FORMAL BAYES ESTIMATION WITH APPLICATION TO A
RANDOM EFFECTS ANALYSIS OF VARIANCE MODEL

BY
STEPHEN L. PORTNOY

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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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Introduction

In this paper we will consider some techniques of formal Bayes estimation and apply them to the estimation of components of variance in the one way layout random effects model of the analysis of variance. In particular, we will consider the following problem in canonical form: we observe

\[ \bar{Y} \sim \mathcal{N}(\mu, (\sigma_e^2 + J\sigma_a^2)/IJ), \quad S_1 \sim \sigma_e^2 \chi^2_{I(J-1)} \text{, and} \quad S_2 \sim (\sigma_e^2 + J\sigma_a^2)\chi^2_{I-1} \]

where \( I \) and \( J \) are positive integers (the number of treatments and replications respectively), \( \mu \) is real, and \( \sigma_e \) and \( \sigma_a \) are positive. We want to find estimates for \( \sigma_e \) and \( \sigma_a \) using essentially mean squared error as a measure of performance.

The problem of estimating \( \sigma_a \) and \( \sigma_e \) is not new, and minimum variance unbiased estimates and maximum likelihood estimates are well known. However, one can look for improvements; and in the estimation of \( \sigma_a \) a special problem arises: the unbiased estimate may be negative and the maximum likelihood estimate may be exactly zero. This particular problem has been considered recently by a number of investigators (see, for example, Scheffe (1961) and Thompson (1962)) and various interpretations for such estimates have been suggested. However, in problems where estimates are really desired, use of such estimators seems to me to be unacceptable. To solve this problem we will consider the use of formal Bayes estimators (i.e. Bayes estimators versus priors which are not necessarily finite), which will be strictly positive. We will show that certain formal Bayes estimators both of \( \sigma_a \) and \( \sigma_e \) have good
mean squared error properties and can be seriously recommended.

Considerations of \( \sigma_e \) and \( \sigma_a \) from Bayesian viewpoint have
been recently presented by Box and Tiao (1967), Hill (1965), Stone
and Springer (1965), and Tiao and Tan (1965). Tiao and Tan and other
authors have used the prior \( \frac{\sigma_e}{\sigma_e + J \sigma_a} \) and computed the posterior distri-
butions directly. We will see here that using an invariant loss function
with normalizing constant \( \frac{1}{(\sigma_e + J \sigma_a)^2} \) and computing the Bayes estimator
is the same as computing the posterior mean with respect to the prior
\( \frac{1}{(\sigma_e + J \sigma_a)^3} \). Thus, one should actually consider posterior distributions
with respect to the prior \( \frac{1}{(\sigma_e + J \sigma_a)^3} \) instead of \( \frac{1}{\sigma_e + J \sigma_a} \) if one wants
the posterior distribution to be properly centered. That is, using an
invariant prior in problems that are not truly invariant might very well
lead to erroneous conclusions.

The techniques used here are special cases of more general consider-
ations applicable whenever the statistical problem is invariant under a
group of transformations which does not act transitively on the parameter
space (i.e. in problems where there is not a unique best invariant
procedure). The analysis of variance problem considered here is easily
seen to be invariant under location and scale changes (if invariant
loss functions are used); that is, under transformations on the parameter
space \((\mu, \sigma_e, \sigma_a) - \infty < \mu < \infty, \sigma_e > 0, \sigma_a > 0\) described by

\[ \mu \to a \mu + b, \quad \sigma_e \to a^2 \sigma_e, \quad \sigma_a \to a^2 \sigma_a. \]

However, here invariance only reduces the parameter space to the range
of the maximal invariant which we will take to be \( \gamma = \frac{\sigma_e}{\sigma_e + J \sigma_a} \). In
general, any invariant procedure will have a risk function which is a function only of the maximal invariant (in our case, the risk will be a function of $\gamma$). Thus, we can assume that there is a prior distribution on the space of the maximal invariant and ask for the invariant procedure minimizing the expected risk under the assumed prior distribution. Zidek (1967) shows that such Bayes invariant procedures are actually formal Bayes procedures with respect to a prior measure constructed from the assumed prior and right Haar measure on the group. In chapters 2 and 3 we will consider such formal Bayes estimators of $\sigma_0$ and $\sigma_a$ and will describe some reasonable optimality properties like admissibility (in certain classes) and minimaxity. In chapter 1 we give a general multi-parameter admissibility theorem which we later apply. This theorem gives sufficient conditions for admissibility and seems to be particularly applicable in proving admissibility of these Bayes invariant estimators when the group is one dimensional.

We conclude by giving some numerical calculations of mean squared errors of the formal Bayes estimators, and comparing them with values calculated by Klotz and Milton (1967). These tables should clarify the behavior of the formal Bayes estimators and should justify my recommendation of them.
Chapter 1

A Sufficient Condition for Admissibility

The proofs in this section are only slight generalizations of the work of Stein (1965) and Zidek (1967). Lemma 1.1 has been known for some time and is essentially the necessary and sufficient condition for admissibility of Stein (1955). The proof of lemma 1.2 essentially appears in Stein (1959) and that of lemma 1.3 is closely related to work in Zidek (1967). The proof in theorem 1.1 comes from Stein (1965) and Zidek (1967). Although the methods of proof are known, the result of theorem 1.1 appears to be new and as useful as such admissibility results can be. The proofs are presented for the sake of completeness.

Lemma 1.1 can be formulated in a general decision theoretic framework: there are measurable spaces $\mathcal{X}$, the observation space, $\mathcal{R}_\gamma$, the parameter space, and $\mathcal{A}$, the action space. We assume there is a jointly measurable loss function, $L: \mathcal{R} \times \mathcal{A} \rightarrow [0, \infty]$, and densities $p: \mathcal{X} \times \mathcal{R} \rightarrow [0, \infty)$ satisfying

\begin{equation}
\int p_\theta(x) d\mu(x) = 1 \quad \text{for all } x \in \mathcal{X}
\end{equation}

where $\mu$ is a $\sigma$-finite measure on $\mathcal{X}$. We consider non-randomized decision rules $\phi: \mathcal{X} \rightarrow \mathcal{A}$ measurable, and define the risk of $\phi$ to be

\begin{equation}
R(\phi, \theta) = \int L(\theta, \phi(x)) p_\theta(x) d\mu(x)
\end{equation}

The decision rule, $\phi$, is said to be admissible if there is no decision rule $\phi^*$ such that
(1.3) \[ R(\varphi^*, \theta) \leq R(\varphi, \theta) \text{ for all } \theta \]

with strict inequality for some \( \theta_0 \). If \( \parallel \) is a measure on \( \mathcal{G}_\varphi \), a rule, \( \varphi \), is said to be almost admissible with respect to \( \parallel \) if there is no \( \varphi^* \) for which (1.3) holds, with strict inequality on a set of positive \( \parallel \) measure. We will later assume that the loss function \( L \) is strictly convex. It will follow that the restriction to non-randomized decision rules is legitimate; and that, under conditions, every almost admissible rule is admissible.

Lemma 1.1. Let \( \parallel \) be a \( \sigma \)-finite measure on \( \mathcal{G}_\varphi \) and let \( \varphi_0 \) be an arbitrary decision rule. Let \( \mathcal{C} \subset \mathcal{G}_\varphi \) be a countable covering of \( \mathcal{G} \) by sets with finite \( \parallel \) measure. If for every \( C \in \mathcal{C} \) and every \( \epsilon > 0 \) there is a function \( f: \mathcal{G} \to [0, \infty) \) satisfying (1.4), (1.5), and (1.6) below then \( \varphi_0 \) is almost admissible with respect to \( \parallel \):

(1.4) \[ f(\theta) \geq 1 \text{ for } \theta \in C \]

(1.5) \[ \int R(\varphi_0, \theta)f(\theta)d\parallel(\theta) < \infty \]

(1.6) \[ K(f) = \inf_{\varphi} \int R(\varphi, \theta)f(\theta)d\parallel(\theta) < \epsilon \]

Proof: Suppose the contrary; that is, suppose there is \( \varphi^* \) such that (1.3) holds with strict inequality on a set of positive \( \parallel \)-measure. In particular, there is a set \( A \subset \mathcal{G} \) such that \( \parallel(A) > 0 \) and

(1.7) \[ R(\varphi_0, \theta) - R(\varphi^*, \theta) \geq b_1 > 0 \text{ for } \theta \in A. \]

Choose \( C_0 \in \mathcal{C} \) such that \( \parallel(C_0 \cap A) = b_2 > 0 \) and choose \( \epsilon_0 = b_1 b_2 \). Then for any \( f \) satisfying (1.4) (for \( C_0 \)) and (1.5),
\[(1.8) \quad \int R(\varphi, \theta) f(\theta) d\Pi(\theta) - \inf_\varphi \int R(\varphi, \theta) f(\theta) d\Pi(\theta) \\
\quad \geq \int [R(\varphi, \theta) - R(\varphi^*, \theta)] f(\theta) d\Pi(\theta) \\
\quad \geq \int [\Pi(\varphi, \theta) - \Pi(\varphi^*, \theta)] f(\theta) d\Pi(\theta) \\
\quad \geq b_1 \Pi(C_0 \cap A) = b_1 b_2 = \epsilon_0 .
\]

That is (1.6) cannot hold for this \( \epsilon_0 \); and, hence, the lemma follows by contradiction.

We now make the following specializations: Let \( A = R \) and let

\[(1.9) \quad L(\theta, a) = v(\theta)(a - g(\theta))^2
\]

where \( g: \mathcal{X} \rightarrow R \) and \( v: \mathcal{X} \rightarrow (0, \infty) \). It follows that we can restrict ourselves to non-randomized decision rules (see Ferguson (1967), p. 78). Furthermore, if for given \( \Pi \) we assume that

\[(1.10) \quad p_\theta(x) > 0 \quad \text{for all} \ x \in \mathcal{X} \ \text{and all} \ \theta \in \mathcal{X}
\]

then the following proof shows that an almost admissible rule is admissible:

Suppose \( \varphi_0 \) is inadmissible and that \( \varphi^* \) is better. By strict convexity of \( L \),

\[(1.11) \quad L(\theta, \frac{1}{2}\varphi_0(x) + \frac{1}{2}\varphi^*(x)) \leq \frac{1}{2}L(\theta, \varphi_0(x)) + \frac{1}{2}L(\theta, \varphi^*(x))
\]

with strict inequality on a set of positive measure under some \( p_{\theta_0} \); hence, by (1.10), on a set of positive measure under each \( p_\theta \). Therefore integrating (1.11) with respect to \( p_\theta d\mu \),
\[(1.12) \quad R\left(\frac{1}{2}\varphi_0 + \frac{1}{2}\varphi^*, \theta\right) < \frac{1}{2}R(\varphi_0, \theta) + \frac{1}{2}R(\varphi^*, \theta) \leq R(\varphi_0, \theta).\]

That is, \(\frac{1}{2}\varphi_0 + \frac{1}{2}\varphi^*\) has uniformly smaller risk than \(\varphi_0\); which implies that \(\varphi_0\) is not almost admissible with respect to any non-zero measure.

Furthermore, the specialization \((1.9)\) implies that
\[
\inf_{\varphi} \int R(\varphi, \theta)f(\theta)d\|\theta\| = \inf_{\varphi} \int \frac{\int \varphi(\theta)g(\theta)p_\theta(x)f(\theta)d\theta}{\int \varphi(\theta)p_\theta(x)f(\theta)d\theta}\|\theta\|
\]
whenever the numerator and denominator are finite almost everywhere \((\mu)\). (Note: if the denominator vanishes, so does the numerator.)

