UNBIASED HYPOTHESIS TESTS FOR FREE-RECALL VERBAL LEARNING

BY

MICHAEL WOODROOFFE

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1. Introduction and Summary.

In 1952 Miller and McGill proposed a model for free recall verbal learning which describes a subject by three parameters, a initial probability of recall $p$, a learning rate parameter $\gamma$, and a learning asymptote $\lambda$. (See also Bush and Mosteller (1955) for a discussion of this model.) Ten years later Lorge and Solomon (1962) used the Miller-McGill model to study the relation between group performance and the performance of the individuals which comprise the group. During the course of their investigation (Lorge and Solomon (1962)), a question arose as to the significance of differences between some estimated group $\gamma$'s and products of the estimated individual $\gamma$'s. The present paper is essentially an outgrowth of this question.

One way of answering the question would be to calculate the sampling distributions of the estimates involved. However, as Solomon (1964) points out, the distribution problem is distinctly non trivial and has, to the best of the author's knowledge, remained unsolved. Another approach, the one to be followed in this paper, is through the general theory of hypothesis testing. Indeed, the Miller-McGill model lends itself quite readily to this approach, and we have been able to find optimal tests of, not only the Lorge-Solomon hypothesis, but also a number of other interesting hypotheses which arise from it. To describe our results more fully we must first review the model and introduce some terminology.
The experiment and model are briefly as follows. On each of a
sequence of trials a subject is shown the same list of I words in a
random order and asked to write down as many as he can recall; his
response is recorded and the experiment proceeds to the next trial.
The mathematical model assumed is the following. Define random variables
\( Z_{i,n} \) by

\[
0: \text{ if the word } i \text{ is recalled on the } n\text{th trial}
\]

\[
Z_{i,n} =
\]

\[
1: \text{ otherwise}
\]

for \( i = 1, \ldots, I \) and \( n = 1, 2, \ldots \). It is assumed that the subject's
responses to the different words are independent of each other — i.e.
that the families of random variables

\[
\{Z_{1,n}: n \geq 1\}, \ldots, \{Z_{I,n}: n \geq 1\}
\]

are mutually independent. It is also assumed that for each \( i, 1 \leq i \leq I, \{Z_{i,n}: n = 1, 2, \ldots\} \) is a Markov process with

\[
\Pr(Z_{i,1} = 1) = p
\]

(1.1)

\[
\Pr(Z_{i,n+1} = 1|Z_{i,n} = 1) = \Pr(Z_{i,n} = 1)
\]

\[
\Pr(Z_{i,n+1} = 1|Z_{i,n} = 0) = \gamma \Pr(Z_{i,n} = 1)
\]

for \( n = 1, 2, \ldots \). Here \( \gamma \) and \( p, 0 < \gamma, p < 1 \) are unknown parameters
concerning which we wish to make inferences. We assume the learning
asymptote to be one, thus simplifying the model considerably. This
assumption appears to be satisfied by the Lorge-Solomon data (Solomon
(1964)).
The statistical terminology used below may be found in any honest introduction to statistical theory -- e.g. Hogg and Craig (1965). A statistical test is unbiased iff its power at any alternative is at least equal to its significance level; it is uniformly most powerful (U.M.P.), unbiased iff at every alternative its power exceeds or equals that of any other unbiased test. The property of unbiasedness seems desirable when one is dealing with multi-parameter hypotheses in which the importance of one parameter dominates that of the other(s).

In section two of the present paper, we develop U.M.P., unbiased tests for composite hypotheses of the form

\[(1.2a) \quad H: \gamma = \gamma_0, \quad 0 < p < 1, \]

where \(\gamma_0\) is specified, against the unrestricted alternatives

\[(1.2b) \quad K: \gamma \neq \gamma_0, \quad 0 < p < 1, \]

for the idealized case in which the entire sequence of trials is observed. We also describe the related confidence intervals for \(\gamma\), which will probably be of more interest in applications than the test itself. While the idealized case can, of course, never be realized, it should be approximated rather closely if the experiment is continued until the learning asymptote is reached.

In section 3 we consider problems of comparison. First we give the U.M.P., unbiased test, again in the idealized case, of

\[(1.3) \quad H: \gamma_1 \leq \gamma_2, \quad 0 < p_1, p_2 < 1 \]

\[K: \gamma_1 > \gamma_2, \quad 0 < p_1, p_2 < 1\]
where \( \gamma_i \) and \( p_i \), \( i=1,2 \) are the parameters from two experiments of the type described above. (1.3) would be of interest if, for example, one wished to assess the effect of some treatment on the learning parameter \( \gamma \). Finally, we consider the problem of comparing individual and group performance in learning experiments of the type described above. In particular, the U.M.P., unbiased test is found, again in the idealized case, for a generalized version of the hypothesis considered by Lorge and Solomon. A procedure for approximating the tests of sections 2 and 3 is outlined in section 4, and the necessary formulae are listed in an appendix.

Repeated use will be made below of a general theorem of statistical theory, Theorem 3 of Chapter 4, Lehmann (1959), which describes the optimal, unbiased hypothesis tests for exponential families of distributions (Lehmann (1959), pp. 134-138). Indeed, most of sections 2 and 3 of the present paper are devoted to showing that the assumptions of this theorem, which will be referred to as Theorem 3 hereafter, are satisfied and to describing the resulting tests. Also we have and will continue to use the relationship between hypothesis tests and confidence sets discussed in Lehmann's Book.

