rho_s, \]

\[ \frac{1}{2} \sigma^2(\nu) \|t\|^2 + \frac{1}{z} \log \phi(\|t\|; \nu) = U(t, z), \]

(2.7), and the arbitrariness of \( t \).

In terms of the moments

\[ \alpha(v_1, \ldots, v_p) = E[X_1^{v_1} \cdots X_p^{v_p}], \quad 0 \leq \sum_{i=1}^p v_i \leq s \]

\[ \alpha_v(t) = E[t'X]^v, \quad t \in \mathbb{R}_p, \quad 0 \leq v \leq s, \]

the first two \( P_k(\cdot) \)'s may be expressed as follows. Clearly

\[ \alpha_v(t) = \sum_{(v)} \frac{v_1! \cdots v_p!}{v_1! \cdots v_p!} \alpha(v_1, \ldots, v_p) \prod_{i=1}^p t_i^{v_i} \]

where \( \sum_{(v)} \) denotes summation over all vectors \( (v_1, \ldots, v_p) \) with non-negative, integral components whose sum is \( v \); and

\[ P_1(it) = \frac{-1}{3!} \alpha_3(it) \]

\[ P_2(it) = \frac{1}{2^4} (\alpha_4(it) - 3\alpha_2(it)^2) + \frac{1}{72} \alpha_3(it)^2. \]
In general we may write

\[(2.8) \quad P_k(\text{i}t) = \sum_{k=1}^{k} \sum_{k+2}^{p} c_k(\nu_1, \ldots, \nu_p) \prod_{i=1}^{p} (\text{i}t_i)^{\nu_i}\]

where the $c_k(\cdot)$'s are polynomials in the moments $X$. Let

\[\hat{\varphi}_s(t) = \exp(-\frac{1}{2}t't)\{1 + \sum_{k=1}^{s-3} P_k(\text{i}t)n_k^{k/2}\};\]

then the fact that

\[\frac{1}{(2\pi)^p} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\text{i}t_1)^{\nu_1} \cdots (\text{i}t_p)^{\nu_p} \exp(-\frac{1}{2}t't)e^{-\text{i}t'x} dt \]

\[(2.9) \quad = (-1)^\nu \frac{\partial^\nu}{\partial \nu_1 \cdots \partial \nu_p} \varphi(y; \bar{z})|_{y=x}, \quad x \in \mathbb{R}^p,\]

where $0 \leq \sum_{i=1}^{p} \nu_i = \nu \leq s$, shows that the Fourier Transform of $\hat{\varphi}_s(\cdot)$ is

\[\varphi_s(x) = \frac{1}{(2\pi)^p} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{\varphi}_s(t)e^{-\text{i}t'x} dt \]

\[(2.10) \quad = \varphi(x; \bar{z}) + \sum_{k=1}^{s-3} P_k(-\varphi(x; \bar{z}))n_k^{k/2}, \quad x \in \mathbb{R}^p.\]

Here for $k = 1, \ldots, s-3$ $P_k(-\varphi(x; \bar{z}))$ is obtained by replacing $\prod_{i=1}^{p}(\text{i}t_i)^{\nu_i}$ by the right-hand-side of (2.9) in (2.8). Finally, we let

\[\phi_s(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_p} \varphi_s(u)du, \quad x \in \mathbb{R}^p\]

be the indefinite integral of $\varphi_s(\cdot)$.
In closing this section we observe an inequality which will be needed in the next. Let \( X(\cdot, \ldots, \cdot) \) denote the cumulant function of \( X \)

\[
X(v_1, \ldots, v_p) = i^{-v} \frac{\partial^v}{\partial t_1^{v_1} \cdots \partial t_p^{v_p}} \log \psi(t) \bigg|_{t=0},
\]

for \( v_i \geq 0, i=1, \ldots, p, \sum_i v_i = v \leq s \). Leonov and Shiraev (1959) have given an explicit formula for \( X(\cdot, \ldots, \cdot) \) in terms of the moment function \( \alpha(\cdot, \ldots, \cdot) \). From this formula, Hölder's Inequality, and some routine estimation, it follows that

\[
|X(v_1, \ldots, v_p)| \leq C_2(p) \prod_{i=1}^p \beta_{s,i}^{v_i/s}
\]

for all \( v_1, \ldots, v_p \) of interest, where \( C_2(p) \) depends only on \( p \). Expanding \( U(\cdot; \cdot) \) in a Taylor Series it is then easy to see that

\[
(2.12) \quad |c_k(v_1, \ldots, v_p)| \leq C_3(p) \prod_{i=1}^p \beta_{s,i}^{v_i/s}
\]

for all \( v_1, \ldots, v_p \) of interest, where \( C_3(p) \) depends only on \( p \).