**Lemma 1.2.** Let \(\|\) be a \(\sigma\)-finite measure on \(\mathcal{B}_\Theta\). If \((1.5)\) holds and \(f: \Theta \to [0, \infty)\) is such that \(\varphi_{f,\|}\) is given by \((1.13)\), then (for \(K(f)\) defined in \((1.6)\)),
\[
(1.14) \quad K(f) = \delta_{\mu}(x) \frac{\int (\varphi_0(x) - g(\theta'))v(\theta')p_{\theta'}(x)f(\theta')d\|\theta')}{\int \varphi(\theta)p_{\theta}(x)f(\theta)d\|\theta)}.
\]

Proof: We have, using \((1.9)\) in \((1.6)\) and Fubini's theorem,
\[
(1.15) \quad K(f) = \int f(\theta)[\int \varphi_0(x)p_{\theta}(x)v(\theta)[(\varphi_0(x) - g(\theta))^2 - (\varphi_{f,\|}(x) - g(\theta))^2]d\|\theta)]
\]
\[
- \int \delta_{\mu}(x)f_{\theta}(x)v(\theta)[(\varphi_0(x) - g(\theta))^2 - (\varphi_{f,\|}(x) - g(\theta))^2]d\|\theta]
\]
\[
= \int \delta_{\mu}(x)f_{\theta}(x)v(\theta)[(\varphi_0(x) - \varphi_{f,\|}(x))^2 - 2(\varphi_0(x) - \varphi_{f,\|}(x))\theta(x) - g(\theta) - \varphi_{f,\|}(x)]d\|\theta)
\]
\[
= \int \delta_{\mu}(x)f_{\theta}(x)v(\theta)[(\varphi_0(x) - \varphi_{f,\|}(x))^2d\|\theta)
\]

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since the cross-product term vanishes by definition of $\Phi$. Therefore, using (1.13)

$$\tag{1.16} K(x) = \int \phi(x) \exp \left( \frac{\int (\phi(x) - g(x')) \nu(x') \mathbb{P}_{\theta'}(x) \nu(\theta') d\nu(\theta')} {\int \nu(x') \mathbb{P}_{\theta'}(x) \nu(\theta') d\nu(\theta')} \right)^2$$

from which the result follows. $\|\$

Now let $\mathcal{Y} = (\theta, \bar{\theta}) \times \mathcal{F}_0$ where $(\theta, \bar{\theta})$ is an interval (not necessarily finite) on the real line and $(\mathcal{F}_0, \mathcal{F}_0)$ is a measurable space. Consider a prior $\sigma$-finite measure $\nu$ on $\mathcal{F}$ of the form

$$\tag{1.17} d\nu(\theta) = \pi(\theta_1, \theta_2) d\nu(\theta_2) d\theta_1$$

where $\nu$ is a $\sigma$-finite measure on $\mathcal{F}_0$. Let $\Phi_0 = \Phi_\|\| \Phi_\mathcal{F}$ and define, for $\theta_1 \in (\theta, \bar{\theta})$ and $x \in \mathcal{X}$,

$$\tag{1.18} h_1(\theta_1, x) = \int_{\mathcal{F}_0} (\phi(x) - g(\theta_1', \theta_2')) \mathbb{P}(\theta_1', \theta_2')(x) \nu(\theta_1', \theta_2') \pi(\theta_1', \theta_2') d\nu(\theta_2') d\theta_1'$$

$$\tag{1.19} h_2(\theta_1, x) = \int_{\mathcal{F}_0} \mathbb{P}(\theta_1, \theta_2)(x) \nu(\theta_1, \theta_2) \pi(\theta_1, \theta_2) d\nu(\theta_2)$$

$$\tag{1.20} \lambda(\theta_1) = \int_{\mathcal{F}_0} \mathbb{E}(\theta_1, \theta_2) \left( \frac{h_1(\theta_1, x)}{h_2(\theta_1, x)} \right)^2 \nu(\theta_1, \theta_2) d\nu(\theta_2)$$.

**Lemma 1.3.** Suppose lemma 1.2 holds and suppose

$$\tag{1.21} h_2(\theta_1, x) > 0 \text{ for all } x \in \mathcal{X}, \ \theta \in (\theta, \bar{\theta})$$,

$$\tag{1.22} \lambda(\theta_1) \text{ is a continuous function on } (\theta, \bar{\theta}),$$

$$\tag{1.23} f(\theta_1, \theta_2) \text{ is a function of } \theta_1 \text{ alone, is continuous,}$$

and is differentiable except at a finite number of points in $(\theta, \bar{\theta})$. 

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Then, letting $q(\theta_1) = \sqrt{f(\theta_1, \theta_2)}$ ($q'(\theta_1)$ denotes the derivative)

\begin{equation}
K(f) \leq \frac{4}{\theta} \lambda(\theta_1)[q'(\theta_1)]^2 d\theta_1 .
\end{equation}

\textbf{Proof:} Evaluate the numerator, $N$, of the integrand in (1.14):

\begin{equation}
N = \int_\theta \left[ \int_0^\theta \left( \varphi(x) - g(\theta_1, \theta_2) \right) \nu(\theta_1, \theta_2) \pi(\theta_1, \theta_2) d\nu(\theta_2) \right] c^2(\theta_1) d\theta_1
\end{equation}

\begin{align*}
&= 2 \int_\theta h_1(\theta_1, x) q(\theta_1) q'(\theta_1) d\theta_1
\end{align*}

where we have integrated by parts; note that $h_1(\theta, x) = 0$ by definition of $\varphi(x)$ and $h(\theta, x) = 0$ by integrability conditions. Continuing and using Schwarz inequality,

\begin{equation}
N = 2 \int_\theta \frac{h_1(\theta_1, x)}{h_2(\theta_1, x)} q'(\theta_1) q(\theta_1) h_2(\theta_1, x) d\theta_1
\end{equation}

\begin{equation}
N^2 \leq 4 \left[ \frac{h_1(\theta_1, x)}{h_2(\theta_1, x)} \right]^2 [q'(\theta_1)]^2 h_2(\theta_1, x) d\theta_1 \left[ \frac{q^2(\theta_1)}{h_2(\theta_1, x)} h_2(\theta_1, x) d\theta_1 \right]
\end{equation}

\begin{align*}
&= 4 \int_\theta \left[ \frac{h_1(\theta_1, x)}{h_2(\theta_1, x)} \right]^2 \left[ q'(\theta_1) \right]^2 \nu(\theta_1, \theta_2) \pi(\theta_1, \theta_2) d\nu(\theta_2) d\theta_1
\end{align*}

\begin{align*}
&\times \int_\theta \left[ \frac{p(\theta_1, \theta_2)}{\nu(\theta_1, \theta_2)} \right]^2 \nu(\theta_1, \theta_2) \pi(\theta_1, \theta_2) d\nu(\theta_2) d\theta_1
\end{align*}

Therefore, inserting (1.27) in (1.14) and interchanging orders of integration, we get the desired result. ||

Before going to theorem 1.1, note that (1.10) implies that

$h_2(\theta_1, x) > 0$ for all $\theta_1 \in (\underline{\theta}, \theta)$ and all $x \in \mathfrak{X}$. Also recall that condition (1.10) implies that any rule almost admissible with respect to $\parallel$ is admissible.
We first summarize sufficient conditions for lemma 1.3 to hold:

\[(1.28) \quad d\Pi(\theta) = d\Pi(\theta_1, \theta_2) = \pi(\theta_1, \theta_2) d\nu(\theta_2) d\theta_1\]

\[(1.29) \quad \varphi_\Pi(x) = \frac{\int \nu(\theta) g(\theta) p_\theta(x) d\Pi(\theta)}{\int \nu(\theta) p_\theta(x) d\Pi(\theta)}\]

(the numerator and denominator being finite)

\[(1.30) \quad P(\theta_1, \theta_2)(x) > 0 \quad \text{for all} \quad \theta_1 \in (\theta, \overline{\theta}), \theta_2 \in \mathcal{F}_0, x \in \mathcal{X}\]

\[(1.31) \quad \lambda(\theta_1) \quad \text{is a continuous function of} \quad \theta_1 \quad \text{on} \quad (\theta, \overline{\theta})\]

\[(1.32) \quad f(\theta) = f(\theta_1, \theta_2) \quad \text{is a function of} \quad \theta_1 \quad \text{alone. It is continuous and differentiable except at a finite number of points on} \quad (\theta, \overline{\theta}).\]

**Theorem 1.1.** Suppose (1.28), (1.29), (1.30) and (1.31) hold.

Suppose further that for every compact sub-interval \([a_0, b_0] \subset (\theta, \overline{\theta})\)

\[(1.33) \quad \int_{a_0}^{b_0} \int_{\mathcal{F}_0} R(\varphi_\Pi, (\theta_1, \theta_2)) \pi(\theta_1, \theta_2) d\nu(\theta_2) d\theta_1 < \infty\]

and, also, that for every \(c \in (\theta, \overline{\theta})\) conditions (A) and (B) hold:

(A) if

\[(1.34) \quad \int_c^\overline{\theta} \int_{\mathcal{F}_0} R(\varphi_\Pi, (\theta_1, \theta_2)) \pi(\theta_1, \theta_2) d\nu(\theta_2) d\theta_1 = \infty,\]

then

\[(1.35) \quad \int_{\theta}^{c} \frac{d\theta_1}{\lambda(\theta_1)} = \infty.\]

(B) if

\[(1.36) \quad \int_{\theta}^{c} \int_{\mathcal{F}_0} R(\varphi_\Pi, (\theta_1, \theta_2)) \pi(\theta_1, \theta_2) d\nu(\theta_2) d\theta_1 = \infty,\]
then

\[(1.37) \quad \int_{\theta}^{c} \frac{d\theta}{\lambda(\theta)} = \infty.\]

Then \(\Phi\) is admissible.

Proof: We will apply lemma 1.1 with \(\mathcal{C} = \{[a_n, b_n] \times \mathcal{C}_0 : [a_n, b_n] \text{ a compact interval, } a_n \downarrow \theta, b_n \uparrow \theta \text{ as } n \to \infty\}. \) Let \(C = [a, b] \times \mathcal{C}_0 \in \mathcal{C}\) and let \(\epsilon > 0\) be given. In terms of \(a, b,\) and \(\epsilon\) we will construct a function \(f(\theta_1)\) satisfying the conditions for lemma 1.1. By lemma 1.3, it is sufficient to define \(q(\theta_1) = \sqrt{f(\theta_1)}\) so that

\[(1.38) \quad 4\int_{\theta}^{b} \lambda(\theta_1)[q'(\theta_1)]^2 d\theta_1 = 4\int_{\theta}^{a} \lambda(\theta_1)[q'(\theta_1)]^2 d\theta_1 + \int_{a}^{b} \lambda(\theta_1)[q'(\theta_1)]^2 d\theta_1 + \int_{b}^{\theta} \lambda(\theta_1)[q'(\theta_1)]^2 d\theta_1 < \epsilon\]

and so that (1.4) and (1.5) hold. First define \(q(\theta_1) = 1\) for \(\theta_1 \in [a, b].\) Then condition (1.4) holds and the middle integral in (1.38) is zero. We will now define \(q\) on \((b, \theta);\) its construction on \((\theta, a)\) will be entirely similar. First if

\[(1.39) \quad \int_{b}^{\theta} \int_{\mathcal{C}_0} R(\Phi, (\theta_1, \theta_2)) \pi(\theta_1, \theta_2) d\nu(\theta_2) d\theta_1 < \infty\]

then we can define \(q(\theta_1) = 1\) for \(\theta_1 \in (b, \theta)\) and satisfy all desired conditions here. Otherwise, define \(q(\theta_1)\) by

\[(1.40) \quad q(\theta_1) = \begin{cases} 1 - \frac{y}{K} & 0 \leq y \leq K \\ 0 & \text{otherwise} \end{cases}\]

where \(y\) is a function of \(\theta_1\) defined for \(\theta_1 > b\) by
(1.41) \[ y = \int_{b}^{\theta} \frac{d\theta_1}{\lambda(\theta_1)} \]

and \( K > \frac{8}{\epsilon} \). Because of condition (1.39), condition (A) implies that \( y \) goes monotonically from 0 to \( \infty \) as \( \theta_1 \) goes from \( b \) to \( \bar{\theta} \). So \( q \) is defined on all of \((b, \bar{\theta})\), vanishes at some \( b^* < \bar{\theta} \) and is differentiable (since \( \lambda \) is continuous). Furthermore

(1.42) \[ \frac{dy}{d\theta_1} = \frac{1}{\lambda(\theta_1)}. \]

Therefore,

(1.43) \[ q'(\theta_1) = \begin{cases} -\frac{1}{K\lambda(\theta_1)} & 0 \leq y \leq K \\ 0 & \text{otherwise} \end{cases} \]

and \( q \) is differentiable except perhaps at \( \theta_1 = b \) and \( y = K \). Now, changing variables from \( \theta_1 \) to \( y \) in (1.38)

(1.44) \[ \int_{b}^{\theta_1} \lambda(\theta_1)[q'(\theta_1)]^2 d\theta_1 = \int_{0}^{K} \left(-\frac{1}{K\lambda(\theta_1)}\right)^2 \lambda(\theta_1) \frac{d\theta_1}{dy} dy = \frac{1}{K^2} \int_{0}^{K} dy = \frac{1}{K} \leq \frac{\epsilon}{8}. \]

Therefore, the integral over \((b, \bar{\theta})\) in (1.38) is less than \( \frac{\epsilon}{2} \).