2. The case of a single experiment.

In this section we derive the U.M.P., unbiased test of (1.2) and describe the related confidence intervals for \( \gamma \) in the case of a single experiment. Toward this end it is convenient to transform the parameter space as follows: let
\[ \omega_1 = \log p \]
\[ \omega_2 = \log \gamma . \]

The new parameter space is then

\[ \Omega = \{\omega = (\omega_1, \omega_2) : -\infty < \omega_1, \omega_2 < 0 \} . \]

Define

\[ Z_i = (Z_{i,1}, Z_{i,2}, \ldots), \ i=1,\ldots,I \]
\[ Z = (Z_1, \ldots, Z_I) , \]

and let \( \tilde{\mathcal{I}} \) denote the common sample space of \( Z_1, \ldots, Z_n \). Thus \( Z \) is a random finite, ordered, array of infinite ordered sequences of zeros and ones, \( \tilde{\mathcal{I}} \) is the space of all such sequences and

\[ \tilde{\mathcal{I}} = \{z=(z_1, \ldots, z_I) : z_i \in \tilde{\mathcal{I}}, i=1,\ldots,I\} , \]

the Cartesian product of \( \tilde{\mathcal{I}} \) with itself \( I \) times, is the sample space of \( Z \). Let \( P_{\omega} \{ \cdot \} \) denote the probability distribution induced in \( \tilde{\mathcal{I}} \) by (1.1) and (2.1) and let \( \Phi \) be the family of all such distributions as \( \omega \) varies over \( \Omega \). We wish to demonstrate that \( \Phi \) is an exponential family (Lehmann (1959), pp. 134-138), for then we will be able to apply the powerful theorem mentioned in the introduction to (1.2).

An important role in this demonstration will be played by the waiting times \( Y_{i,j} \) \( i=1,\ldots,I, \ j=0,1,2,\ldots \) defined by

\[ Y_{i,0} = Y_{i,0}(Z) = \text{least non-negative integer } y \text{ such that } Z_{i,y+1} = 0 \]
\[ Y_{i,j+1} = Y_{i,j+1}(Z) = \text{least non-negative integer } y \text{ such that } Z_{i,[(Y_{i,j})+y+1]} = 0 . \]
Thus $Y_{i,j}$ is the number of trials on which the $i^{th}$ word is not recalled that occur between its $j^{th}$ and $(j+1)^{th}$ recalls. It is easily seen that the random variables $Y_{i,j}$, $i=0,\ldots,I$, $j=0,1,2,\ldots$ are completely independent and that each $Y_{i,j}$ has the geometric distribution with parameter $(\gamma^j)$ -- i.e.

$$(2.2) \quad P_\omega[Y_{i,j} = y_{i,j}] = (1-\gamma^j p)(\gamma^j p)^{y_{i,j}}$$

for $i=1,\ldots,I$ and $j=0,1,2,\ldots$. In (2.2) we have tacitly introduced two conventions which we will follow throughout the remainder of the paper. Namely, we have suppressed the dependence of the $Y_{i,j}$'s on $Z$, and we have distinguished random variables from the values they assume at a particular $z \in Z$ by using capital and lower case letters respectively. Let $Z_{\omega}$ be the set of $z \in Z$ for which only finitely many of the numbers $y_{i,j} = Y_{i,j}(z)$, $i=1,\ldots,I$, $j=0,1,2,\ldots$ differ from zero. Then it is an easy consequence of (2.2) and the Borel-Cantelli Lemmas (Feller (1959), p. 188) that $P_\omega(Z_{\omega}) = 1$, for all $\omega \in \Omega$; for

$$P_\omega[Y_{i,j} \neq 0] = (\gamma^j p) ,$$

$$\sum_{j=0}^{\infty} \gamma^j p = p(1-\gamma)^{-1} < \infty .$$

Since $z \in Z_{\omega}$ implies $z_{i,j} = 0$, $i=1,\ldots,I$, for all but a finite number of $j$, $\omega \in Z_{\omega}$ is clearly countable. Thus, the measure-theoretic requirements in the definition of an exponential family will be satisfied.

Moreover,
\[ P_w(z) = \prod_{i=1}^{\infty} \prod_{j=0}^{\infty} (py_j)^{y_{i,j}}(1-ry_j) \]
\[ = \Phi_s \exp(\omega_1 s + \omega_2 t) \]

where \( \Phi = \prod_{j=0}^{\infty} (1-ry_j) \) and

\[ s = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} y_{i,j} \]

\[ t = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} j y_{i,j} \]

This shows that \( \Theta \) is an exponential family.

Now consider the hypothesis (1.2). In terms of the new parameterization (2.1) it becomes

\[ H: \ \omega_2 = \omega_{2,0}, \ \infty < \omega_1 < 0 \]

\[ \text{vs. } K: \ \omega_2 \neq \omega_{2,0}, \ \infty < \omega_1 < 0 \]

where \( \omega_{2,0} = \log \gamma_0 \). In view of (2.3) it follows directly from