3. **Asymptotic Expansions.**

The condition we will have to impose on the T.A. (1.1) in addition to the i.i.d. property, (1.2), (1.3), and the existence of higher moments is the following. Let \( \psi_n(\cdot) \) denote the c.f. of \( X^{(n,1)} \) for \( n = 1, 2, \ldots \). We require the condition \( C_1 \) that for every \( \delta > 0 \), there exist \( \eta < 1 \) for which

\[
|\psi_n(t)| \leq \eta
\]
for all $t$ of norm exceeding $8$ and all $n$. It is easily seen that if

$$\psi_n(\cdot) = \psi(\cdot)$$

is independent of $n$, so that we are actually dealing with a sequence of i.i.d. random vectors, then $C_1$ reduces to Cramér's condition $C$ (Cramér 1937):

$$\limsup_{||t|| \to \infty} |\psi(t)| < 1.$$

**Theorem 3.1.** Let $(X(n, k))$ be i.i.d. by rows and satisfy (1.2) and (1.3); let

$$\beta_{n, s}^{1/s} \leq O(k_n^{c_n}),$$

as $n \to \infty$.

for some $c < 1/6$; and let $C_1$ be satisfied. If $F_n(\cdot)$ denotes the d.f. of $(1/\sqrt{k_n})S(n)$ and $\phi_{n, s}(\cdot)$ is defined by (2.11) (with $X(n, l)$ replacing $X$), then

$$|F_n(x) - \phi_{n, s}(x)| \leq \Delta \left( \frac{\beta_{n, s}^{3/s}}{\sqrt{k_n}} \right)^{-2} \left[ \log k_n \right]^{p-1}$$

uniformly on $R_p$.

In the statement of Theorem 3.1, we have introduced two conventions. First we have appended the subscript $n$ on the notations of the previous section to indicate that $X = X(n, l)$ in their definition. Also we have used $\Delta$ to denote a positive constant which is independent of $n$ and may change from one usage to the next. Our $\Delta$'s will also be independent of the parameters $A$ and $\epsilon$ when they are introduced. The proof of the theorem begins with two lemmas, the first of which is proved by Bergström (1945).
Lemma 3.1. Let $F(\cdot)$ be a distribution function on $\mathbb{R}_p$ and let $G(\cdot)$ have bounded continuous first order derivatives and agree with $F(\cdot)$ at $(+\infty, \ldots, +\infty)$. If for $\epsilon > 0$

\begin{equation}
(3.2) \quad \left| \frac{1}{(2\pi)^p} \int_{-\infty}^{\infty} \cdots \int (F(x-y) - G(x-y)) e^{-\frac{1}{2\epsilon} \|y\|^2} dy \right| \leq M(\epsilon)
\end{equation}

for all $x \in \mathbb{R}_p$, then

\[ |F(x) - G(x)| \leq \max \{ \tau(p) M(\epsilon), B \epsilon \tau(p) \} \]

for all $x \in \mathbb{R}_p$, where $B$ is an upper bound for the first order derivatives of $G(\cdot)$ and $\tau(p)$ depends only on $p$.

Remark. The left-hand side of (3.2) defines the convolution of $F(\cdot) - G(\cdot)$ and $\phi(\cdot; \epsilon I)$, the isotropic normal distribution with variance parameter $\epsilon$. In the sequel it will be denoted by $(F-G) * \phi^\epsilon$. We will also use the notation $[-A, A]$ to denote the set of $x \in \mathbb{R}_p$ with $|x_i| \leq A$, $i = 1, \ldots, p$ and $\int_{-A}^A \cdots \int$ to denote integration over $[-A, A]$.

Lemma 3.2. Let $A > 0$. Then under the assumptions of Theorem 3.1, the following inequality holds for all $x \in [-A, A]$:

\begin{equation}
(3.3) \quad |(F_n - \Phi_n, s) * \phi^\epsilon(x)| \leq \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \cdots \int h_A(x, t)|\psi_n(\frac{tk_n}{\epsilon}, \frac{k_n}{\epsilon})| \Phi_n, s(t)| e^{-\frac{\epsilon \|t\|^2}{2}} dt
\end{equation}

\[ + \Delta \Phi_n, s \rightarrow \mathbb{R}_p \]

where $h_A(\cdot, \cdot)$ is continuous and
\[ h_A(x, t) = \prod_{i=1}^{p} \frac{-it_{x_i} - e^{it_{x_i}A}}{it_i}, \quad x, t \in \mathbb{R}^p \text{ and } t \neq 0. \]

**Proof.** Define

\[ g(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{x-y}{2\pi\sigma^2}} d(F_{n-1}^n, s)(y), \]

Then, it is an easy consequence of (2.12) and Theorem 1 of Brillinger (1962) that

\[ \sum_{i=1}^{p} \int_{|x_i| \geq A} |g(x)| dx \leq \Delta_{n, s}^{-\delta}, \quad n \geq 1. \]