Furthermore, \( q(\theta_1) \) vanishes for \( \theta_1 \geq b^* \) and is bounded above by 1. So, by (1.33),

(1.45) \[ \int_{b}^{\theta_1} R(\varphi, (\theta_1, \theta_2)) \pi(\theta_1, \theta_2) [q^2(\theta_1)] dv(\theta_2) d\theta_1 \]

\[ \leq \int_{b}^{b^*} R(\varphi, (\theta_1, \theta_2)) \pi(\theta_1, \theta_2) dv(\theta_2) d\theta_1 < \infty. \]

Thus, defining \( q(\theta_1) \) similarly on \((\theta, a)\), we see that \( f(\theta_1) = q^2(\theta_1) \) satisfies the desired conditions for lemma 1.1. Admissibility follows from the remarks just before theorem 1.1 using condition (1.30).
Chapter 2

Estimation of the "Between" Component of Variance
in a Random Effects Analysis of Variance Model

2.1 The Basic Problem

We now apply theorem 1.1 to the estimation of the components of variance in the one way random effects model (Model II) in analysis of variance. In particular we consider the following statistical problem in canonical form: we observe

\[(2.1.1) \quad \bar{Y} \sim \mathcal{N}(\mu, (\sigma_e^2 + J\sigma_a^2)/IJ), S_1 \sim \sigma_e^2 X^2_{I(J-1)}, S_2 \sim (\sigma_e^2 + J\sigma_a^2)X^2_{I-1}\]

where \(\mu\) is real, \(\sigma_e > 0, \sigma_a > 0\) and \(I\) and \(J\) are positive integers. We wish to estimate \(\sigma_a\) with loss proportional to squared error; and we will consider formal Bayes estimators, which are known to be strictly positive.

Throughout this chapter we will consider primarily location and scale invariant procedures; that is, estimators not depending on \(\bar{Y}\). Although the work of Stein (1965a) indicates that such procedures are likely to be inadmissible and Zacks (1968) shows that location and scale invariant estimates of \(\sigma_e\) are, in fact, inadmissible, we know that consideration of \(\bar{Y}\) can not lead to substantial improvements (\(\bar{Y}\) can contribute at most one additional degree of freedom). Furthermore, the estimators considered here can be expressed in the form of tabulated functions; and this could not be done if we did not restrict ourselves to location and scale invariant estimators.
2.2. A Class of Formal Bayes Estimators Admissible

in the Class of Location Invariant Estimators

Consider the reduced problem where we observe

\[(2.2.1) \quad S_1 \sim \sigma_e \chi^2_{I-1} \quad \text{and} \quad S_2 \sim (\sigma_e + J\sigma_a) \chi^2_{I-1}.\]

For ease in integrating we will use the following parameterization:

\[(2.2.2) \quad \alpha = \frac{1}{2\sigma_e}, \quad \beta = \frac{1}{2(\sigma_e + J\sigma_a)}, \quad \gamma = \frac{\beta}{\alpha}.

Then the densities of $S_1$ and $S_2$ become

\[(2.2.3) \quad p(\alpha, \beta)(S_1, S_2) = \alpha^{\frac{1}{2}(I-1)} \beta^{\frac{1}{2}(I-1)} e^{-\alpha S_1 - \beta S_2} \quad \gamma^\frac{1}{2}(I-1) e^{-\gamma(S_1 + \gamma S_2)}

with respect to the measure $\mu$ given by

\[(2.2.4) \quad d\mu(S_1, S_2) = S_1^{\frac{1}{2}(I-1)} S_2^{\frac{1}{2}(I-1)} dS_1 dS_2.

We take the parameter space to be

\[(2.2.5) \quad \Omega = \{ (\alpha, \gamma) : 0 < \alpha < \infty, 0 < \gamma < 1 \}.

We wish to estimate $\sigma_a = \frac{1}{2J\alpha\gamma} - 1$; so we take the loss function

\[(2.2.6) \quad L(\alpha, \gamma, \varphi) = \alpha^2 \gamma^2 (\varphi - \frac{1}{2J\alpha\gamma} - 1)^2.

Then the problem is invariant under scale changes; that is, under the transformations

\[(2.2.7) \quad S_1 \rightarrow \frac{1}{c} S_1, \quad S_2 \rightarrow \frac{1}{c} S_2; \quad \alpha \rightarrow c\alpha,

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for \( c > 0 \). Thus, we can think of \( \alpha \) as a scale parameter and of \( \gamma \) as a parameter indexing the orbits. Zidek's result then indicates that we can find Bayes scale invariant procedures by considering prior measures with densities of the form \( \frac{1}{\alpha^c} \gamma^d \). For ease of computation, we will take for the prior density

\[
(2.2.8) \quad \pi(\alpha, \gamma) = \alpha^a \gamma^b
\]

(with respect to Lebesgue measure restricted to \( \gamma \)) and we will calculate the Bayes rules \( \Phi_n = \Phi(a, b) \). We expect (from Zidek's result) that \( \Phi(a, b) \) should be reasonable (in this reduced problem) for \( a = -1 \) (the invariant Haar measure) and \( b > -1 \).

We will use (1.13) to calculate \( \Phi(a, b) \). First note that (with \( v(\alpha, \gamma) \) denoting \( \alpha^c \gamma^d \))

\[
(2.2.9) \quad p(\alpha, \gamma)(s_1, s_2)\pi(\alpha, \gamma)v(\alpha, \gamma) \propto \alpha^c \gamma^d e^{-\alpha(s_1 + s_2)}
\]

where

\[
(2.2.10) \quad c = \frac{1}{2}(IJ - 1) + a + 2, \quad d = \frac{1}{2}(I - 1) + b + 2.
\]

Then, \( \Phi(a, b) \), the Bayes rule versus \( \Pi \), is given by

\[
(2.2.11) \quad \Phi(a, b)(s_1, s_2) = \frac{\int_0^{\frac{1}{\alpha}(\frac{1}{\gamma} - 1)} \alpha^{-1} \gamma^{-1} e^{-\alpha(s_1 + s_2)} d\alpha d\gamma}{\int_0^{\frac{1}{\alpha}} \alpha^c \gamma^d e^{-\alpha(s_1 + s_2)} d\alpha d\gamma}
\]

\[
= \frac{\int_0^{\frac{1}{\alpha}(\frac{1}{\gamma} - 1)} \gamma^d \frac{1}{(s_1 + s_2)^c} dy}{\int_0^{\frac{1}{\alpha}} \gamma^d \frac{1}{(s_1 + s_2)^{c+1}} dy}.
\]

Now, integrating by parts, we have (for \( d > 0, c - d - 1 > 0 \))
\[(2.2.12) \int_0^\gamma \frac{d-1}{(S_1+\gamma S_2)^c} \, d\gamma = \frac{1}{d} (S_1+\gamma S_2)^{c-1} \frac{cS_2}{d} \frac{1}{(S_1+\gamma S_2)^{c+1}} d\gamma \]

and,

\[(2.2.13) \int_0^\gamma \frac{d}{(S_1+\gamma S_2)^c} \, d\gamma = \frac{1}{(S_1+\gamma S_2)^{c-1}} \frac{1}{d-c+1} \frac{cS_1}{d-c+1} \frac{1}{(S_1+\gamma S_2)^{c+1}} d\gamma.\]

Inserting (2.2.12) and (2.2.13) in (2.2.11),

\[(2.2.14) \varphi_{(a,b)}(S_1, S_2) = \frac{1}{2J} \left\{ \frac{S_2}{a} - \frac{S_1}{a-d-l} + \frac{(c-1)}{cd(c-d-l)} \cdot \frac{(S_1+S_2)}{F_{c+1, d}(A)} \right\},\]

where \( A = \frac{S_1}{S_1+S_2} \) and

\[(2.2.15) F_{c+1, d}(A) = \frac{1}{(A+\gamma(l-A))^{c+1}} = \frac{(l-A)^{c-d} (l-1-A)^{d+1}}{(1-A)(c-d+1)} \beta(c-d,d+1)\]

where \( \beta(\cdot, \cdot) \) is the incomplete Beta function (see Pearson (1934));

\( I_x(p,q) = \int_0^x u^{p-1}(1-u)^{q-1} \, du / \beta(p,q) \) and \( \beta(\cdot, \cdot) \) is the complete Beta function. Note that, from (2.2.11) and (2.2.15) we can directly write

\[(2.2.16) \varphi_{(a,b)}(S_1, S_2) = \left( \frac{S_1+S_2}{2Jc} \right) \frac{F_{c,d-l}(A)}{F_{c+1, d}(A)} = \frac{1}{2J} \left[ \frac{S_2}{d} \frac{1-I_A(c-d,d)}{1-A(c-d,d+1)} - \frac{S_1}{c-d-l} \frac{1-I_A(c-d-1,d+1)}{1-A(c-d,d+1)} \right] \]

which may be easier to compute from tables of the incomplete Beta function (e.g. Pearson (1934)).

To prove admissibility of the appropriate \( \varphi_{(a,b)} \), we will not need explicit expressions for \( \varphi_{(a,b)} \). We will only require the following inequality:

\[(2.2.17) \varphi_{(a,b)}(S_1, S_2) \leq M(S_1 + S_2)\]
where $M$ is a constant depending only on $a$, $b$, $I$, and $J$, and is finite for $d > 0$ and $(c - d - 1) > 0$.

This inequality follows from (2.2.14) and the inequality (see (2.2.15))

$$(2.2.18) \quad F_{c+1,d}(A) \geq \int_0^1 \gamma^d d\gamma = \frac{1}{d+1}$$

(which follows since $A + \gamma(1 - A) \leq 1$ for $0 < \gamma < 1$). Further properties of $\Phi(a,b)$, particularly for $a = b = -1$, will be discussed in section 2.3. We will now prove

**Theorem 2.1:** Consider the statistical problem described by formulas (2.2.1), (2.2.5), (2.2.6) and (2.2.8). If

$$(2.2.19) \quad a = -1 \quad \text{and}$$

$$(2.2.20) \quad -1 < b < \frac{1}{2}I(J - 1) - 2$$

then $\Phi(a,b)$ defined by (2.2.11) is admissible. That is, if (2.2.19) and (2.2.20) hold, the Bayes invariant rule $\Phi(a,b)$ is admissible in the class of location invariant estimators in the original problem.