Theorem 3 that for any significance level \( \alpha, \ 0 < \alpha < 1 \), there exists

a U.M.P. unbiased, level \( \alpha \) test, \( \Psi \) say, of (2.5) which may be

described as follows. Let \( U \) be a uniformly distributed random variable

over \([0,1]\) which is independent of the experimental outcome \( Z \) and

let \( u \) denote an observation on it. The value \( u \) may be obtained from

a table of random numbers, for example. Then for each \( s \) there exists

numbers \( c_1(s;\omega_{2,0}) \) and \( c_2(s;\omega_{2,0}) \) for which
\( (2.6) \quad \psi = \psi(s, t, u) = \begin{cases} 0: & \text{if } c_1(s; \omega_{2,0}) \leq t + u \leq c_2(s; \omega_{2,0}) \\ 1: & \text{otherwise} \end{cases} \)

where \( \psi = 0 \) means accept \( H \) and \( \psi = 1 \) means reject \( H \). The cut off points \( c_1(s; \omega_{2,0}) \) and \( c_2(s; \omega_{2,0}) \) are determined by the conditions

\[
E_{\omega_{2,0}} (\psi(s, T, U) | S = s) = \alpha
\]

\[
E_{\omega_{2,0}} (T | S = s) = \alpha E_{\omega_{2,0}} (T | S = s)
\]

where \( S \) and \( T \) are the random variables

\[
S = \sum_{i=1}^{I} \left\{ \sum_{j=0}^{\infty} Y_{i, j} \right\} = \sum_{i=1}^{I} S_{i},
\]

\[
T = \sum_{i=1}^{I} \left\{ \sum_{j=1}^{\infty} j \cdot Y_{i, j} \right\} = \sum_{i=1}^{I} T_{i},
\]

and \( E_{\omega_{2,0}} (\cdot | S = s) \) denotes conditional expectation given \( S = s \) when the hypothesis (2.4) is true. In (2.7) we have used the fact that when \( \omega_2 = \omega_{2,0} \) \( S \) is sufficient for \( \omega_1 \) (Lehmann (1959), p. 52) so that this conditional expectation does not depend on \( \omega_1 \).

From \( \psi \) we obtain, in the usual way (Lehmann (1959), pp. 175-180), confidence intervals for \( \omega_2 \). In fact, we have

\[
(2.9) \quad P_{\omega_{2}} (\lambda(s; T+U) \leq \omega_2 \leq u(s, T+U) | S = s) = 1 - \alpha
\]
for all $s$ and $\omega_2$. Here $f(s; \cdot)$ is the inverse of the function $c_2(s; \cdot)$, which may be shown to be strictly monotone increasing. (Lehmann (1959), p. 179). Specifically

$$f(s; c_2(s; t)) = t, \text{ for all } t.$$ 

Similarly, $u(s; \cdot)$ is the inverse of $c_1(s; \cdot)$. The confidence interval in (2.9) has two interesting properties. First they are unbiased in the sense that

$$(2.10) \quad P(\omega_1, \omega_2) \{ \omega_2' \in f(S, T+U), u(S, T+U) \} \leq 1-\alpha$$

for all $\omega_1, \omega_2$ and $\omega_1' \neq \omega_2'$. Moreover, it is uniformly most accurate among all unbiased confidence intervals in the sense that for any confidence interval $I(Z)$ which satisfies (2.10), we have

$$P(\omega_1, \omega_2) \{ \omega_2' \in f(S, T+U), u(S, T+U) \} \leq P(\omega_1, \omega_2) \{ \omega_2' \in I(Z) \}$$

for all $\omega_1, \omega_2$ and $\omega_1' \neq \omega_2'$.

Finally, it should be noted that an analogous treatment may be given to the parameter $p$ and to one-sided hypotheses--e.g. $\gamma \leq \gamma_0$.

3. Comparison of Subjects.

Now suppose that instead of one learning experiment we observe two and that we wish to compare some aspect of the performance of the two subjects -- for example, we might wish to compare the learning rates of children from different environments. We would then be interested in hypotheses of the form (1.3). Below we will find the U.M.F. unbiased test of (1.3) under the additional assumptions and with the new notation given in the next paragraph.
Let us agree to superscript the observations and \( \omega \)'s associated with the \( v^{th} \) experiment with the superscript \( (v) \), \( v = 1, 2 \). Thus
\[
\omega_1^{(v)} = \log p_v
\]
is the probability that the \( v^{th} \) subject will not recall a specific work on the first trial, \( y_{1,0}^{(v)} \) is the number of non-recalls of the \( i^{th} \) word before its first recall in the \( v^{th} \) experiment, and
\[
s(v) = \sum_{i=1}^{I} \sum_{j=0}^{\infty} y_{1,j}^{(v)}
\]
\[
t(v) = \sum_{i=1}^{I} \sum_{j=1}^{\infty} y_{1,j}^{(v)}
\]
for \( v = 1, 2 \). We assume that the experiments are performed independently in the sense that if
\[
\omega = (\omega^{(1)}, \omega^{(2)})
\]
and if \( A^{(v)} \subset \Omega \), \( v = 1, 2 \), then
\[
(3.1) \quad P_{\omega}(Z^{(v)} \in A^{(v)}) = \prod_{v=1}^{2} P_{\omega}^{(v)}(Z^{(v)} \in A^{(v)})
\]
which may be paraphrased by such phrases as "the experimental errors in the experiments are independent" and "the experiments are conditionally independent given the parameters."