Therefore,

\[ |(F_{n-1}^n, s)^{\epsilon}(x)| \leq \int_{-A}^{X_1} \int_{A}^{X_p} g(y) dy + \Delta_{n, s}^{-\delta} \]

for all \( x \in [-A, A] \). Since the Fourier Transform of \( g(\cdot) \) is

\[ \hat{g}(t) = (\psi_n(tk_n^{-1/2}) \hat{\phi}_{n, s}^\epsilon(t)) e^{-\frac{\|t\|^2}{2}}, \quad t \in \mathbb{R}^p, \]

which is absolutely integrable, we have

\[ g(x) = \frac{1}{(2\pi)^p} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{g}(t)e^{-it'x} dt, \quad x \in \mathbb{R}^p. \]

Substitution of (3.6) into (3.5) and a simple application of Fubini's Theorem now yield (3.3).
Next we estimate the integral, I say, on the right-hand-side of (3.3) for \( x \in [-A,A] \). It is obviously \( I_1 + I_2 + I_3 \) where

\[
I_1 = \frac{1}{(2\pi)^p} \int \cdots \int_{\|t\| \leq T_n} h_A(x,t) |\psi_n(tk_n^{-1/2})|^{k_n} - \phi_{n,s}(t) |dt
\]

\[
I_2 \leq \frac{A^p}{(2\pi)^p} \int \cdots \int_{\|t\| \geq T_n} |\psi_n(tk_n^{-1/2})|^{k_n} \exp(-\frac{c}{2}\|t\|^2) dt
\]

\[
I_3 \leq \frac{A^p}{(2\pi)^p} \int \cdots \int_{\|t\| \geq T_n} |\phi_{n,s}(t)| dt
\]

with \( T_n = \sqrt{k_n} / 8s_0 n_0^3 \). (Recall our conventions!)

Consider first \( I_1 \) and for each subset \( \omega \subset \{1, \ldots, p\} \) let

\[
S_n(\omega) = \{ t \in \mathbb{R}^p : \|t\| < T_n, \|t_i\| \leq A^{-1}, i \notin \omega, \|t_i\| > A^{-1}, i \in \omega \}.
\]

Then from Lemma 2.1, (1,2), and elementary linear algebra we find

\[
(\frac{\sqrt{n}}{\beta_3/\sigma})^{s-2} I_1 \leq \Delta \sum_{\omega} \int_{S_n(\omega)} \cdots \int h_A(x,t)(\|t\|^s + \|t\|^3(s-2)) e^{-\frac{1}{\sigma} \lambda_0 \|t\|^2}
\]

Now again using (1.2) we find

\[
\int_{S(\phi)} \leq \Delta \sum_{j=1}^{p} \int_{S(\phi)} \prod_{i \neq j}^p t_i^{-1} t_j e^{-\frac{1}{\sigma} \lambda_0 \|t\|^2}
\]

\[
\leq \Delta_1 ([\log A]^{p-1})
\]

where \( \phi \) denotes the null set and that for \( \omega \neq \phi \)
\begin{equation}
\int_{S_n^w(\omega)} \int_{S^w(\omega)} (\prod_{j \neq \omega} t_j^{-1})e^{-\frac{1}{s} \lambda_0 \|t\|^2} dt \leq \Delta_1 \left( (\log A)^{p-1} \right)
\end{equation}

Therefore,

\begin{equation}
I_1 \leq \Delta \left( \frac{\beta_{n, s}}{\sqrt{\nu_n}} \right)^{s-2} (\log A)^{p-1}
\end{equation}

for all $x \in [-A, A]$ and all $n$.

To estimate $I_2$, we first observe that (1.3) implies the equicontinuity of the second order derivatives of $\{\psi_n(\cdot)\}$. Therefore, since $\gamma_n \rightarrow \gamma_0$ as $n \rightarrow \infty$, there exists $\delta, \eta > 0$ for which

$$|\psi_n(t)| \leq 1-\eta \|t\|^2$$

for all $t$ of norm $\leq \delta$ and all $n$. And by condition $C_1$ there exists a $\eta' < 1$ for which

$$|\psi_n(t)| \leq \eta'$$

for all $t$ of norm exceeding $\delta$ and all $n$. Finally, (3.1) and (1.2) imply the existence of $\eta'' > 0$ for which

$$T_n = \sqrt{k_n/\delta \sup_{n, s} \rho_{n, s} \gamma_0} \geq \eta'' \kappa_n, \quad n \geq 1,$$

where $\gamma = 1/2 - 3\alpha > 0$. It follows that for $\|t\| \geq T_n$ and $n$ sufficiently large
\[ |\psi_n\left(\frac{t_k}{n^{1/2}}\right)|^k_n \leq \exp(-\eta' \cdot \eta'' k_n 2^\gamma) \]

and therefore \[ I_2 \leq \Delta(A^{-1})^p \exp(-\eta' \cdot \eta'' k_n 2^\gamma), \quad n \geq 1. \] Similarly, we find \[ I_3 \leq \Delta(A^{-1})^p \exp(-\eta''' k_n 2^\gamma) \] for some \( \eta''' > 0 \).