**Proof:** We will apply theorem 1.1 directly with $\theta_1 = \alpha$, $\theta_2 = \gamma$, $(\bar{\theta}, \theta) = (0, \infty)$, $\mathcal{Y}_0 = (0,1)$, $\pi(\alpha, \gamma) = \alpha^a \gamma^b$, $\nu(\alpha, \gamma) = \alpha^2 \gamma^2$, $\mathcal{X} = \{S_1, S_2\}$, $p_\theta(x)$ given by (2.2.9), and $\Phi$ given by (2.2.11). In theorem 1.1, conditions (1.28), (1.29), and (1.30) are immediate. Condition (1.33) can be checked as follows: By (2.2.17) and (2.2.1),

$$(2.2.21) \quad E_{(\alpha, \gamma)} L(\Phi, (\alpha_1, \gamma)) = E_{(\alpha, \gamma)} \alpha^2 \gamma^2 \left(\Phi_{(S_1, S_2)} - \frac{1}{2J}\alpha(1 - \gamma - 1)\right)^2$$

$$\leq \alpha^2 \gamma^2 E_{(\alpha, \gamma)} \Phi_{(S_1, S_2)} + \frac{1}{4J^2}(1 - \gamma)^2$$
\[ \leq M \alpha^2 \gamma^2 E(\alpha, \gamma)(S_1 + S_2)^2 + \frac{1}{4\gamma^2} \]

\[ \leq M \alpha^2 \gamma^2 [\sigma^2 \Gamma(1) + (\sigma + \sigma')^2 \Gamma(2)] + \frac{1}{4\gamma^2} \]

\[ \leq M' \gamma^2 + M'' + \frac{1}{4\gamma^2} \]

\[ \leq M^* \]

where \( M^* \) depends only on \( a, b, I, \) and \( J. \) Therefore, since \( \pi \) is integrable on \( C \times (0, 1) \) (with \( C \) compact) for \( b > -1 \), the integrated risk is also finite on such sets. It remains to investigate the function \( \lambda(\alpha) \) given by (1.20). From (1.18) and (1.19) we have

\[ (2.2.22) \quad \frac{h_1(\alpha,S_1,S_2)}{h_2(\alpha,S_1,S_2)} = \frac{\alpha^{-1} (\varphi(S_1,S_2) - \frac{1}{2} \alpha'^{-1}) \gamma c' d' - \alpha'(S_1 + \gamma' S_2) \gamma' d' \alpha'}{\int_1^c \gamma c' d' - \alpha(S_1 + \gamma' S_2) \gamma' d' \alpha'} \]

where \( c \) and \( d \) are given by (2.2.10). Changing variables, let \( T_1 = \alpha S_1 \) and \( T_2 = \alpha \gamma S_2. \) Then \( T_1 \sim \frac{1}{2} X_{(J-1)}^2, \ T_2 \sim \frac{1}{2} X_{1-1}^2, \) and

\[ (2.2.23) \quad E(\alpha, \gamma) \left\{ \frac{h_1(\alpha,S_1,S_2)}{h_2(\alpha,S_1,S_2)} \right\}^2 = E(1,1) \left\{ \frac{h_1(\alpha,S_1,S_2)}{h_2(\alpha,S_1,S_2)} \right\}^2 \]

Furthermore, changing variables from \((\alpha', \gamma')\) to \(x = \frac{\alpha'}{\alpha}, y = \frac{\gamma'}{\gamma},\) and using the fact that \( \varphi \) is scale invariant for any \( a \) and \( b, \)

\[ (2.2.24) \quad \frac{h_1(\alpha,S_1,S_2)}{h_2(\alpha,S_1,S_2)} = \frac{\alpha}{\gamma} \int_0^\infty (\varphi(\gamma T_1, T_2) - \frac{1}{2} \alpha'/\gamma) \gamma' d' e^{-(T_1 + \gamma' T_2)} dy \frac{dy}{\gamma} \]

Furthermore from the definition of \( \varphi, \) the integral over \((0, \infty)\) in the numerator of (2.2.24) is zero. So we also have
\[
(2.2.25) \quad \frac{h_1(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})}{h_2(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})} = \frac{1}{\gamma} \int_0^\alpha \left(1 - \frac{\alpha}{\gamma} \right) x e^{-\left(T_1 + \gamma T_2\right)} dx dy
\]

Note that \( \frac{h_1(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})}{h_2(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})} \) is independent of \( \alpha \). Hence, \( E_0(\alpha, \gamma) \left( \frac{h_1(\alpha, S_1, S_2)}{h_2(\alpha, S_1, S_2)} \right)^2 \)

is a function of \( \gamma \) alone (i.e., is independent of \( \alpha \)). Therefore, from (1.20),

\[
(2.2.26) \quad \lambda(\alpha) = \int_0^1 E(\alpha, \gamma) \left( \frac{h_1(\alpha, S_1, S_2)}{h_2(\alpha, S_1, S_2)} \right)^2 \alpha^{a+2} \gamma^{b+2} d\gamma
\]

\[
= \alpha^{a+2} \int_0^1 E(\alpha, \gamma) \left( \frac{h_1(\alpha, S_1, S_2)}{h_2(\alpha, S_1, S_2)} \right)^2 \gamma^{b+2} d\gamma
\]

and \( \lambda(\alpha) \) is a continuous function of \( \alpha \) whenever the integral in (2.2.26) is finite. I will later use (2.2.24) and (2.2.25) to show that

\[
(2.2.27) \quad \frac{h_1(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})}{h_2(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})} \leq \frac{1}{\gamma} (M_1 + M_2 T_1 + M_3 T_2) \quad \text{for } c - d - 1 > 0, \ d > 0
\]

where \( M_1, M_2, M_3 \) are constants not depending on \( \alpha \) and \( \gamma \). From (2.2.27), it follows that

\[
(2.2.28) \quad E(1, 1) \left( \frac{h_1(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})}{h_2(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})} \right)^2 \leq \frac{1}{\gamma} M^* \]

and, from (2.2.23), that

\[
(2.2.29) \quad \lambda(\alpha) \leq M^* \alpha^{a+2} \int_0^1 \gamma^{b+2} d\gamma = \alpha^{a+2} \frac{M^*}{b+1}
\]

which is finite for \( b > -1 \). To prove \( \varphi \) admissible it remains to check conditions (A) and (B) of theorem 1.1. From (2.2.14),
\[ \varphi(S_1, S_2) \geq \frac{S_2}{d} - \frac{S_1}{c-d-1}; \] so calculating expectations,

\[
(2.2.30) \quad R(\varphi, (\alpha, \gamma) = \alpha^2 \gamma^2 \mathbb{E}(\alpha, \gamma) \left( \varphi(S_1, S_2) - \frac{1}{2(\alpha \gamma)(1 - \gamma)} \right)^2 \\
\geq \frac{(I-1)\gamma^2 + I(I-1)}{2d^2} \frac{1}{2(c-d-1)^2}
\]

Therefore, combining this with (2.2.21),

\[
(2.2.31) \quad \int_0^1 R(\varphi, (\alpha, \gamma)) \pi(\alpha, \gamma) d\gamma \approx M** \alpha^a.
\]

For conditions (A) and (B) we want \( \frac{1}{\mathbb{E}(\alpha)} \) to be non-integrable whenever \( \int_0^1 R(\varphi, (\alpha, \gamma)) \pi(\alpha, \gamma) d\gamma \) is; that is, we want \( \frac{1}{\alpha^2+2} \) to be non-integrable whenever \( \alpha^a \) is. But this holds if and only if \( a = -1 \).

Thus, \( \varphi \) is admissible (by theorem 1.1) for \( a = -1 \), \( b > -1 \) and \( d > 0 \), \( c - d - 1 > 0 \). These conditions, by (2.2.10), are just (2.2.19) and (2.2.20); hence, to complete the proof of this theorem we need only prove (2.2.27), for which we will give the following rather technical and lengthy argument:

First note that for \( s, t > 0 \)

\[
(2.2.32) \quad \int_s^\infty u^n e^{-tu} du = \frac{1}{t^{n+1}} \int_{st}^\infty v^n e^{-v} dv \\
= \frac{1}{t^{n+1}} \int_{st}^\infty \left( v^{n+2} e^{-v} \right) dv \\
\leq \frac{1}{t^{n+1}} \left( st \right)^{n+2} e^{-st} \int_{st}^\infty \frac{1}{v^2} dv \quad \text{for } st \geq n+2 \\
= s^{n+1} e^{-st} \quad \text{(for } st \geq n+2). 
\]

Therefore, taking the outer integral (over \( x \)) in the numerator of (2.2.24), we have, for \( T_1 \geq c + 2 \)
\[(2.2.33) \quad \gamma \cdot \text{(numerator)} \leq \int_0^\infty \left( \Phi(\gamma T_1, T_2) + \frac{1}{1/2} y^d e^{-\gamma T_2} \right) dy \]

So

\[(2.2.34) \quad \gamma \left| \frac{h_1(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})}{h_2(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})} \right| \leq \Phi(\gamma T_1, T_2) + \frac{1}{1/2} y^d e^{-\gamma T_2} \frac{1}{\int_0^\infty y^d e^{-\gamma T_2} dy} . \]

Integrating by parts, we can bound the last term in (2.2.34) by

\[(2.2.35) \quad \frac{T_2}{2Jd} + \frac{1}{2Jd} \frac{1/2 d e^{-T_2/\gamma}}{\int_0^\infty y^d e^{-\gamma T_2} dy} , \]

and since

\[(2.2.36) \quad \int_0^\infty y^d e^{-\gamma T_2} dy \geq e^{-\gamma T_2} \int_0^\infty y^d dy = \frac{1}{d+1} e^{-\gamma T_2} \frac{1}{(1/\gamma)^{d+1}} \geq \frac{T_2}{2Jd} e^{-\gamma T_2/2} , \]

we have, for \( T_1 \geq c + 2, \ d > 0 \) (using (2.2.17)),

\[(2.2.37) \quad \gamma \left| \frac{h_1(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})}{h_2(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})} \right| \leq M_1 + M_2 T_1 + M_2 T_2 . \]

Now, for \( T_1 < c + 2 \), we will use (2.2.25). Since \( e^{-T_1 x} \leq 1 \),

taking absolute values, we have

\[(2.2.38) \quad \gamma \left| \frac{h_1(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})}{h_2(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})} \right| \leq e^{c+2} \left( \int_0^\infty \left[ \Phi(\gamma T_1, T_2) - 1/2 y^d \left( \int_0^\infty xy^d c^{-d} e^{-\gamma T_2 y} dy dx \right) \right] \right) \]

Letting \( z = xy \) in the inner integral of the numerator,

\[(2.2.39) \quad \gamma \left| \frac{h_1(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})}{h_2(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha})} \right| \leq e^{c+2} \left( \int_0^\infty \left[ \Phi(\gamma T_1, T_2) + \frac{1}{2/2} z^d e^{-\gamma T_2 z} dz \right] \right) \]

The integrand is maximized over \( 0 \leq x \leq 1 \) at \( x = 1 \) (for \( c - d - 1 > 0 \)). So
\[ (2.2.40) \quad \gamma \left| \frac{h_1(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha^2})}{h_2(\alpha, \frac{T_1}{\alpha}, \frac{T_2}{\alpha^2})} \right| \leq e^{c+2} \frac{1}{\int_0^1 \left( \frac{1}{\gamma r(T_1, T_2)} \right) e^{-T_2 y} dy} \cdot e^{\frac{1}{2} \int_0^1 y e^{-T_2 y} dy} \]

\[ \leq M_1' + M_2^T T_1 + M_2^T T_2 \]

where we have used the same argument as we used to get (2.2.37). Therefore, (2.2.27) follows from (2.2.37) and (2.2.40) and the proof of the theorem is complete. 

