ON A MONOTONE CHARACTER OF SOME INVARIANT STATISTICS

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GOVIND S. MUDHOLKAR

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1. Introduction and Summary.

Let \( X = (X_1, X_2, \ldots, X_n) \) be a random vector with the probability density function \( h(x|\theta) \). The purpose of this article is to report an extension of some results [7] concerning a variation of the distributions and moments for a class of statistics \( T(X) \) with respect to changes in the location parameter vector \( \theta \) and to discuss some particular cases including the case in which the components of \( X \) are independent identically distributed random variables arising out of simple random sampling. These results generalize the monotonically increasing (decreasing) nature of the expectation \( E_\theta(T) \) and the monotonically decreasing (increasing) nature of the distribution function \( F_\theta(t) \) — where both are considered as functions of \( \theta \) — of a symmetric convex (concave) function \( T \) of a unimodal random variable \( X \) with the probability density function \( f(x|\theta) \) symmetric about \( \theta \). The monotonicity properties, in turn, are among the various consequences of an elementary inequality,

\[ \int_{-a+\theta}^{a+\theta} f(x) dx \leq \int_{-a+\varphi}^{a+\varphi} f(x) dx, \text{ if } |\theta| \geq |\varphi|, \tag{1} \]

satisfied by the symmetric unimodal function \( f \). Anderson [1] generalized this inequality by proving the following Theorem 1, which has been exploited by many authors, including Das Gupta, Anderson, Mudholkar [3],
Jorgeco [4], Sidak [10], Wolfowitz [11]. We obtain the results of this article as consequences of a generalization of Theorem 1.

**Theorem 1.** Let $E$ be a convex set in $\mathbb{R}^n$, symmetric about the origin. Let $h(x) \geq 0$ be a function such that

(i) $h(x) = h(-x)$,

(ii) $\{x|h(x) \geq u\} = K_u$ is convex for every $u$, $0 \leq u \leq \infty$, and

(iii) $\int_E h(x)dx < \infty$ (in the Lebesgue sense). Then for any $x$,

\[
(2) \quad \int_E h(x+ky)dx \geq \int_E h(x)dx,
\]

for $0 \leq k \leq 1$.

In section 2 a generalization of Theorem 1 is outlined in brief and the monotone character of the expected values and the distribution functions is developed. Some particular cases of the results of section 2 are discussed in section 3. In section 4 we give some examples of populations and statistics to which the inequalities of the previous sections can be applied. The final section contains a few miscellaneous observations and comments.

2. **The Monotone Character.**

In this section we develop the main result of this article. The result extends a generalized monotone character of the distribution functions and moments of invariant statistics of random vectors with invariant unimodal probability density functions obtained as a consequence of the following Theorem 2 due to Mudholkar [6] which is a generalization of Theorem 1.
Theorem 2. Let $G$ be a group of linear Lebesgue measure-preserving transformations of $\mathbb{R}^n$ onto $\mathbb{R}^n$. Let a real-valued function $h(x)$ on $\mathbb{R}^n$ satisfy (i) the $G$-invariance condition: $h(gx)$ for all $x$ and $g \in G$, (ii) unimodality condition: $\{x : h(x) \geq u\} = K_u$ is convex for each $u$, $0 \leq u \leq \infty$, and (iii) $\int_{E} h(x) dx < \infty$, in the Lebesgue sense.

Let a convex subset $E$ of $\mathbb{R}^n$ be $G$-invariant, i.e., $x \in E$ implies $gx \in E$, all $g \in G$. Then for any fixed $\theta$ in $\mathbb{R}^n$ and any $\varphi$ in the convex hull of the $G$-orbit of $\theta$.

\[(3) \quad \int_{E+\theta} h(x) dx \geq \int_{E+\varphi} h(x) dx .\]

Theorem 2 was further generalized by Mudholkar [8]. We briefly outline this generalization for the sake of completeness. Let $G$ be a group of linear Lebesgue measure-preserving transformations of $\mathbb{R}^n$ onto $\mathbb{R}^n$. Let $E$ and $F$ be two compact, convex, $G$-invariant sets of $\mathbb{R}^n$. Consider the convolution $\psi(y) = \chi_{E^*F}(y)$ of the indicator functions of the sets. A crucial step in the proofs of Theorem 1 and its generalizations is the use of the Brunn-Minkowski inequality.

Appealing to this inequality and recalling the attributes of the group $G$, it is easy to show that the set $\{y : \psi(y) \geq u\}$ is compact, convex, and $G$-invariant.