Together with (3.7), the last two inequalities yield

\[
(3.8) \quad I \leq \Delta\left(\frac{\beta_{n,s}}{\sqrt{k_n}}\right)^{3/4} \left(\log A\right)^{p-1} + \Delta(A^{-1})^p \exp(-\eta iv k_n 2^\gamma)
\]

for all \( x \in [-A, A] \) and all \( n \). Substitution of (3.8) into (3.3) and an application of (3.4) now show that \( \left| (F_n - \phi_n, s)^* \beta^\epsilon (x) \right| \) is bounded by

\[
\Delta\left(\frac{\beta_{n,s}}{\sqrt{k_n}}\right)^{3/4} \left(\log A\right)^{p-1} + \Delta(A^{-1})^p \exp(-\eta iv k_n 2^\gamma) + \Delta_{\beta_n, s} A^{-s}
\]

for all \( x \in \mathbb{R}_p \) and all \( n \). Thus, the theorem follows from Lemma 3.1 by choosing \( A = \sqrt{k_n} \) and \( \epsilon = \left(\sqrt{k_n}\right)^{-s}. \)

4. **Large Deviations.**

In this section we will assume that the T.A. \( \{X^{(n,k)}\} \) consists of bounded random variables. More precisely we will assume that for each \( n \)

\[
|X^{(n,1)}| \leq A_n \quad \text{w.p. one}
\]

\[ A_n^2 = o(k_n), \quad \text{as} \ n \to \infty. \]
A vector \( \mu = (\mu_1, \ldots, \mu_p) \) will be called admissible with respect to (w.r.t.) a positive definite matrix \( \Sigma = (\sigma_{ij}) \) iff

\[
\mu_i > 0, \quad i = 1, \ldots, p
\]

(4.2)

\[
\sum_{j=1}^{p} \mu_j \sigma_{i,j} > 0, \quad i = 1, \ldots, p
\]

where \( (\sigma_{i,j}) \) is the inverse of \( \Sigma \).

**Theorem 4.1.** Let \( X^{(n,k)} \) be a T.A. which is i.i.d. by rows and satisfies (1.2) and (4.1) and let \( \mu \) be an admissible vector. Let \( \{\lambda_n\} \) be a sequence of real numbers for which \( \lambda_n \to \infty \) and \( \lambda_n^2 \alpha_n = o(k_n) \) as \( n \to \infty \) with \( q = \max(3, p) \). Then

\[
\Pr[k_n^{-1/2} S_i(n) \geq \lambda_n \mu_i^2, \quad i = 1, \ldots, p] \sim \int_{\lambda_n \mu_i}^{\infty} \cdots \int_{\lambda_n \mu_i}^{\infty} d\phi(y; \Sigma_n)
\]

where the integration extends over \( y \) with \( \lambda_n \mu_i \leq y_i \leq \infty, \quad i = 1, \ldots, p \).

**Proof.** By (4.1) the moment generating function (m.g.f.)

\[
M_n(t) = E[\exp(t' X^{(n,1)})]
\]

exists finitely for all \( t \in \mathbb{R}_p \). We may therefore define a distribution function \( G_n(\cdot; t) \) by

\[
G_n(y; t) = \frac{1}{M(t)} \int_{-\infty}^{y} \cdots \int_{-\infty}^{y} \exp(t' x) \, dF_n(x),
\]

where \( F_n(\cdot) \) denotes the d.f. of \( X^{(n,1)} \). and, as above, the limits of integration are meant componentwise. The first and second order moments of \( G_n(\cdot; t) \) are then
\[
\theta_{i}^{(n)}(t) = \frac{\partial}{\partial t_{i}} L_{n}(t) , \quad i = 1, \ldots, p ,
\]
(4.3)

\[
\psi_{i,j}^{(n)}(t) = \frac{\partial^{2}}{\partial t_{i} \partial t_{j}} L_{n}(t) , \quad i, j = 1, \ldots, p .
\]

where \( L_{n}(\cdot) = \log M_{n}(\cdot) \). If we now let

\[
H_{n}(y;t) = G^{*}_{n}(y \sqrt{k_{n}} + \theta^{(n)}(t)k_{n};t)
\]

where \( \theta^{(n)}(t) = (\theta_{1}^{(n)}(t), \ldots, \theta_{p}^{(n)}(t))^{\prime} \), and \( * \) denotes convolution, we have

\[
Pr(S_{i}^{(n)} \leq y_{i} , i = 1, \ldots, p)
\]
(4.4)

\[
= \exp(-k_{n}t^{\prime} \theta^{(n)}(t))M_{n}(t)k_{n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-t^{\prime}x \sqrt{k_{n}})dH_{n}(x;t)
\]

which holds identically for \( t, y \in \mathbb{R}^{p} \). (4.4) is most easily established by showing that the c.f.'s of the d.f.'s defined by its left and right-hand sides to be the same. It has been used by several authors -- e.g. Cramer (1938) and Bahadur and Rao (1960) in the case \( p = 1 \) and by Woodroofe (1966) for \( p = 2 \). The proof for general \( p \) requires only trivial modifications and will therefore be omitted. Since (4.4) is an identity, we are free to use any \( t \) we choose, in particular,