2.3 A Minimax, Formal Bayes Estimate of \( \varphi_a \)

Admissible Among Scale and Location Invariant Estimators

In section 2.2 we discussed a class of formal Bayes estimators \( \varphi(-1, b) \) admissible among location invariant procedures. In this section we consider the limiting case as \( b \to -1 \). In particular we will consider the estimator \( \varphi(-1, -1) \), which, as is often the case with limits of Bayes rules, we will show to be admissible among fully invariant procedures and minimax (in the original problem). Since we are now taking \( a = -1 \) in \( \varphi(a, b) \), we will suppress dependence on \( a \) throughout this section and refer to \( \varphi_b \) instead of \( \varphi(a, b) \).

We now show that \( \varphi(-1) \) is admissible among location and scale invariant estimators. To do this, we will reduce the problem by invariance and apply lemma 1.1 and formula (1.15) directly. We first discuss the reduced statistical problem.

If \( \varphi \) is scale (and location) invariant, there is a function \( \psi: \mathbb{R} \to \mathbb{R} \) such that \( \varphi \) can be written
\[(2.3.1) \quad \psi(S_1, S_2) = (S_1 + S_2) \psi \left( \frac{S_1}{S_1 + S_2} \right). \]

(Note: we let \( \psi_b \) denote the appropriate function of the maximal invariant corresponding to \( \phi_b \).) Then, the risk of any invariant rule is given by

\[(2.3.2) \quad R(\psi, (\alpha, \gamma)) = \text{E} \gamma^2 (\phi(S_1, S_2) - \frac{1}{2J \alpha \gamma^2} (1 - \gamma))^2 - \text{E}((T_1 + T_2) \psi \left( \frac{T_1}{T_1 + T_2} \right) - \frac{1}{2 \gamma^2} (1 - \gamma))^2 \]

where

\[(2.3.3) \quad T_1 \sim \frac{2 \chi^2}{2 (J - 1)} \quad \text{and} \quad T_2 \sim \frac{1 \chi^2}{2 (1 - 1)}. \]

If we let

\[(2.3.4) \quad U = T_1 + T_2, \quad V = \frac{T_1}{T_1 + T_2} \]

then the joint distribution of \((U, V)\) can be found to have densities

\[(2.3.5) \quad p_{\gamma, U,V}(u, v) \propto \left( \frac{1}{2} \right)^{\frac{1}{2} (J-1)} u^{\frac{1}{2} (J-1)} - 1 v^{\frac{1}{2} (J-1)} - 1 (1 - v)^{\frac{1}{2} (1 - 1)} - 1 e^{\gamma (V + \gamma (1 - V))}. \]

That is, the conditional distribution of \(U\) given \(V\) is given by

\[(2.3.6) \quad U|V \sim \frac{\gamma}{2 (V + \gamma (1 - V))} \chi^2_{(J-1)}. \]

Therefore, from (2.3.2)

\[(2.3.7) \quad R(\psi, (\alpha, \gamma)) = \text{E} \left( \psi(V) - \frac{1}{2J} (1 - \gamma)^2 | U \right) \]

\[= \text{E} \left\{ \frac{\gamma^2 (J - 1) (J + 1)}{4 (V + \gamma (1 - V))^2} \psi(V) - \frac{(1 - \gamma) \gamma (J - 1)}{2 J (V + \gamma (1 - V))} \psi(V) + \frac{(1 - \gamma)^2}{4 J^2} \right\} \]
\[ \mathbb{E}\left[ \frac{Z^{2(J-1)(J+1)}}{4(V+\gamma(1-V))^2} \left[ \frac{\psi(V) - (V+\gamma(1-V))(1-\gamma)}{J\gamma(J+1)} \right]^2 + \frac{(1-\gamma)^2}{4J^2} (1-\gamma)^2 \frac{(I-1)}{2(I+1)} \right]. \]

We are thus led to consider the statistical problem where we observe \( V \) having densities (parametrized by \( \gamma, 0 < \gamma < 1 \))

\begin{equation}
(2.3.8) \quad p_\gamma(V) = \frac{2^{\frac{1}{2}(I+1)\gamma(\frac{1}{2}I(J+1)-1)(1-V)\frac{1}{2}(I-1)-1}}{(V+\gamma(1-V))^{\frac{1}{2}(I+1)}}
\end{equation}

(with respect to Lebesgue measure restricted to \( \{ V : 0 \leq V \leq 1 \} \)), and we wish to estimate \( \frac{(V+\gamma(1-V))(1-\gamma)}{J\gamma(J+1)} \) with loss

\begin{equation}
(2.3.9) \quad L(\psi, \gamma, V) = \frac{\gamma^{2(J-1)(J+1)}}{4(V+\gamma(1-V))^2} \left[ \frac{\psi(V) - (V+\gamma(1-V))(1-\gamma)}{J\gamma(J+1)} \right]^2.
\end{equation}

It is clear from (2.3.7) that \( \Phi(S_1, S_2) \) is admissible among invariant procedures (with squared error as loss) if and only if the corresponding estimate \( \psi(V) \) is admissible in the above problem.

We remark that \( \psi_b \), corresponding to the Bayes invariant procedure \( \Phi_b \), is actually the Bayes procedure (in the reduced problem) with respect to the prior distribution with density \( \gamma^b \). Also note that the proof of lemma 1.1 does not depend on whether or not the loss function depends on the sample point; the proof only considers the risk function as a function of the parameter. Thus, to prove admissibility of \( \psi(-1) \), and, hence, of \( \Phi(-1) \), we can apply lemma 1.1 with parameter \( \theta = \gamma (\sigma = (0, 1)) \), \( \| \) equal to Lebesgue measure (restricted to \( (0, 1) \)), \( \Phi_b \) given by (2.3.8) and \( L \) given by (2.3.9) (and we will use \( f(\gamma) = \gamma^b \)). To apply lemma 1.1, we will need an expression for

\[ K(f) = R(\psi(-1), \gamma) - R(\psi, \|, \gamma), \]

where \( \psi, \| \) will just be \( \psi_b \). The
expression we will use is that given by equation (1.15), for which we will need a bound for \((\Psi_{-1} - \Psi_{-1})^2\). The following technical lemma provides this bound:

**Lemma 2.1:** Let \(\Psi_b = \frac{S_1}{S_1 + S_2}\), where \(\Psi_b\) is given by (2.2.14) (with \(a = -1\)). Then there are \(p_0 > 0\), and constants \(K_1\) and \(K_2\) (independent of \(V\) and \(p\)) such that, for \(p < p_0\),

\[
|\Psi_{-1}(V) - \Psi_{-1+p}(V)| \leq p(K_1 V + K_2 (1 - V))
\]

for all \(V, 0 < V < 1\).

**Proof:** From (2.2.14) we have, with \(V = \frac{S_1}{S_1 + S_2}\),

\[
\Psi_{-1+p}(V) = \frac{1-V}{2J(d+p)} - \frac{V}{2J(c-d-l-p)} + \frac{(c-l)}{2J(d+p)(c-d-l-p)} \cdot \frac{1}{F(c+d+p)(V)}
\]

where \(c\) and \(d\) are given by (2.2.10) with \(a = b = -1\). (Note that, since \(I\) and \(J\) are integers, \(c - d - l \geq 0\) implies that \(c - d - l \geq \frac{1}{2}\); so we can consider \(\Psi_{-1+p}\) for \(p < \frac{1}{2}\); and, also, the steps below hold.) Using (2.2.15), we now show that

\[
\frac{1}{F(c+1,d-p)(V)} = \frac{V^{c-d-p}(1-V)^{d+p+1}}{(1-I_V(c-d-p,d+p+1))\beta(c-d-p,d+p+1)} \leq M_1 V^{c-d-p}.
\]

Inequality (2.3.12) follows from considering the function

\[
f(V) = \frac{(1-V)^{d+p+1}}{(1-I_V(c-d-p,d+p+1))\beta(c-d-p,d+p+1)} = \frac{(1-V)^{d+p+1}}{\int_V^1 u^{c-d-p-1}(1-u)^{d+p} du}.
\]

Differentiating \(f\) we have (for the numerator)

\[
f'(V)(\int_V^1 u^{c-d-p-1}(1-u)^{d+p} du)^2 = -(d+p+1)(1-V)^{d+p} \int_V^1 u^{c-d-p-1}(1-u)^{d+p} du.
\]
\[ + (1 - \nu)^{2d + 2p + l} c^{-d - p - l} \]

\[ \leq - (1 - \nu)^{d + p} c^{-d - p - l} (d + p + l) \int_{0}^{1} (1 - u)^{d + p} du + (1 - \nu)^{2d + 2p + l} c^{-d - p - l} \]

\[ = - c^{-d - p - l} (1 - \nu)^{2d + 2p + l} + (1 - \nu)^{2d + 2p + l} c^{-d - p - l} \]

\[ = 0 . \]

Thus \( f(\nu) \) is decreasing and, hence, is maximized at \( \nu = 0 \) where

\[(2.3.15) \quad f(0) = \frac{1}{\beta(c - d - p, d + p + l)} = \frac{\Gamma(c + l)}{\Gamma(c - d - p + l + d + p + l)} . \]

Therefore (2.3.12) holds, and, in fact, we can take

\[(2.3.16) \quad M_{\nu} = \frac{\Gamma(c + l)}{\Gamma(c - d - p + l + d + p + l)} . \]

Furthermore, since \( c - d - l \geq \frac{1}{2} \), for any \( \epsilon, 0 < \epsilon < \frac{1}{2} \), for \( p \) small enough, we have (from (2.3.12))

\[(2.3.17) \quad \frac{1}{F(c + l, d + p)(\nu)} \leq M_{\nu}(1 + \epsilon) . \]

Now, calculating directly,

\[(2.3.18) \quad \psi(-1)(\nu) - \psi(-1+p)(\nu) = \frac{1 - \nu}{2 \nu} \left( \frac{1}{d} - \frac{1}{d + p} \right) \frac{\nu}{2J(c - d - 1) - \frac{1}{c - d - 1} - \frac{1}{c - d - 1 - p}} \]

\[ + \frac{(c - l)}{2J\nu(c - d - 1)} \cdot \frac{1}{F(c + l, d)(\nu)} - \frac{(c - l)}{2J\nu(c - d - 1 + d + p)} \cdot \frac{1}{F(c + l, d + p)(\nu)} \]

Thus, adding and subtracting \( \frac{(c - l)}{2J\nu(c - d - 1)} \cdot \frac{1}{F(c + l, d + p)} \), we have

\[(2.3.19) \quad |\psi(-1)(\nu) - \psi(-1+p)(\nu)| \leq \frac{1}{2J\nu(d + 1)} + \frac{\nu}{2J(c - d - 1)} \]

\[ + \frac{(c - l)}{2J\nu(c - d - 1)} \left| \frac{1}{F(c + l, d)(\nu)} - \frac{1}{F(c + l, d + p)(\nu)} \right| \]

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\[
+ \frac{1}{F(c+1,d+p)(\nu)} \frac{(c-1)}{2Jc} \frac{1}{d(c-a-1)} - \frac{1}{(d+p)(c-d-p-1)} \cdot
\]

Therefore, using (2.3.17) (with \( \epsilon = 0 \)), we have

\[
(2.3.20) \quad |\psi(-\nu) - \psi(-1+p)| \leq p(M_2V + M_3(1-V)) + M_4 \left| \frac{1}{F(c+1,d+p)} - \frac{1}{F(c+1,d)} \right|
\]

where the \( M_4 \) are independent of \( V \). Now using (2.3.17) again, we have

\[
(2.3.21) \quad \left| \frac{1}{F(c+1,d+p)} - \frac{1}{F(c+1,d)} \right| = \frac{1}{F(c+1,d+p)} \left| \frac{1}{F(c+1,d+p)} - \frac{F(c+1,d+p)}{F(c+1,d)} \right| \leq M_4\nu^{1+\epsilon} \left| \frac{1}{F(c+1,d+p)} - \frac{1}{F(c+1,d)} \right|
\]

Using (2.2.15) to evaluate \( F \), we have

\[
(2.3.22) \quad \frac{F(c+1,d+p)}{F(c+1,d)} = \frac{1}{\int_0^1 (V+x(1-V))c+1 \ dx} \leq 1
\]

Now, we also have the inequality (for any fixed \( \epsilon > 0 \))

\[
(2.3.23) \quad x^p \geq 1 - M_5x^\epsilon
\]

where \( M_5 \) does not depend on \( x \) or \( p \) for \( p \) small enough. This inequality follows from considering the function

\[
(2.3.24) \quad g(u) = u^{p+\epsilon} - u^\epsilon.
\]

Differentiating \( g \) twice, we see that \( g \) is minimized at \( u^* = \left( \frac{\epsilon}{p+\epsilon} \right)^{\frac{1}{p}} \).