Let \( \delta = \omega_2^{(1)} - \omega_2^{(2)} \); it is expedient to consider a generalization of (1.3)--namely
\[
H': \delta \leq \delta_0, \quad -\infty < \omega^{(1)}_1, \omega^{(2)}_1, \omega^{(2)}_2 < 0
\]
\[
(3.2) \quad K': \delta > \delta_0, \quad -\infty < \omega^{(1)}_1, \omega^{(2)}_1, \omega^{(2)}_2 < 0
\]
for which we will find a U.M.P. unbiased test. From (2.3) and (3.1) we may readily deduce

\[ P_{\Omega} \{ Z^{(1)} = z^{(1)}, Z^{(2)} = z^{(2)} \} \]

\[ = \frac{2}{1} c(\omega(v)) \exp(\sigma(v)\omega_{\frac{1}{2}} + t(v)\omega_{\frac{1}{2}}) \]

\[ = K(\Omega) \exp(t^{(1)}\delta + \sum_{v=1}^{s} \lambda_{v}v_{v}) \]

where \( K(\Omega) = c(\omega^{(1)}(1))c(\omega^{(1)}(2)) \), and

\[ v_{j} = s(j) \quad \text{and} \quad \lambda_{j} = \omega_{\frac{1}{2}}(j), j=1,2 \]

\[ v_{j} = t^{(1)} + t^{(2)} \quad \text{and} \quad \lambda_{j} = \omega_{\frac{1}{2}}(2). \]

Since (3.3) clearly defines an exponential family as \( \Omega \) varies over \( \Omega \times \Omega \), there exists by Theorem 3 a U.M.P., unbiased, level-\( \alpha \) test \( \varphi \) of (1.3) for any significance level \( \alpha, 0 < \alpha < 1 \). It is given by

\[ \varphi(v, t^{(1)}, u) = \begin{cases} 1: \text{if } t^{(1)} + u \geq c(v; \delta_{o}) \\ 0: \text{if } t^{(1)} + u < c(v; \delta_{o}) \end{cases} \]

where \( u \) is as in (2.6) and \( v = (v_{1}, v_{2}, v_{3}) \). The cut-off point \( c(v; \delta_{o}) \) is determined by the condition

\[ E_{\delta_{o}}(\varphi(v, T^{(1)}, U)|V=v) = 1-\alpha. \]

The confidence set corresponding to (3.5) is one-sided. In fact, we have
\[ P_{\delta}(\delta \leq u_{1}(v; T^{(1)} + U)|V=v) = 1-\alpha \]

for all \( v \) and \( \delta \) where \( u_{1}(v; \cdot) \) is the inverse function to \( c(v; \cdot) \). Moreover, analogues of (2.10) and (2.11) are also true.

The final hypothesis to be considered in this paper is the one relating groups to individuals, which was discussed in the introduction. For this hypothesis we observe four experiments of which 3 are performed with different individuals acting as subjects and the other is performed with the group of all three individuals acting as subject. If we label the individuals as subjects 1, 2, and 3 and label the group as subject 0, then we may continue to use the notation introduced above with the exceptions that now \( v = 0, 1, 2, 3 \) instead of \( v = 1, 2 \) and that now

\[ \omega' = (\omega^{(0)}, \ldots, \omega^{(3)}) . \]

Also, we assume the analogue of (3.1)—namely, that if \( A^{(v)} \leq \gamma \), \( v = 0, \ldots, 3 \) then

\[ (3.6) \quad P_{\omega'}(Z^{(v)} \epsilon A^{(v)}, v=0,\ldots,3) = \prod_{v=0}^{3} P_{\omega(v)}(Z^{(v)} \epsilon A^{(v)}) \]

which may be interpreted as was (3.1).

The hypothesis to be considered is

\[ \gamma^{(o)} = (\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}) \Delta \]

\[ (3.7) \quad \omega^{(o)} = \Delta \sum_{v=1}^{3} \omega^{(v)} \]

where \( \Delta \geq 0 \) is specified. The alternatives are all distributions for which (3.7) is not true. (3.7) may be handled by essentially the same methods as were (1.2) and (1.3). From (2.3) and (3.6) we find
\[ P_{\omega'}(z(v) = z(v), v=0,1,2,3) \]
\[
= \prod_{v=0}^{3} c(\omega(v)) \exp(\omega_1(v) s(v) + \omega_2(v) t(v)) \\
= K_{\omega'} \exp(\sum_{v=0}^{3} \omega_1(v) s(v) + \sum_{v=0}^{3} \omega_2(v) t(v))
\]

where \( K_{\omega'} = \prod_{v=0}^{3} c(\omega(v)) \). If we make the transformation

\[
\theta = \omega_2^{(0)} - \Delta \sum_{v=1}^{3} \omega_2(v)
\]
\[
\lambda_v = \omega_1(v-1), \ v=1,2,3,4
\]
\[
\lambda_v = \omega_2(v-4), \ v=5,6,7
\]
\[
v_j' = s^{(j-1)}, \ j=1,\ldots,4
\]
\[
v_{j+4}' = t^{(j)} + \Delta t^{(0)}, \ j=1,\ldots,3
\]

then (3.7) becomes

\[(3.7') \quad \theta = 0, \]

while the right-hand side of (3.8) becomes

\[(3.8') \quad K_2(\theta, \lambda) \exp(\theta t^{(0)} + \sum_{j=1}^{7} \lambda_j v_j')
\]

where \( K_2(\theta, \lambda) = K_{\omega'} \). It is expedient to test the more general hypothesis

\[(3.7'') \quad \theta = \theta_0 \]

of which (3.7') is a special case. Since (3.8') is again an exponential density, there is for any \( \alpha, 0 < \alpha < 1 \), a U.M.P., unbiased, level-\( \alpha \) test of (3.7'') which is given by
\[ \varphi' = \begin{cases} 
0: & c_1'(v'; \theta_0) < t^{(o)} + u < c_2'(v'; \theta_0) \\
1: & \text{otherwise} 
\end{cases} \]

where \( u \) is as in (2.6) and \( v' = (v'_1, \ldots, v'_n) \). The cut off points \( c_1'(v'; \theta_0) \) and \( c_2'(v'; \theta_0) \) are determined by

\[ \mathbb{E}_{\theta_0}(\varphi'(v', T^{(o)}, U)|V' = v') = \alpha \]

(3.11)

\[ \mathbb{E}_{\theta_0}(T^{(o)} \varphi'(v', T^{(o)}, U)|V' = v') = \alpha \mathbb{E}_{\theta_0}(T^{(o)}|V' = v') \]

for all \( \theta_0 \) and all \( v' \). Here we have let \( V' \) be the random vector whose values are \( v' \), and we have used the fact that the conditional distribution of \( T^{(o)} \) given \( V' = v' \) depends only on \( \theta_0 \).