\[
t^{(n)} = \frac{\lambda_{n}}{\sqrt{k_{n}}} \nu^{(n)} ,
\]
(4.5)

where

\[
\nu_{i}^{(n)} = \sum_{j=1}^{p} \mu_{i}^{(n)} \sigma_{i,j}^{(n)} , \quad i = 1, \ldots, p ,
\]

and \( (\sigma_{n}^{i,j}) = \chi_{n}^{-1} \), which exists for \( n \) sufficiently large by (1.2).
(Observe that for \( n \) sufficiently large, say \( n \geq n_0 \), \( \Sigma_n^{-1} \) will not only exist, but also satisfy
\[
\sum_{j=1}^{p} \mu_j \sigma_n^{i,j} \geq \Delta > 0, \quad i=1,\ldots,p.
\]

In the sequel it will be tacitly assumed that \( n \geq n_0 \).) We have then
\[
\Pr(S_i^{(n)} \geq \lambda_n \mu_i, \quad i=1,\ldots,p)
\]
(4.6)
\[
= \exp(-k_n \theta(n)^{-1}) k_n \int_{z(n)}^{\infty} \exp(-y\cdot w(n) \lambda_n) dH_n(y)
\]

where \( \theta(n) = \theta(n; t(n)) \), \( H_n(\cdot) = H_n(\cdot; t(n)) \), and
\[
z_i^{(n)} = \lambda_n \mu_i - \epsilon_i^{(n)} \sqrt{k_n}, \quad i=1,\ldots,p.
\]

Lemma 4.1. Under the hypotheses of Theorem 4.1
\[
\beta_n = \max_{0 \leq t_i \leq A^{-1}_n} \max_{v_1,\ldots,v_p} \frac{\partial^3}{\partial t_1 \partial t_2 \partial t_p} |I_n(t)| \leq \Delta_n
\]

The lemma is proved in the case \( p = 2 \) by Woodroofe (1966); its extension to general \( p \) presents no difficulty. Let
\[
\Psi_n = (\psi_{i,j}^{(n)}(t(n)));
\]

then by the theorem of Bergström (1945) mentioned in the introduction
\[ H_n(x) = \phi(x; \Psi_n) + R_n(x), \quad x \in \mathbb{R}^p, \]

(4.7)

\[ |R_n(x)| \leq \Delta_n \kappa_n^{-1/2}, \quad x \in \mathbb{R}^p, \]

for all large \( n \). We are now in a position to exploit (4.6). By (4.7)

\[
\int_{z(n)}^\infty \int \exp(-y^t w^{(n)} \lambda_n) dR_n(y)
\]

(4.8)

\[
= \lambda_n^p \prod_{i=1}^p w_i^{(n)} \int_{z(n)}^\infty \int \exp(-y^t w^{(n)} \lambda_n) dR(y) - R_n(z(n)) \exp(-y^t w^{(n)} \lambda_n) dy
\]

\[ = o(\lambda_n^{-p}), \quad \text{as} \quad n \to \infty. \]

To estimate the integral with respect to \( \phi(\cdot; \Psi_n) \) we observe first that from Lemma 4.1, Taylor's Theorem and the fact that \( \hat{z}^{(n)}_w \equiv \mu \) we have

\[ |z_i^{(n)}| \leq \Delta_n^2 \kappa_n^{-1/2}, \quad i = 1, \ldots, p, \quad n \geq 1, \]

from which it follows readily that as \( n \to \infty \)

(4.9)

\[
\left| \left\{ \int_{z(n)}^\infty \int - \int_0^\infty \int \exp(-y^t w^{(n)} \lambda_n) d\phi(y; \Psi_n) \right\} \right| = o(\lambda_n^{-p}).
\]

Since also by (1.2) and Lebesgue's Dominated Convergence Theorem

\[
\lambda_n^p \int_0^\infty \int \exp(-x^t w^{(n)} \lambda_n) \phi(x; \Psi_n) dx
\]

(4.10)

\[
= \int_0^\infty \int \exp(-x^t w^{(n)} \lambda_n^{-1}) \phi(x; \Psi_n) dx \to \phi(0; \Psi_n) \prod_{i=1}^p w_i \quad \text{as} \quad n \to \infty
\]

as \( n \to \infty \), we find
\[ \Pr(S_1^{(n)} > \lambda_n u_1, \ i=1, \ldots, p) \]
\[ \sim \exp(-k_n \theta(n, t(n)) M_n(t) \int_0^\infty \int \exp(-y' W(n)^{\lambda_n} d\phi(y; \Psi_n)) \]
\[ = \exp(-k_n [L_n(t(n)) \theta(n, t(n)) + \frac{1}{2} t(n)' \Psi_n t(n)]) \int_0^\infty \int d\phi(y; \Psi_n), \]

as \( n \to \infty \). Now by (4.3), (4.5), Lemma 4.1, and the Mean Value Theorem

\[ |L_n(t(n)) \theta(n, t(n)) + \frac{1}{2} t(n)' \Psi_n t(n)| \leq \Delta k \lambda_n^{3/2} = o(k_n^{-1}); \]

therefore the theorem would follow from

(4.11) \[ \int_{\lambda_n \Psi_n}^{\infty} d\phi(y; \Psi_n) \sim \int_{\lambda_n \Psi_n}^{\infty} d\phi(y; \Psi_n), \ 	ext{as} \ n \to \infty. \]

To prove (4.11) we prove a lemma which will also be useful in applications of the theorem.