Therefore,

\[
(2.3.25) \quad g(u) = u^{p+\epsilon} - u^\epsilon \geq (\frac{\epsilon}{p+\epsilon})^{\frac{p+\epsilon}{p}} - (\frac{\epsilon}{p+\epsilon})^\epsilon \geq (1 + \frac{\epsilon}{p+\epsilon})^{\frac{p}{p+\epsilon}}(\frac{\epsilon}{p+\epsilon} - 1)
\]

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\[ = - \frac{p}{p+\epsilon} (1 + \frac{p}{\epsilon})^{-\epsilon} \]
\[ \geq - M_{\epsilon} p \quad \text{(for } p \text{ small enough)} \]

since \( \frac{1}{p+\epsilon} (1 + \frac{p}{\epsilon})^{-\epsilon} \to \frac{1}{\epsilon} \) as \( p \to 0 \). From (2.3.25), we have

\[ (2.3.26) \quad u^{p+\epsilon} \geq u^\epsilon - M_{\epsilon} p \]

from which (2.3.23) follows.

Inserting (2.3.23) in the numerator of (2.22)

\[ (2.3.27) \quad \frac{F(c+1, \alpha+p)}{F(c+1, \alpha)} \geq 1 - M_{\epsilon} p \cdot \frac{\int_0^1 x^{-\epsilon} dx}{\int_0^1 (x+1)^{c+1} dx} \]

\[ = 1 - M_{\epsilon} p (1 + W^\epsilon) f(W) \]

where \( W = \frac{1-W}{V} \) and

\[ f(W) = \frac{1}{(1+W^\epsilon)} \cdot \frac{\int_0^1 x^\epsilon dx}{\int_0^1 (1+Wx)^{c+1} dx} \]

(2.3.28)

First note that \( f(W) \) is independent of \( p \), and it is continuous on \((0, \infty)\). Next, note that

\[ (2.3.29) \quad \lim_{W \to 0} f(W) = 1 \cdot \frac{\int_0^1 x^\epsilon dx}{\int_0^1 x^\epsilon dx} = \frac{d+1}{\alpha+1-\epsilon} . \]

We also have, changing variables to \( y = Wx \),

\[ (2.3.30) \quad \lim_{W \to \infty} f(W) = \lim_{W \to \infty} \left( \frac{W^\epsilon}{1+W^\epsilon} \right) \cdot \frac{\int_0^1 y^\epsilon dy}{\int_0^1 (1+y)^{c+1} dy} = \frac{\beta(c-d+\epsilon, d+1-\epsilon)}{\beta(c-d, d+1)} . \]
(where $\beta(\cdot, \cdot)$ is the Beta function). Thus, $f(W)$ is continuous on $[0, \infty]$; and, hence, achieves its maximum. That is,

\[(2.3.31) \quad f(W) \leq M_6\]

where $M_6$ does not depend on $W$ or $p$. Therefore, from (2.3.21), (2.3.22), (2.3.27), and (2.3.31), it follows that

\[(2.3.32) \quad \left| \frac{1}{F(c+1,d+p)} - \frac{1}{F(c+1,d)} \right| \leq M_1 M_2 M_3 (1 + V^{-\epsilon})V^{1+\epsilon} \leq 2M_4 V \]

(since $W = \frac{1-V}{V} \leq \frac{1}{V}$).

The lemma follows from (2.3.20) and (2.3.32).

We are now ready to prove that $\Phi_{(-1)}$ is admissible among scale and location invariant estimators.

\textbf{Theorem 2.2:} If $\Phi_{(-1)}$ is defined by (2.2.11) (for $a = b = -1$), then it is admissible among scale and location invariant estimators in the original problem of estimating $\sigma_\alpha$ with squared error loss.

\textbf{Proof:} As previously remarked, it is sufficient to prove that $\Phi_{(-1)}$ is admissible in the reduced problem (described by equations (2.3.8) and (2.3.9)). To do this, we will apply lemma 1.1 with $\parallel$ equal to Lebesgue measure (restricted to $(0, 1)$) and the covering of the parameter space consisting of just one set, $\{\Phi\}$. In particular, we will show that for any $\epsilon > 0$, conditions (1.4), (1.5) and (1.6) are satisfied by $f(\gamma)$ of the form $f(\gamma) = \gamma^b$ for $-1 < b < -\frac{1}{2}$. First note that, for $-1 < b < -\frac{1}{2}$, $\gamma^b \geq 1$ (for $0 < \gamma < 1$); and, hence, (1.4) is satisfied. Furthermore, we showed in section 2.2 (see (2.2.21))
that \( R(\psi_{(a,b)}, (\alpha, \gamma)) \) was bounded for any \( a, b, I, \) and \( J \) for which \( c - d - 1 > 0 \) and \( d > 0 \). Hence, if \( I > 2, J > 2 \) (which is nearly needed for the problem to make sense), \( R(\psi_{(-1)}, (\alpha, \gamma)) \) is bounded. Therefore, from (2.3.7) it follows (since the expectation of the last two terms in the final expression are bounded) that \( E_{\gamma} L(\psi_{(-1)}, \gamma, V) \) is bounded where \( L \) is given by (2.3.9) (and the distribution by (2.3.8)). That is, the risk function in the reduced problem is bounded. Therefore, since \( f(\gamma) \) is integrable, it follows that condition (1.5) holds.

To check condition (1.6) we evaluate \( K(f) = R(\psi_{(-1)}, \gamma) - R(\psi_{b}, \gamma) \) using equation (1.15). First note that the reasoning in equation (1.15) holds even if we allow the functions \( v(\theta) \) and \( g(\theta) \) to depend on \( x \) as well as \( \theta \). The final step in (1.15) only requires that \( \psi_{f} \|_{b} \) (which is \( \psi_{b} \) here) be Bayes versus \( f(\theta) \delta_{\gamma}(\theta) \) (which is \( \gamma^{b} \delta_{\gamma} \) here). But the fact that \( \psi_{b} \) is actually Bayes versus \( \gamma^{b} \) can be checked directly from (2.3.8) and (2.3.9). Therefore, inserting (2.3.9) in (1.15), with the appropriate substitutions, we have, for \( b = -1 + p, \)

\[
(2.3.35) \quad K(f) = \int_{0}^{1} \int_{0}^{1} \gamma^2 (V) \left[ \gamma^2 (\psi_{(-1)} - \psi_{(-1+p)}(V)) \right] \gamma^p (\gamma) d\gamma dV
\]

\[
\leq K_p \frac{2}{3} \int_{0}^{1} \gamma^p \left[ \frac{\gamma^2 (K_1 V + K_2 (1-V))^2}{(V + \gamma (1-V))^2} \right] \gamma^p d\gamma
\]

where the last inequality uses lemma 2.1. Thus,

\[
(2.3.34) \quad K(f) \leq K_p \frac{2}{3} \int_{0}^{1} \gamma^p d\gamma = K_p \leq \epsilon
\]

for \( p \) small enough. Therefore, condition (1.6), and, hence, lemma 1.1
hold. The admissibility of $\Psi\{\gamma\}$ (and, hence, $\Phi\{\gamma\}$) follows from lemma 1.1 and the reasoning following lemma 1.1, using the fact that $p_{\gamma}(v) > 0$ (with $p_{\gamma}$ given by (2.3.8)) and that $L(v, \gamma, \nu)$ is strictly convex (with $L$ given by (2.3.9)).

We now turn to the question of the minimax character of the estimator $\Phi\{\gamma\}$. Unlike the case of admissibility, the question of minimaxity depends strongly on the loss function. In particular, if a procedure is admissible in a statistical problem with a certain loss function, it is admissible in any problem whose loss function is a product of the original loss function and any arbitrary strictly positive function of the parameter. However, if a procedure is minimax in one problem it is not necessarily minimax in any other; and, in fact, unless it is uniformly best, there will always be a loss function (the original loss function times a function of the parameter) for which the procedure is not minimax.

In the problem of estimating $\sigma^2_a$, it is fairly clear that there is some loss function for which $\Phi\{\gamma\}$ is minimax. In particular, if we divide the loss function given by (2.2.6) by $R(\Phi\{\gamma\}, \gamma)$, then (in the new problem) $\Phi\{\gamma\}$ would be a constant risk admissible rule; and, hence, minimax among fully invariant procedures. We could then apply Kiefer's theorem (Kiefer (1957)) to show that $\Phi\{\gamma\}$ is minimax among all procedures. However, in the remainder of this section we will actually show that $\Phi\{\gamma\}$ is minimax for a fairly large class of loss functions, some of which are related to (2.2.6); and we will replace the use of Kiefer's theorem by a direct proof which
should clarify the nature of the minimaxity of \( \Phi(-1) \).

In order to show that \( \Phi(-1) \) is minimax for some loss function, even among invariant procedures, we will need to prove that its risk function is maximized at \( \gamma = 0 \) (\( \sigma_e = 0 \)). We will see, in the proof below, that the loss function given by (2.2.6) does not have this property.

**Lemma 2.2:** If \( \Phi(-1) \) is given by (2.2.11) (with \( a = b = -1 \)), then there is a constant \( M (M > 1) \) such that the risk function \( R(M)(\Phi(-1), \gamma) \) is maximized at \( \gamma = 0 \), where the loss function is given by

\[
L^M(\varphi, (\alpha, \gamma)) = \frac{4\alpha^2 \gamma^2}{(1 + \gamma(M-1))^2} (\varphi - \frac{1-\gamma}{2M\alpha \gamma})^2
\]

(i.e., \( L^M(\varphi, (\sigma_e, \sigma_a)) = \frac{1}{(M\sigma_e + J\sigma_a)^2}(\varphi - \sigma_a)^2 \).