The confidence intervals corresponding to (3.6) are, unfortunately, confidence intervals for \( \theta \) not \( \Delta \). They may be used to estimate \( \Delta \), however, in the following manner. From the general theorem which led to (2.9) we may infer that for each fixed \( v \)

\[ P_{\theta}(\ell'(v; T^{(o)} + U) \leq \theta \leq u'(v, T^{(o)} + U)|V = v) = 1 - \alpha \]

where \( \ell'(v; \cdot) \) and \( u'(v; \cdot) \) are inverse functions to \( c_2'(v; \cdot) \) and \( c_1'(v; \cdot) \) respectively. One possible estimate of \( \theta \) is then

\[ \hat{\theta} = \frac{1}{2}(\ell'(V, T^{(o)} + U) + u'(V, T^{(o)} + U)) \]

Recalling the definition of \( \theta \), one might now estimate \( \Delta \) by
\[ \Delta = (\hat{\omega}_2^{(o)} - \bar{\theta}) / \sum_{v=1}^{3} \hat{\omega}_2^{(v)} \]

where \( \hat{\omega}_2^{(o)}, \ldots, \hat{\omega}_2^{(3)} \) are estimates of \( \omega_2^{(o)}, \ldots, \omega_2^{(3)} \), respectively. For example, the \( \hat{\omega}_2^{(v)} \)'s could be the maximum likelihood estimates (Bush and Mosteller (1955), Ch. 10) or the midpoints of the confidence intervals (2.9).

4. **Approximations.**

We now consider the approximation of (2.6), (3.5), (3.10), and the related confidence intervals. Our approach is based on some asymptotic expansions. Let

\[
\mu(\omega) = \begin{pmatrix} \mu_1(\omega) \\ \mu_2(\omega) \end{pmatrix} = E \begin{pmatrix} S_1 \\ T_1 \end{pmatrix}
\]

\[
\zeta(\omega) = \begin{pmatrix} \sigma_1^2(\omega) & \sigma_{12}(\omega) \\ \sigma_{21}(\omega) & \sigma_2^2(\omega) \end{pmatrix}
\text{the covariance matrix of } (S_1, T_1)
\]

where \( S_1 \) and \( T_1 \) are as in (2.8). If \( r \) is an integer, \( r \geq 4 \), and if we use the notation \( O(I^{-r}) \) to denote a remainder term \( R_I \) for which \( I^{-r}R_I \to 0 \) as \( I \to \infty \), then as \( I \to \infty \):

\[
\sqrt{IP}_\omega \{ S = s \} = \varphi(y) + \sum_{k=1}^{r-3} P_k \{-\varphi(y); \omega\} \frac{k}{2} + O(I^{-\frac{r-2}{2}})
\]

(4.1)

\[
y = \frac{1}{\sigma_1(\omega) \sqrt{I}} (s - \mu_1(\omega))
\]

uniformly in \( s \); and
\[ IP_{\omega}(S=s, T=t) = \varphi(x; \tilde{\varphi}(\omega)) + \sum_{k=1}^{r-2} Q_k(-\varphi(x; \tilde{\varphi}(\omega)); \omega) \frac{k}{2} + O\left(1 \cdot \frac{r-2}{2}\right) \]

\[(4.2)\]

\[ x = (x_1, x_2) = (t^{-1/2} \cdot (s - \mu_1(\omega)), t^{-1/2} \cdot (t - \mu_2(\omega))) \]

uniformly in \( s, t = 0, 1, 2, \ldots \). Here \( \varphi(\cdot) \) and \( \varphi(\cdot; \tilde{\varphi}) \) denote the standardized, univariate normal density and the standardized, bivariate normal density with covariance matrix \( \tilde{\varphi} \) respectively; the \( P_k(\cdot; \omega) \)'s and \( Q_k(\cdot; \omega) \)'s are polynomials whose coefficients depend on \( \omega \).

\[ P_k(\lambda; \omega) = \sum_{\ell=1}^{k} c_{k, \ell}(\omega) \lambda^{k+2\ell}, -\infty < \lambda < \infty \]

\[ Q_k(\xi; \omega) = \sum_{\ell=1}^{k} \sum_{j=0}^{k+2\ell} c_{k, \ell, j}(\omega) \xi_1^{j} \xi_2^{k+2\ell-j}, \xi \in \mathbb{R}_2; \]

and \( Q_k(-\varphi(x; \tilde{\varphi}(\omega)); \omega) \) and \( P_k(-\varphi(y); \omega) \) are understood in a symbolic sense--i.e.

\[ (-1)^k P_k(-\varphi(y); \omega) = \sum_{\ell=1}^{k} c_{k, \ell}(\omega) \frac{d^{k+2\ell}}{d\lambda} \varphi(\lambda) \big|_{\lambda=y}. \]

\[ (-1)^k Q_k(-\varphi(x; \tilde{\varphi}(\omega)); \omega) = \sum_{\ell=1}^{k} \sum_{j=0}^{k+2\ell} c_{k, \ell, j}(\omega) \frac{\partial^{k+2\ell}}{\partial \xi_1^j \partial \xi_2^{k+2\ell-j}} \varphi(\xi; \tilde{\varphi}(\omega)) \big|_{\xi=x}. \]

\[(4.1)\]

(4.1) follows directly from the fact that the \( S_i \)'s, \( i=1, \ldots, I \) are independent, identically distributed, integer valued random variables. (See the local limit theorem in Kolmogorov and Gnedenko (1954), p. 241)

\[(4.2)\]

is a straightforward generalization of (4.1).