**Lemma 4.2.** Let \( \{B_n^i : n = 0, 1, 2, \ldots \} \) be a sequence of covariance matrices for which \( B_n \to B_0 \) (pos. def.); let \( v = (v_1, \ldots, v_p)' \) be an admissible vector with respect to \( B_0 \); and let \( \lambda_n \to \infty \) as \( n \to \infty \). Then

\[ \int_{v^{\lambda_n}}^{\infty} d\phi(x; B_n) \sim \frac{[B_0]}{\left(\sqrt{2n}\right)^p \lambda_n^p} \left( \frac{1}{\prod_{i=1}^{p} \left( \sum_{j=1}^{p} v_j B_0^{-1} v_j \right)} \right), \ 	ext{as} \ n \to \infty. \]

**Proof.** For every \( n \) we have

\[ \int_{v^{\lambda_n}}^{\infty} d\phi(x; B_n) = \exp(- \frac{1}{2} v' B_n^{-1} v) \int_0^{\infty} \int \exp(-\lambda_n v' B_n^{-1} x) d\phi(x; B_n). \]
Moreover, for $n$ sufficiently large we have (as $B_n \to B_0$ as $n \to \infty$)

$$
\sum_{i=1}^{p} b_{i,j} v_{j} \geq \delta_{i} > 0 , \\
x'B_n^{-1}x \leq \Delta \|x\|^2 , \quad x \in \mathbb{R}_p .
$$

Therefore for $n$ large

$$
|\lambda_n^p \int_0^{\infty} \int_0^{\infty} \exp(-\lambda_n v'B_n^{-1}x) d\phi(x;B_n) - \frac{|B_n|^{-\frac{1}{2}}}{(2\pi)^{\frac{p}{2}}} \int_0^{\infty} \exp(-v'B_n x) dx |
$$

$$
\leq \frac{|B_n|^{-\frac{1}{2}}}{(2\pi)^{\frac{p}{2}}} \int_0^{\infty} (1 - \exp(-\frac{1}{2\lambda_n^2} \Delta \|x\|^2)) \exp(-\delta_{i} \sum_{i=1}^{p} x_i) dx
$$

which tends to zero as $n \to \infty$ by Lebesgue's Dominated Convergence Theorem. The Lemma is an easy consequence since clearly

$$
\int_0^{\infty} \int_0^{\infty} \exp(-v'B_n^{-1}x) dx \to \int_0^{\infty} \int_0^{\infty} \exp(-v'B_0^{-1}x) dx
$$

$$
= \left( \sum_{i=1}^{p} \sum_{j=1}^{p} b_{i,j} \delta_{i,j} \right)^{-1} , \text{as } n \to \infty .
$$

(4.11) follows easily from Lemmas 4.1 and 4.2, the Mean Value Theorem, and the fact that $\chi_{n,n}^{(n)} = \mu$ since

$$
\int \int \Phi(x;\Phi_n^p) \sim \frac{|\chi_0^p|^{-1/2} e^{-\lambda_0^p(n) \Phi(W_{n}^p)}}{(2\pi)^{p/2} \prod_{i=1}^{p} (\Sigma_{j=1}^{p} \mu_j \sigma_{i,j}^2)}
$$

$$
\sim \frac{1}{2} e^{-\lambda_0^p(n) \chi_{n,n}^{(n)}}
$$

$$
\frac{1}{(2\pi)^{p} \prod_{i=1}^{p} (\Sigma_{j=1}^{p} \mu_j \sigma_{i,j}^2)} \sim \int \int d\phi(y;\chi_n^p) .
$$
Theorem 4.2. Let \( \{X_{n,k}\} \) be i.i.d. by rows and satisfy (1.2), (4.1), (3.1) with \( \alpha = 0 \) and \( s = p+3 \), and condition \( C_1 \); let \( \mu \) be an admissible vector (w.r.t. \( Z_{i}^o \)); and let \( \lambda_n \to \infty \) with \( \frac{\lambda_n^{p-1}}{n} = o(k_n) \) as \( n \to \infty \). Then as \( n \to \infty \)

\[
\Pr[k_n^{-1/2} S_i(n) \geq \mu_i \lambda_n, i=1,\ldots,p] \\
\sim \int_{\mu \lambda_n}^{\infty} d\phi(x; \frac{Z_{i}^o}{n}) \left[ 1 + \sum_{k=1}^{p} q(n,k)n^{-\frac{k}{2}} \right].
\]

where for \( n = 1,2,\ldots, P_{n,1}(\cdot), \ldots, P_{n,p}(\cdot) \) are obtained by expanding \( \psi_n(\cdot) \) as in Lemma 2.1 and

\[
q(n,k) = \frac{1}{\phi(0; \frac{Z_{i}^o}{n})} \frac{\partial^p}{\partial x_1\cdots \partial x_p} \left| \frac{P_{n,k}(-\phi(x; \frac{Z_{i}^o}{n}))}{x=0} \right|.
\]