**Proof:** Define \( f(\gamma) \) to be the risk of \( \Phi(-1) \) for the loss function given by (2.2.6) (multiplied by 4); that is

\[
f(\gamma) = E(\alpha, \gamma)L^1(\varphi(-1), (\alpha, \gamma)).
\]

By invariance, we have

\[
f(\gamma) = E(\varphi(-1)) \left( \gamma T_1, T_2 \right) - \frac{1}{3}(1 - \gamma))^2
\]

where

\[
T_1 \sim \chi^2_{I(J-1)} \text{ and } T_2 \sim \chi^2_{I-1}.
\]

We now compute \( f(0) \) and \( f'(0) \). Copying (2.2.14),
\[(2.3.39) \quad \Phi_{(-1)}(S_1, S_2) = \frac{S_2}{2Jd} - \frac{S_1}{2J(c-d-1)} + \frac{(c-1)(S_1+S_2)}{cd(c-d-1)F(c+1,d)(A)}\]

where, in this case,

\[(2.3.40) \quad d = \frac{1}{2}(I+1), c = \frac{1}{2}(I-1) + 1, \text{ and } A = \frac{S_1}{S_1+S_2}.\]

Now (from (2.2.17)) \(\Phi_{(-1)}(\gamma T_1, T_2)\) is bounded by an integrable function; and, using (2.3.12)

\[(2.3.41) \quad \lim_{\gamma \to 0} \Phi_{(-1)}(\gamma T_1, T_2) = \frac{T_2}{2Jd} = \frac{T_2}{J(I+1)}.\]

Therefore, by the dominated convergence theorem,

\[(2.3.42) \quad f(0) = E\left(\frac{T_2}{J(I+1)} - \frac{1}{J}\right)^2 = \frac{2}{J^2(I+1)}.\]

From (2.3.39) (taking derivatives inside the beta integrals)

\[(2.3.43) \quad \Phi'_{(-1)}(S_1, S_2) = -\frac{1}{2J(c-d-1)} + \frac{(c-1)}{2Jcd(c-d-1)}\left\{ \frac{1}{F(c+1,d)(A)} + \frac{(1-A)^{c+2}}{\int_0^1 u^{c+1}du} \right\}\]

where \(\Phi'_{(-1)}(S_1, S_2)\) denotes \(\frac{\partial}{\partial S_1} \Phi_{(-1)}(S_1, S_2)\). Again using (2.3.12), we see that \(\Phi'_{(-1)}(S_1, S_2)\) is uniformly bounded; and that

\[(2.3.44) \quad \lim_{\gamma \to 0} \Phi'_{(-1)}(\gamma T_1, T_2) = -\frac{1}{2J(c-d-1)}.\]

Thus, we can interchange differentiation and integration and obtain,

(again also using dominated convergence)

\[(2.3.45) f'(0) = 2 \lim_{\gamma \to 0} E[\Phi_{(-1)}(\gamma T_1, T_2) - \frac{1}{J}(1-\gamma)(\Phi'_{(-1)}(\gamma T_1, T_2) + \frac{1}{J})].\]
\[
= 2\mathbb{E}\left[\frac{T^2}{2J_d} - \frac{1}{J}\right] \left(- \frac{T}{2J(c-d-1)} + \frac{1}{J}\right)
= \frac{2}{J^2} \left(\frac{I-1}{I+1} - 1\right) \left(- \frac{I(J-1)}{I(J-1)^2} + 1\right)
= \frac{8}{J^{2}(I+1)(I(J-1)-2)}
\]

Now consider the risk, \( R^{(M)} \), for the loss function, \( L^{(M)} \), given by (2.3.35). We have directly

\[(2.3.46) \quad R^{(M)}(\varphi_{(-1)}, \gamma) = \frac{1}{(1+\gamma(M-1))^2} f(\gamma). \]

Therefore,

\[(2.3.47) \quad \frac{d}{d\gamma} R^{(M)}(\varphi_{(-1)}, \gamma) = - \frac{2(M-1)}{(1+\gamma(M-1))^3} f(\gamma) + \frac{1}{(1+\gamma(M-1))^2} f'(\gamma). \]

So letting \( \gamma \to 0 \), we have

\[(2.3.48) \quad \frac{d}{d\gamma} R^{(M)}(\varphi_{(-1)}, 0) = -2(M-1)f(0) + f'(0) < 0 \]

for \( M \) sufficiently large (in fact for \( M > \frac{I(J-1)}{I(J-1)-2} \)). So there is \( \gamma_0 \) such that \( R^{(M)}(\varphi_{(-1)}, \gamma) \) is decreasing for \( 0 \leq \gamma \leq \gamma_0 \). Furthermore, \( f(\gamma) \) is bounded (say by \( K_0 \)); so, for \( \gamma > \gamma_0 \),

\[(2.3.49) \quad R^{(M)}(\varphi_{(-1)}, \gamma) \leq \frac{1}{(1+\gamma_0(M-1))^2} K_0 \leq R^{(M)}(\varphi_{(-1)}, 0) \]

for \( M \) large enough (depending on \( K_0 \) and \( \gamma_0 \)). Therefore, \( R^{(M)}(\varphi_{(-1)}, \gamma) \) is maximized at \( \gamma = 0 \).
We now show that $\Phi(-1)$ is minimax for loss, $L^{(M)}$, given by (2.3.35) for $M$ large enough. It will be clear from the proof below, that $\Phi(-1)$ is actually minimax for any loss function proportional to squared error for which the risk is maximized at $\sigma_e = 0$ ($\gamma = 0$).

Theorem 2.3: Consider the statistical problem where we observe $(\overline{Y}, S_1, S_2)$ with distributions given by (2.1.1), and we wish to estimate $\sigma_a$. If the loss, $L^{(M)}$, is given by (2.3.35), then for $M$ large enough so that lemma 2.2 holds, $\Phi(-1)$ (given by (2.2.11) with $a = b = -1$) is minimax.

Proof: First note that by the minimax theorem (with $\theta = (\sigma_e, \sigma_a, \mu)$),

$$\inf_{\Phi} \sup_{\theta} R^{(M)}(\varphi, \theta) = \inf_{\Phi} \sup_{\varphi} r^{(M)}(\varphi, \Pi) = \sup_{\Pi \in \mathcal{P}} \inf_{\varphi} r^{(M)}(\varphi, \Pi)$$

where $\mathcal{P}$ is the class of all prior probability measures, $\Pi$, concentrated on compact subsets of $\{ (\sigma_e, \sigma_a, \mu) : 0 < \sigma_e < \infty, 0 < \sigma_a < \infty, -\infty < \mu < \infty \}$, and $r^{(M)}(\varphi, \Pi)$ is the expected risk under $\Pi$; i.e.

$$r^{(M)}(\varphi, \Pi) = \int R^{(M)}(\varphi, \theta) d\Pi(\theta).$$

That the conditions for the minimax theorem hold in this problem can be seen from LeCam (1955), for example. Thus, it is sufficient to consider Bayes rules versus priors concentrated on compact sets. Let $\Pi$ be such a prior. Then, since $\Phi_{\Pi}$ minimizes

$$\frac{1}{(M\sigma_e^2 + J\sigma_a^2)} E(\varphi(S_1, S_2, \overline{Y}) - \sigma_a)^2 d\Pi(\sigma_e, \sigma_a, \mu),$$

it is clear that $\Phi_{\Pi}$ is bounded within the support (under $\Pi$) of $\sigma_a$;
that is, $\varphi_{\Pi}$ is uniformly bounded. Furthermore, solving for $\varphi_{\Pi}$, we see that $\varphi_{\Pi}$ is a ratio of Laplace transforms; and, hence, continuous. Furthermore, since $\Pi$ has compact support, it can be shown that $\varphi_{\Pi}(S_1, S_2, \overline{Y})$ is continuous as $S_2 \to 0$. Therefore, as $\sigma_e \to 0$,

\begin{align*}
(2.3.53) \quad \frac{1}{(M\sigma_e + J\sigma_a)^2} E(\sigma_e, \sigma_a, \mu)(\varphi_{\Pi}(S_1, S_2, \overline{Y}) - \sigma_a)^2 \\
= \frac{1}{(M\sigma_e + J\sigma_a)^2} E(1, 1, 0)(\varphi_{\Pi}((\sigma_e + J\sigma_a)S_2^{\#}, \sigma_e S_2^{\#} \sqrt{\frac{E + J\sigma_a}{I_0}} (\overline{Y} + \mu)) - \sigma_a)^2 \\
\to \frac{1}{J^2\sigma_a^2} E(1, 1, 0)(\varphi_{\Pi}(J\sigma_a S_2^{\#}, \sqrt{\frac{J\sigma_a}{I_0}} (\overline{Y} + \mu)) - \sigma_a)^2 \\
= \frac{1}{J^2\sigma_a^2} E(\mu, \sigma_a)(\varphi^*(S_1^{\#}, \overline{Y}^{\#}) - \sigma_a)^2
\end{align*}

where

\begin{equation}
(2.3.54) \quad \overline{Y}^{\#} \sim \mathcal{N}(\mu, \sigma_a) \quad \text{and} \quad S_1^{\#} \sim \sigma_a \chi_{I+1}^2
\end{equation}

and where $\varphi^*$ is the appropriate restriction of $\varphi_{\Pi}$. But it is well known that $S_1^{\#}$ is minimax in the problem of estimating $\sigma_a$ with distributions given by (2.3.54) and loss equal to $\frac{1}{\sigma_a^2}(\varphi - \sigma_a)^2$.

Furthermore, the risk of $\frac{S_1}{I+1}$ is just $\frac{2}{I+1}$. Therefore, (from (2.3.53))

\begin{equation}
(2.3.55) \quad \sup_{(\sigma_e, \sigma_a^2)(M\sigma_e + J\sigma_a)^2} E(\varphi_{\Pi}(S_1, S_2, \overline{Y}) - \sigma_a)^2 \geq \frac{2}{J^2(I+1)}
\end{equation}

Hence, by the minimax theorem,

\begin{equation}
(2.3.56) \quad \inf_{\varphi} \sup_{\theta} R^{(M)}(\varphi, \theta) \geq \frac{2}{J^2(I+1)}
\end{equation}
But, by lemma 2.2, \( R^{(M)}(\varphi_{(-1)}, \gamma) \) is maximized at \( \gamma = 0 \); and its value is (by (2.3.46) and (2.3.42)) \( f(0) = \frac{2}{J^2(I+1)} \). Therefore, theorem 2.3 follows. ||

Note that since \( f'(0) > 0 \), the risk is not maximized at \( \gamma = 0 \) for loss given by (2.2.6); and, hence, \( \varphi_{(-1)} \) is not minimax for this loss function. For similar reasons, \( \varphi_{(-1)} \) is not minimax for the loss function \( L(\varphi, (\sigma_e, \sigma_a)) = \frac{1}{(\sigma_e + \sigma_a)^2} (\varphi - \sigma_a)^2 \) used in the Appendix. Nonetheless, the proofs in this section do indicate that \( \varphi_{(-1)} \) has a fairly strong minimax property. In particular, it does as well as can be done (with respect to any loss proportional to squared error) at \( \gamma = 0 \) (i.e., at \( \sigma_e = 0 \)). Thus, there is no rule with risk uniformly smaller by a fixed amount than the risk of \( \varphi_{(-1)} \). Furthermore, calculations similar to those leading to (2.3.46) show that the derivative of \( R(\varphi_b, \gamma) \) at \( \gamma = 0 \) is minimized at \( b = -1 \); that is, \( \varphi_{(-1)} \) is better than any other \( \varphi_b \) near \( \gamma = 0 \). These reasons, together with the numerical calculations presented in the Appendix, certainly indicate that \( \varphi_{(-1)} \) is a reasonable estimator in terms of mean squared error.
Chapter 3
Some Related Estimates and Conclusions

3.1 Related Estimates

We now discuss the problem of estimating \( \sigma_e \) in the previously described analysis of variance problem. As before, we observe

\[
S_1 \sim \frac{1}{2\sigma} \chi^2_{I-1} \quad \text{and} \quad S_2 \sim \frac{1}{2\alpha} \chi^2_{I-1}
\]

where we are using the same parameterization

\[
\alpha = \frac{1}{2\sigma} \quad \text{and} \quad \gamma = \frac{\sigma}{\sigma + J\sigma_a}.
\]