(4.1) and (4.2) may be used to approximate the conditional distribution of \( T \) given \( S \) in the following manner. Let
\[ \sqrt{T} \ p(s; \omega) = \varphi(y) + \sum_{k=1}^{r-3} p_k(-\varphi(y); \omega) R_k \]

\[ (4.3) \]

\[ I q(s, t; \omega) = \varphi(x; \xi(\omega)) + \sum_{k=1}^{r-3} q_k(-\varphi(x; \xi(\omega)); \omega) I \]

where \( y \) and \( x \) are as in (4.1) and (4.2) respectively. Then if \( \omega = (\omega_1, \omega_2) \)

\[ (4.4) \quad p(t|s, \omega) = q(s, t; \omega)/p(s; \omega) \]

will differ from \( P_{\omega_2}(T=t|S=s) \) by an error term which is \( 0(I - \frac{r-2}{2}) \) as \( I \to \infty \). In terms of (4.3) we may also write approximations to \( P_\delta(T(1)=t(1)|V=v) \) and \( P_\delta(T(0)=t(0)|V'=v') \). To see this in the former case we observe that, inverting the transformation (3.4),

\[ P_{\omega}(T(1)=t(1), V_j=v_j, j=1,2,3) \]

\[ = P_{\omega}(T(1)=t(1), S(1)=v_1) P_{\omega}(T(2)=v_2-t(1), S(2)=v_2) \]

\[ = q(v_1, t(1); \omega(1)) q(v_2, v_2-t(1); \omega(2)) + o(I - \frac{r-2}{2}) \]

as \( I \to \infty \). It seems natural, then, to approximate the conditional distribution, \( P_\delta(T(1)=t(1)|V=v) \) by

\[ (4.5) \quad p'(t(1)|v, \omega) = q'(v, t(1); \omega)/\sum_{t=0}^{\infty} q'(v, t; \omega) \]

where

\[ q'(v, t; \omega) = q(v_1, t; \omega(1)) q(v_2, v_2-t; \omega(2)) \]
and $\omega = (\omega(1), \omega(2))$ is any parameter vector for which $\omega_2(1) - \omega_2(2) = 0$.

Similarly, we are led to approximate $P_\theta(T^{(0)} = t^{(0)} | V' = v')$ by

\begin{equation} \tag{4.6} P''(t^{(0)} | v', \omega') = q''(v', t^{(0)}; \omega') \sum_{t=0}^{\infty} q''(v', t; \omega') \end{equation}

where

\[ q''(v', t^{(0)}; \omega') = q(v'_1, t^{(0)}; \omega^{(0)}) \cdot \prod_{j=1}^{3} q(v'_{j+1}, v'_{j+4}; \Delta t^{(0)}; \omega(j)) \]

and $\omega' = (\omega^{(0)}, ..., \omega^{(3)})$ is any parameter vector for which

$\omega_2^{(0)} - \Delta \sum_{v=1}^{3} \omega_2(v) = 0$.

Using the approximations (4.4), (4.5), and (4.6) we may approximate $c_1(s; \omega_2)$ and $c_2(s; \omega_2)$, $c(v; \delta)$ and $c_1(v; \theta)$ and $c_2(v; \theta)$ by a straightforward, if tedious, solution of the equations defining them.

For example, to approximate $c(v; \delta)$ we would solve for $c$ the equation

\[ \sum_{t=0}^{[c] - 1} p'(t | v, \omega) + (c-[c])p'([c] | v, \omega) = 1 - \alpha \]

where $\omega = (\omega(1), \omega(2))$ is any parameter vector for which $\omega_2(1) - \omega_2(2) = 0$, and $[c]$ denotes the greatest integer which is $\leq c$. The confidence intervals may then be obtained by graphical methods.

The perceptive reader will have noticed that the parameters $\omega = (\omega_1, \omega_2)$ $\tilde{\omega} = (\omega(1), \omega(2))$, and $\omega = (\omega^{(0)}, ..., \omega^{(3)})$ appear in (4.4), (4.5) and (4.6), while the conditional distributions of $T$ given $S = s$, $T^{(1)}$ given $V = v$, and $T^{(0)}$ given $V' = v'$ depend only on $\omega_2$, $\delta$ and $\theta$ respectively. There is, consequently, some indeterminacy in the procedure described above. While a bit disconcerting at first, this
indeterminacy is a real boon for two reasons, which we illustrate in the case of (4.4). First, we are free to use any value of $\omega_1$ which we choose, and it may well be the case that a particular choice may be preferable to others. For example, we would caution against choosing $p = \exp(\omega_1)$ close to either 0 or 1 since (4.1) and (4.2) break down as $p \to 0$ or 1. Also, our freedom to choose $\omega_1$ provides a check on the approximation procedure itself. That is to say, we want any two values of $p_1(t|s,\omega)$ computed with the same value of $\omega_2$ and values of $p$ not too close to 0 or 1 to differ by an amount which is negligible. If their difference is not negligible, then something is wrong.