Proof. Observe first that (3.1) with \( \alpha = 0 \) implies (1.3). We proceed exactly as in the proof of the previous theorem until we reach (4.7) where we write

\[
H_n(y) = \phi'_{n,s}(y) + R_n'(y),
\]

(4.12)

\[
R_n'(y) \leq \Delta k_n^{p-1} (\log k_n)^{p-1},
\]

where

(4.13) \[
\phi'_{n,s}(\cdot) = \phi(\cdot; \Psi_n) + \sum_{k=1}^{p} P_{n,k}(-\phi(\cdot; \Psi_n)) n^{-\frac{k}{2}},
\]

and \( P_{n,1}(\cdot), \ldots, P_{n,s-3}(\cdot) \) are obtained by expanding the c.f.

\[
\psi_n(t) = \psi_n(t-it^{(n)})/M_n(t^{(n)}), t \in \mathbb{R}_p,
\]

20
as in Lemma 2.1. The fact that \( \{\psi_n(r)\} \) satisfies condition \( C_1 \), incidentally, is an easy consequence of

\[
M_n(t^{(n)}) \to 1, \text{ as } n \to \infty,
\]

which is true because of (4.5), (1.3) is satisfied for the same reason. Now by (2.12), (4.12) and (3.1) with \( \alpha = 0 \), (4.8) and (4.9) are true with \( R_n'(\cdot) \) and \( \phi_{n,n}(\cdot) \) replacing \( R_n(\cdot) \) and \( \phi(\cdot; \Psi_n) \) respectively. Moreover,

\[
\lambda_n \int_0^{\infty} \int \exp(-x'w(n)_{\lambda_n}) \frac{\partial^p}{\partial y_1 \ldots \partial y_p} P_{n,k}(-\varphi(y; \Psi_n)) \big|_{y=x} \, dx
\]

\[
\left( -\frac{1}{p} \prod_{i=1}^p w_i(n) \right) \left. \frac{\partial^p}{\partial y_1 \ldots \partial y_p} P_{n,k}(-\varphi(x; \Psi_n)) \right|_{x=0}
\]

\[
\leq \int_0^{\infty} \int \left| \frac{\partial^p}{\partial y_1 \ldots \partial y_p} P_{n,k}(-\varphi(y; \Psi_n)) \right|_{y=x \lambda_n^{-1}} \cdot \exp(-x'w(n)) \, dx
\]

which tends to zero as \( n \to \infty \) since

\[
\int_0^{\infty} \int \sum_{i=1}^p |x_i|^s(\exp(t^{(n)}x) - 1) dF_n(x)
\]

\[
\leq \Delta(\exp(\lambda_n \Lambda_n k_n^{-1/2} \sum_{i=1}^p w_i^{(n)}) - 1) \to 0, \text{ as } n \to \infty,
\]

and the coefficients of \( P_{n,k}'(\cdot) \) and \( P_{n,k}(\cdot) \) are polynomials in the moments of \( H_n(\cdot) \) and \( \Gamma_n(\cdot) \) respectively. It follows that
\[
\int_0^\infty \int \exp(-y'_w(n)\lambda_n) d\phi_n(s, y) \\
\sim \int_0^\infty \int \exp(-y'_w(n)\lambda_n) d\phi(y, \bar{V}_n) \{1 + \sum_{k=1}^p q(n, k) n^{-\frac{k}{2}}\}
\]
as \(n \to \infty\). The remainder of the proof is a verbatim repetition of that of Theorem 4.1.

5. Remarks and Extensions.

The requirement (4.1) that the T.A. \(X^{(n,k)}\) consist of bounded random variables is by no means necessary. Indeed, if we let \(S(\alpha)\) be the set of \(t \in \mathbb{R}_p\) for which \(|t_i| \leq \alpha^i, i=1, \ldots, p\) then (4.1) may be replaced (in both Theorems 4.1 and 4.2) by

\[
M_n(t) \leq \Delta, t \in S(\alpha^*_n), \quad n = 1, 2, \ldots,
\]

(5.1)

\[1 \leq \alpha_n^{-2} = o(k_n^\gamma), \quad \gamma < 1, \quad n \to \infty,
\]

provided that the growth of \(\{\lambda_n\}\) is restricted to

(5.2a) \[\lambda_n^2 [\alpha_n^{-2} + \lambda_n^2] = o(k_n^\gamma), \quad \text{in Theorem 4.1},\]