Again we take the loss function to be

\[
L(\varphi, (\alpha, \gamma)) = \alpha^2 \gamma^2 (\varphi - \frac{1}{2\alpha})^2
\]

and consider formal Bayes estimators with respect to prior measures

\[
d\|\gamma = \alpha^{-b} \alpha^{-d} \, d\alpha \, d\gamma.
\]

Again letting

\[
c = \frac{1}{2}(IJ - 1) + a + 2 \quad \text{and} \quad d = \frac{1}{2}(I - 1) + b + 2
\]

we find (as in (2.2.11), and using (2.2.13)),

\[
\varphi(a, b)(S_1, S_2) = \frac{\int_0^{\frac{1}{2\sigma} \chi^2_{I-1}} \alpha^{c} \gamma^{d} e^{-\alpha(S_1 + \gamma S_2)} \, d\alpha \, d\gamma}{\int_0^{\frac{1}{2\alpha} \chi^2_{I-1}} \alpha^{c} \gamma^{d} e^{-\alpha(S_1 + \gamma S_2)} \, d\alpha \, d\gamma}
\]
\[
\frac{1}{2c_0} \left( \frac{1}{(S_1 + S_2)^c} - \int_0^1 \frac{\gamma^d}{(S_1 + S_2)^{c+1}} \, d\gamma \right) = \frac{S_1}{2(c-d-1)} \left[ 1 - \frac{1}{cA^F(c+1,d)(A)} \right]
\]

where \( A = \frac{S_1}{S_1 + S_2} \) and \( F(c+1,d)(A) \) is given by (2.2.15). We also have, directly,

\[(3.1.7) \quad \varphi(a,b)(S_1, S_2) = \frac{S_1}{2(c-d-1)} \left[ \frac{1-\text{I}_A(c-d-1,d+1)}{\text{I}_A(c-d, d+1)} \right]
\]

where \( \text{I}_A(\cdot, \cdot) \) is the incomplete beta function (again, see Pearson (1934)).

The proofs in chapter 2 can now be used to show admissibility of the appropriate \( \varphi(a,b) \). However, in this case there is a slight difference. From (3.1.6) we have

\[(3.1.8) \quad \varphi(a,b)(S_1, S_2) \leq \frac{S_1}{2(c-d-1)}. \]

Therefore, since \( \varphi(a,b) \) is positive, the risk satisfies

\[(3.1.9) \quad R(\varphi(a,b); (a, \gamma)) = \alpha^2 \gamma^2 \mathbb{E}(\varphi(a,b)(S_1, S_2) - \frac{1}{2a})^2 \leq M \gamma^2
\]

where \( M \) is a fixed constant depending on \( a, b, I, J \) and finite whenever \( c - d - 1 > 0 \). That is, the risk is integrable with respect to \( \Pi \) as long as \( b > -3 \) (instead of \( b > -1 \)). Furthermore, using similar reasoning to that in chapter 2, we can find in this case (using notation of chapters 1 and 2),

39
(3.1.10) \[ E_{1^2}(S_1, S_2, \alpha) \]
\[ E_{2^2}(S_1, S_2, \alpha) \]
\[ \leq M^* \]

(in contrast to (2.2.28)). Therefore, theorem 1.1 implies that \( \varphi(a, b) \) is admissible for \( a = -1 \) and \( b > -3 \).

As before, we can show that \( \varphi(-1,-3) \) is admissible among scale invariant estimators. However, in this case, it is not immediately clear which \( \varphi(-1,b) \) to use. Using loss (5.1.3), the risk of any \( \varphi(a, b) \) tends to 0 as \( \gamma \to 0 \); and, hence, \( \varphi(-1,-3) \) which does best at \( \gamma = 0 \) is not particularly indicated. Furthermore, even if we use the loss function

(3.1.11) \[ L^*(\varphi, (\alpha, \gamma)) = \alpha^2(\varphi - \frac{1}{2\alpha})^2 \]

for which \( R^*(\varphi(a, b), \gamma) \) is bounded away from 0, we still find that \( R^*(\varphi(-1,-3), \gamma) \) is larger at \( \gamma = 1 \) than at \( \gamma = 0 \). Thus, I can find no simple satisfactory theoretical reason for preferring one \( \varphi(-1,b) \) over another. However, numerical calculations in the Appendix do seem to indicate that a certain choice of \( b \) should be made for each \( I \) and \( J \) (in particular, that \( b \) minimizing the risk at \( \gamma = 1 \)). These conclusions will be discussed later.

We conclude this section with a few words concerning extensions to higher way layouts. Again, similar considerations should lead to reasonable Bayes invariant estimators. In this case, however, there are two minor problems. First, in higher way designs the formal Bayes estimators will no longer be expressable in terms of tabulated functions. However, this is not too serious considering the availability of high

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speed computers. The second problem may be more serious. It is clear
that we can find a class of formal Bayes estimators which are admissible
(among location invariant estimators). However, choosing a particular
one might be more difficult. It seems likely that limits of rules in
the class should be reasonable for estimating various "between" groups
variances, but I feel that extensive numerical calculations will be
necessary before any formal Bayes estimators can be seriously recommended.

3.2 Conclusions

There is, I think, sufficient evidence to seriously recommend the
use of a formal Bayes rule for estimating $\sigma_a$ in the model II analysis
of variance problem considered here. Although one may claim that point
estimation of $\sigma_a$ in this problem is not a serious or useful statisti-
cal consideration, there are problems, I think, where point estimates
are really desired. For example, one may want to use the data to plan
future experiments, or one may want to compare the data with some other
results and perhaps estimate a correlation coefficient. In either of
these problems, the use of a negative, or even a zero, estimate is
unacceptable. Thus, a Bayes or formal Bayes estimate must be used.
The formal Bayes estimates considered in chapter 2 are scale and loca-
tion invariant estimates with what I feel are adequate mean squared
error properties. The theoretical considerations in chapter 2 certainly
indicate that no estimate can be substantially better than $\Phi(-1)$
(given by (2.2.11) for $a = b = -1$). Numerical calculations (Table I)
indicate that other estimates are better than $\Phi(-1)$ only very near
\[ \frac{\sigma_e}{\sigma + \sigma_a} = 1. \] Elsewhere, \( \varphi(-1) \) is just as good and is actually a substantial improvement for small and moderate sample sizes when \( \frac{\sigma}{\sigma + \sigma_a} \) is between .5 and .8. Thus, unless we really believe \( \frac{\sigma}{\sigma + \sigma_a} \approx 0 \), use of \( \varphi(-1) \) seems strongly indicated. Furthermore, if we actually have strong prior belief that \( \frac{\sigma_a}{\sigma + \sigma_a} \approx 0 \) then we could use one of the other \( \varphi(b) \) for \( b > -1 \), which will do better at \( \frac{\sigma}{\sigma + \sigma_a} = 0 \) in exchange for worse behavior at \( \frac{\sigma_a}{\sigma + \sigma_a} = 1. \)

Recommendation of the formal Bayes estimators of \( \sigma_e \) does not seem quite as strongly indicated, primarily because of the computational difficulty in calculating them. In this case, a single \( \varphi_b \) can not be recommended at all. However, table II of the appendix shows that for each \( I \) and \( J \), there is a \( b \) such that \( \varphi_b \) has smaller mean squared error than any other suggested estimate except very near \( \frac{\sigma}{\sigma + \sigma_a} = 0 \) (i.e., for \( \frac{\sigma}{\sigma + \sigma_a} < .1 \)). Furthermore, use of the appropriate \( \varphi_b \) can lead to improvement of up to 20\% in mean squared error. Thus, one can reasonably recommend the use of a formal Bayes estimator of \( \sigma_e \), although more extensive numerical calculations would be necessary to pinpoint the best one in any particular case (integral values for the coefficients are not actually required). Nonetheless, the maximum likelihood estimate (A.6) or the related estimate (A.7) are nearly as good and somewhat easier to calculate.

In conclusion, formal Bayes invariant procedures appear to offer reasonable improvements over previously considered rules in certain multi-parameter problems; and consideration of them is certainly highly recommended, especially in the estimation of parameters whose values are more restricted than the values of natural estimators.
Appendix

Numerical Calculations of Mean Squared Errors for Estimators of $\sigma_a$ and $\sigma_e$

A computer program was used to calculate expectations with respect to the following distributions

\begin{align*}
S_1 &\sim \sigma_a \chi^2_{I(J-1)} \quad \text{and} \quad S_2 \sim (\sigma_e + J\sigma_a) \chi^2_{I-1}.
\end{align*}

Table I and table II give the following expectation as a function of $\frac{\sigma_a}{\sigma_e + \sigma_a}$ for estimators $\varphi$ of $\sigma_a$ and $\sigma_e$ respectively:

\begin{align*}
\text{Mean Squared Error} &= \frac{1}{(\sigma_e + \sigma_a)^2} E(\varphi, \varphi_0)(\varphi - \sigma)^2
\end{align*}

(where $\sigma = \sigma_e$ in table I and $\sigma = \sigma_a$ in table II). This form of the risk was chosen so that results could be directly compared to those in Klotz and Milton (1967).

In table I, we compare the above risk of formal Bayes estimators $\varphi_b$ defined by (2.2.11) for $a = -1$ and the listed values of $b$ with values for the following estimators given in Klotz and Milton (1967):

\begin{align*}
\varphi^{(1)}(S_1, S_2) &= \frac{1}{J} \left[ \frac{S_2}{I+1} - \frac{S_1}{I(J-1)} \right]^+ \\
\varphi^{(2)}(S_1, S_2) &= \frac{1}{J} \left[ \frac{S_2}{I+2} - \frac{S_1}{I(J-1)+2} \right]^+ \\
\varphi^{(3)}(S_1, S_2) &= \frac{1}{J} \left[ \frac{S_2}{I+1} - \frac{S_1}{I(J-1)-2} \right]^+
\end{align*}

where $[x]^+ = \max(x, 0)$. 

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In table II we compare the risk of formal Bayes estimators $\varphi_b$ defined by (3.1.6) for $a = -1$ and the listed values of $b$ with values for the following estimators:

(A.6) \[ \varphi^{(4)}(s_1, s_2) = \min \left( \frac{s_1}{I(J-1)}, \frac{s_1 + s_2}{IJ} \right) \]

(A.7) \[ \varphi^{(5)}(s_1, s_2) = \min \left( \frac{s_1}{I(J-1)+2}, \frac{s_1 + s_2}{IJ+4} \right) \]

(A.8) \[ \varphi^{(6)}(s_1, s_2) = \min \left( \frac{s_1}{I(J-1)+1}, \frac{s_1 + s_2}{IJ+4} \right) \]

Here (A.6) is the maximum likelihood estimate.
**TABLE I**

Mean Squared Errors of Estimates of $\sigma_a = \frac{1}{\sigma_e + \sigma_a} E(\varphi - \sigma_a)^2$

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### TABLE II

Mean Squared Errors of Estimates of $\sigma_e$: $\frac{1}{(\sigma_e^2 + \sigma_a^2)} E(\varphi - \sigma_e)^2$

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**3. REPORT TITLE**

Formal Bayes Estimation with Application to a Random Effects Analysis of Variance Model

**4. DESCRIPTIVE NOTES (Type of report and inclusive dates)**

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**5. AUTHOR(S) (Last name, first name, initial)**

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Logistics and Mathematical Sciences Branch
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**13. ABSTRACT**

Techniques of formal Bayes estimation are discussed and applied to the estimation of the components of variance in the one way layout random effects ANOVA model. Invariant formal Bayes estimators are presented and proved to be admissible in certain classes of invariant rules and also minimax. To do this a general sufficient condition for admissible in certain multiparameter problems is given and applied to the ANOVA problem.

Numerical calculations of mean squared errors of various estimators are also presented, and these results help justify the recommendation of the formal Bayes estimators.
Formal Bayesian Estimation
Analysis of Variance, Model II
Admissibility
Components of Variance

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