5. **Acknowledgement.**

I wish to thank Dr. Herbert Solomon for suggesting the hypotheses of Section 3.
APPENDIX

In order to actually use the approximation procedure outlined in section 4, the reader will need to know the polynomials appearing in (4.1) and (4.2). In terms of the moments

\[
\beta_\nu(\omega) = E_\omega \left\{ \left( \frac{s_1 - \mu_1(\omega)}{\sigma_1(\omega)} \right) ^\nu \right\}
\]

the first four \( P_k(\cdot;\omega) \)'s are

\[
P_1(\lambda;\omega) = \frac{1}{6} \beta_3(\omega) \lambda^3,
\]

\[
P_2(\lambda;\omega) = \frac{1}{72} \beta_3(\omega)^2 \lambda^6 + \frac{1}{24} (\beta_4(\omega) - 3\beta_2(\omega)^2) \lambda^4,
\]

\[
P_3(\lambda;\omega) = \frac{1}{1296} \beta_3(\omega)^3 \lambda^9 + \frac{1}{144} \beta_3(\omega)(\beta_4(\omega) - 3\beta_2(\omega)^2) \lambda^7,
\]

\[
+ \frac{1}{120} (\beta_5(\omega) \cdot 10\beta_2(\omega) \beta_3(\omega)) \lambda^5
\]

\[
P_4(\lambda;\omega) = \frac{1}{31104} \alpha_3(\omega)^4 \lambda^{12} + \frac{1}{1728} \alpha_3(\omega)^2 (\alpha_4(\omega) - 3\alpha_2(\omega)^2) \lambda^{10}
\]

\[
+ \frac{1}{64} (\alpha_4(\omega)^2 - 6\alpha_4(\omega)\alpha_2(\omega)^2 + 9\alpha_2(\omega)^4) \lambda^8
\]

\[
+ \frac{1}{720} \alpha_3(\omega)(\alpha_5(\omega) - 10\alpha_2(\omega)\alpha_3(\omega)) \lambda^6
\]

\[
+ \frac{1}{25} (\alpha_6(\omega) - 15\alpha_2(\omega)\alpha_4(\omega) - 10\alpha_3(\omega)^2 + 30\alpha_2(\omega)^3) \lambda^6.
\]

The first four \( Q_k(\cdot;\omega) \)'s may be obtained from the first four \( P_k(\cdot;\omega) \)'s as follows: Let

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\[ \alpha(v_1, v_2; \omega) = E_{\omega} \left( (s_1 - \mu_1(\omega))^{v_1} (T_1 - \mu_2(\omega))^{v_2} \right) \]

\[ \alpha_v(\xi; \omega) = E_{\omega} \left( (s_1 - \mu_1(\omega)) + \xi_2 (T_1 - \mu_2(\omega)) \right)^v \]

\[ = \sum_{r=0}^{v} \binom{v}{r} \alpha(r, v-r; \omega) \xi_1^{v-r} \xi_2^r \]

for \( \xi = (\xi_1, \xi_2) \in \mathbb{R}_2 \). Then \( Q_k(\xi; \omega) \) is obtained from \( P_k(1; \omega) \) by replacing \( \beta_v(\omega) \) with \( \alpha_v(\xi; \omega) \). If one wished to approximate the marginal distribution of \( T \), then he would replace \( S, s, \mu_1(\omega), \) and \( \beta_v(\omega) \) by \( T, t, \mu_2(\omega), \) and \( \beta'_v(\omega) \) respectively where

\[ \beta'_v(\omega) = E_{\omega} \left( \frac{T_1 - \mu_2(\omega)}{\sigma_2(\omega)} \right)^v . \]

It remains only to give the first few of the \( \alpha(v_1, v_2; \omega) \)'s; they are given by

\[ m_{1,j}(\omega) = 0 \]

\[ m_{2,j}(\omega) = \gamma^j (1 - \gamma^j)^2 \]

\[ m_{3,j}(\omega) = (\gamma^2 + \gamma^j)(1 - \gamma^j)^3 \]

\[ m_{4,j}(\omega) = (\gamma^3 + 2\gamma^j)(1 - \gamma^j)^4 \]

\[ m_{5,j}(\omega) = (\gamma^4 + 2\gamma^2 + 2\gamma^j)(1 - \gamma^j)^5 \]

\[ m_{6,j}(\omega) = \frac{\gamma^5 + 5\gamma^4 + 16\gamma^3 + 5\gamma^2 + \gamma^j}{(1 - \gamma^j)^6} \]
\[ \alpha(2-v, v; \omega) = \sum_{j=0}^{\infty} j^v \cdot m_{2, j}(\omega), \quad v = 0, 1, 2, \]

\[ \alpha(3-v, v; \omega) = \sum_{j=0}^{\infty} j^v \cdot m_{3, j}(\omega), \quad v = 0, \ldots, 3 \]

\[ \alpha(4, 0; \omega) = \sum_{j=0}^{\infty} \left( m_{4, j}(\omega) - 3m_{2, j}(\omega) \right)^2 + 3 \left( \sum_{j=0}^{\infty} m_{2, j}(\omega) \right)^2 \]