(5.2b) \[\lambda_n^6 [\alpha_n^{-2} + \lambda_n^2] = o(k_n^\gamma), \quad \text{in Theorem 4.2},\]
as \(n \to \infty\).
To see this in the case of Theorem 4.1, let \( 2\beta = 1 - \gamma \) and

\[
A_n = \max \left\{ \frac{1}{2} \alpha_n^{-1}, \lambda_n \left( \log \lambda_n \right), \kappa_n^\beta \right\},
\]

\[
Y_{i_1}^{(n,k)} = \begin{cases} 
X_{i_1}^{(n,k)} : |X_{i_1}^{(n,k)}| \leq A_n \\
0 : \text{ Otherwise}
\end{cases}, \quad i=1, \ldots, p,
\]

\[
Z^{(n,k)} = Y^{(n,k)} - E[Y^{(n,k)}]
\]

for \( k = 1, \ldots, k_n \) and \( n = 1, 2, \ldots \). Then \( \{Z^{(n,k)}\} \) will satisfy the hypotheses of Theorem 4.1 and therefore, denoting the covariance matrix of \( Z^{(n,1)} \) by \( \Psi_n \) and \( k_n^{-1/2} \sum_{k=1}^{k_n} Z^{(n,k)} \) by \( T(n) \) for \( n \geq 1 \), we have

\[
\Pr(T_{i_1}^{(n)} \geq \mu_{i_1} \lambda_n) \sim \int_{\mu \lambda_n}^{\infty} \cdots \int d\Phi(x; \Psi_n)
\]

as \( n \to \infty \). And since

\[
|E[Y_{i_1}^{(n,k)}]| \leq \Delta \exp(-A_n^2), \quad i=1, \ldots, p,
\]

(5.3)

\[
\max_{i_1,j} |\psi_{n,i,j} - \sigma_{n,i,j}| \leq \Delta \exp(-A_n^2)
\]

for \( n \geq 1 \) by (5.1), we have

\[
|\Pr(T_{i_1}^{(n)} \geq \mu_{i_1}, i=1, \ldots, p) - \Pr[k_{n_1}^{-1/2} S_{i_1}^{(n)} \geq \mu_{i_1}, i=1, \ldots, p]| 
\]

\[
\leq \sum_{k=1}^{k_n} \Pr(X^{(n,k)} \neq Y^{(n,k)})
\]

\[
+ \frac{p}{i=1} \Pr(\left| T_{i_1}^{(n)} - \mu_{i_1} \right| \leq 2 \sqrt{k_n} |E[Y_{i_1}^{(n,1)}]|)
\]
which is easily seen to be

$$o(\lambda_n^{-p} \exp(- \frac{1}{2} \lambda_n^2 \mu_n^\prime \mu_n)), \quad \text{as } n \to \infty,$$

by (5.1), (5.3), and Theorem 4.1. Finally, from (5.3) and Lemma 4.2 we may infer

$$\int_{\lambda_n^{\mu_1} \ldots \lambda_n^{\mu_l}} d\phi(x; \overline{V}_n) \sim \int_{\lambda_n^{\mu_1} \ldots \lambda_n^{\mu_l}} d\phi(x; \overline{V}_n), \quad \text{as } n \to \infty,$$

and the assertion follows.

From Theorem 4.2 we must also check (3.1) (with $\alpha = 0$) and condition $C_1$ for the $\{Z^{(n,k)}\}$ array assuming them for the $\{X^{(n,k)}\}$ array. (3.1) is obvious and $C_1$ follows easily from it.

The above remarks by no means exhaust the possibilities of using the theorems of the previous section with a truncation procedure. They were offered as an illustration.
REFERENCES


ASYMPTOTIC EXPANSIONS AND LARGE DEVIATIONS IN MANY DIMENSIONS

Technical Report

Woodroofe, Michael, B.

August 15, 1966

Technical Report No. 118

Distribution of this document is unlimited

Abstract

Let \( \{X(n,k): k=1, \ldots, k_n, n=1,2, \ldots\} \) be a triangular array of independent, identically distributed, \( p \)-dimensional random vectors and let \( S(n) = \sum_{k=1}^{k_n} X(n,k) \). (1) An asymptotic expansion for the distribution function of \( \frac{1}{\sqrt{n}} S_n \) is given which is valid under regularity conditions which are the natural analogues of those required in the one-dimensional case. For a wide class of vectors \( \mu = (\mu_1, \ldots, \mu_n)' \), \( \Pr[\frac{1}{\sqrt{n}} S_n \geq \lambda \mu] \), \( \lambda = 1, \ldots, p \) is estimated under the assumptions that \( |X(n,1)| \leq A_n \) \( \nu \cdot p \) one and \( \frac{\lambda^2 d}{n A_n} = o(k_n) \) as \( n \to \infty \) where \( q = \max(3,p) \). It is shown that by using truncation the assumption that of boundedness may be dropped.
Edgewood Series Expansions
Large Deviations

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