\[ \alpha(3, 1; \omega) = \sum_{j=0}^{\infty} j \cdot \left( m_{4, j}(\omega) - 3m_{2, j}(\omega) \right)^2 + 3 \left( \sum_{j=0}^{\infty} m_{2, j}(\omega) \right) \left( \sum_{j=0}^{\infty} jm_{2, j}(\omega) \right) \]

\[ \alpha(2, 2; \omega) = \sum_{j=0}^{\infty} j^2 \cdot \left( m_{4, j}(\omega) - 3m_{2, j}(\omega) \right)^2 + 2 \left( \sum_{j=0}^{\infty} jm_{2, j}(\omega) \right)^2 \]

\[ + \left( \sum_{j=0}^{\infty} m_{2, j}(\omega) \right) \left( \sum_{j=0}^{\infty} j^2 m_{2, j}(\omega) \right) \]

\[ \alpha(1, 3; \omega) = \sum_{j=0}^{\infty} j^3 \cdot \left( m_{4, j}(\omega) - 3m_{2, j}(\omega) \right)^2 + 3 \left( \sum_{j=0}^{\infty} jm_{2, j}(\omega) \right) \left( \sum_{j=0}^{\infty} j^2 m_{2, j}(\omega) \right) \]

\[ \alpha(0, 4; \omega) = \sum_{j=0}^{\infty} j^4 \cdot \left( m_{4, j}(\omega) - 3m_{2, j}(\omega) \right)^2 + 3 \left( \sum_{j=0}^{\infty} j^2 m_{2, j}(\omega) \right)^2 \]

\[ \alpha(5, 0; \omega) = \sum_{j=0}^{\infty} \left( m_{5, j}(\omega) - 10m_{2, j}(\omega)m_{3, j}(\omega) \right) + 10 \left( \sum_{j=0}^{\infty} m_{2, j}(\omega) \right) \left( \sum_{j=0}^{\infty} m_{3, j}(\omega) \right) \]

\[ \alpha(4, 1; \omega) = \sum_{j=1}^{\infty} j \cdot \left( m_{5, j}(\omega) - 10m_{2, j}(\omega)m_{3, j}(\omega) \right) + 4 \left( \sum_{j=0}^{\infty} jm_{2, j}(\omega) \right) \left( \sum_{j=0}^{\infty} m_{3, j}(\omega) \right) \]

\[ + 6 \left( \sum_{j=0}^{\infty} m_{2, j}(\omega) \right) \left( \sum_{j=0}^{\infty} jm_{3, j}(\omega) \right) \]
\[ \alpha(3,2;\omega) = \sum_{j=1}^{\infty} j^2 \cdot (m_{j, j}(\omega) - 10m_{2, j}(\omega)m_{3, j}(\omega)) + (\sum_{j=1}^{\infty} j^2 \cdot m_{2, j}(\omega))(\sum_{j=0}^{\infty} m_{3, j}(\omega)) \]

\[ + 6(\sum_{j=1}^{\infty} j \cdot m_{2, j}(\omega))(\sum_{j=1}^{\infty} j \cdot m_{3, j}(\omega)) \]

\[ + 3(\sum_{j=0}^{\infty} m_{2, j}(\omega))(\sum_{j=0}^{\infty} j^2 m_{3, j}(\omega)) \]

\[ \alpha(2,3;\omega) = \sum_{j=1}^{\infty} j^3 \cdot (m_{5, j}(\omega) - 10m_{2, j}(\omega)m_{3, j}(\omega)) + 3(\sum_{j=1}^{\infty} j^2 \cdot m_{2, j}(\omega))(\sum_{j=1}^{\infty} j \cdot m_{3, j}(\omega)) \]

\[ + 6(\sum_{j=1}^{\infty} j \cdot m_{2, j}(\omega))(\sum_{j=1}^{\infty} j^2 m_{3, j}(\omega)) \]

\[ + (\sum_{j=0}^{\infty} m_{2, j}(\omega))(\sum_{j=1}^{\infty} j^3 m_{3, j}(\omega)) \]

\[ \alpha(1,4;\omega) = \sum_{j=1}^{\infty} j^4 \cdot (m_{5, j}(\omega) - 10m_{2, j}(\omega)m_{3, j}(\omega)) + 4(\sum_{j=1}^{\infty} j \cdot m_{2, j}(\omega))(\sum_{j=1}^{\infty} j^2 m_{3, j}(\omega)) \]

\[ + 6(\sum_{j=1}^{\infty} j^2 \cdot m_{2, j}(\omega))(\sum_{j=1}^{\infty} j^2 m_{3, j}(\omega)) \]

\[ \alpha(0,5;\omega) = \sum_{j=1}^{\infty} j^5 \cdot (m_{5, j}(\omega) - 10m_{2, j}(\omega)m_{3, j}(\omega)) + 10(\sum_{j=1}^{\infty} j^2 \cdot m_{2, j}(\omega))(\sum_{j=1}^{\infty} j^3 m_{3, j}(\omega)) \]
REFERENCES


In 1952 Miller and McGill proposed a model for free recall verbal learning which describes a subject by three parameters, a initial probability of recall \( p \), a learning rate parameter \( \gamma \), and a learning asymptote \( \lambda \). (See also Bush and Mosteller (1955) for a discussion of this model.) Ten years later Lorge and Solomon (1962) used the Miller-McGill model to study the relation between group performance and the performance of the individuals which comprise the group. During the course of their investigation (Lorge and Solomon (1962)), a question arose as to the significance of differences between some estimated group \( \gamma \)'s and products of the estimated individual \( \gamma \)'s. The present paper is essentially an outgrowth of this question.
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