Now let $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}$ be the convex cones generated by compact, convex, $G$-invariant sets of $\mathbb{R}^n$ and closed, respectively, in the uniform norm $\|f\|_0 = \sup_x |f(x)|$, in the $L_1$-norm $\|f\|_1 = \int |f(x)| dx$ and in the norm $\|f\| = \max(\|f\|_0, \|f\|_1)$. Also let $\psi(x, u)$ be the
indicator function of the set \( \{ y \mid \psi(y) \geq u \} \). Then observing that
\[
\lim_{s \to 0^+} \| \sum_{j=1}^s \psi(x,j\delta) - \psi(x) \| = 0
\]
in either of the three norms, we get the following

**Theorem 3.** The closed convex cones \( C_0, C_1, \) and \( C \) generated by compact, convex, \( G \)-invariant sets of \( \mathbb{R}^n \) are closed with respect to convolution. That is,

\[
(4) \quad C_0 \ast C_0 \subset C_0, \quad C_1 \ast C_1 \subset C_1, \quad C \ast C \subset C .
\]

If \( \omega \) is in the convex-hull of the \( G \)-orbit of \( \varrho \), then \( X_E(\omega) \geq X_E(\varrho) \) for any compact, convex, \( G \)-invariant set \( E \). Furthermore, it is easy to see that \( \int_{E+\varrho} h(x)dx \) is in \( C \) for any function \( h \) on \( \mathbb{R}^n \) satisfying the conditions of Theorem 2, that is, for any unimodal, \( G \)-invariant \( h \geq 0 \). Therefore, the following Theorem 3 may be considered as a generalization of Theorem 2.

**Theorem 4.** For any \( \varphi \in C \), any \( \varrho \in \mathbb{R}^n \) and any \( \omega \) in the convex-hull of the \( G \)-orbit of \( \varrho \), \( \varphi(\omega) \geq \varphi(\varrho) \). In particular, for any \( h \in C \) and any compact, convex, \( G \)-invariant \( E \),

\[
(5) \quad \int_{E+\omega} h(x)dx \geq \int_{E+\varrho} h(x)dx .
\]

The following theorem is an immediate consequence of Theorem 2 and Theorem 3.

**Theorem 5.** Let \( X = (X_1, X_2, \ldots, X_n) \) be a random vector with the probability density function \( h(x-\varrho) \) such that \( h(x) \in C \). Then for any
statistic $T(\bar{x}) \in \mathcal{C}$, any $\varrho \in \mathbb{R}^n$ and any $\omega$ in the convex-hull of the G-orbit of $\varrho$

\[(6)\quad E_{\omega}(T) \geq E_{\varrho}(T).\]

The following theorem summarizes the monotone character of the distribution functions and the expected values.

**Theorem 6.** Let $\bar{x} = (X_1, X_2, \ldots, X_n)$ be a random vector with the probability density function $h(x-\varrho)$ such that $h(\bar{x}) \in \mathcal{C}$. For any G-invariant statistic $T(\bar{x})$, any fixed $\varrho \in \mathbb{R}^n$, and any $\omega$ in the convex-hull of the G-orbit of $\varrho$, the distribution function and the expected value of $T(\bar{x})$, provided that it exists, satisfy

\[(7)\quad F_{\omega}(t) \geq F_{\varrho}(t), \quad -\infty < t < \infty\]

\[(8)\quad E_{\omega}(T) \leq E_{\varrho}(T),\]

if $\{\bar{x} | T(\bar{x}) \leq t\}$ is convex for each $t$, $-\infty < t < \infty$. The inequalities are reversed if the complement of $\{\bar{x} | T(\bar{x}) \leq t\}$ is convex for each $t$, $-\infty < t < \infty$.

**Proof.** The former of the two inequalities follows, trivally, from Theorem 3. The latter of the inequalities follows from the former because of the relation

\[(9)\quad E(T) = \int_{-\infty}^{\infty} (1 - F^T(t))dt - \int_{-\infty}^{0} F^T(t)dt,\]

provided that the expected value exists.
In particular, if a C-invariant statistic $T(X)$ is a monotonically increasing function of a real-valued convex function of $X$, or a monotonically decreasing function of a real-valued concave function of $X$, then $\frac{F^T(t)}{\xi^2} \geq \frac{F^T(t)}{\xi^2}$, for every $t$, and $E_{\xi^2}(T) \leq E_{\xi^2}(T)$. On the other hand, if the C-invariant statistic $T(X)$ is a monotonically increasing function of a real-valued concave function of $X$, or a monotonically decreasing function of a real-valued convex function of $X$, then $\frac{F^T(t)}{\xi^2} \leq \frac{F^T(t)}{\xi^2}$ for every $t$, and $E_{\xi^2}(T) \geq E_{\xi^2}(T)$, provided that the expected value exists.

Thus, for example, if a random variable $X$ has one of the following probability density functions,

$$f(x) = \left(\frac{\sigma}{\sqrt{2\pi}}\right)^{-1} \exp\left\{-\frac{1}{2\sigma^2} (x-\mu)^2\right\}, \quad -\infty < x < \infty,$$

$$f(x) = \frac{1}{2} \exp\left\{-|x-\mu|\right\}, \quad -\infty < x < \infty,$$

$$f(x) = 1, \text{ if } |x-\mu| \leq \frac{1}{2}, f(x) = 0 \text{ otherwise },$$

$$f(x) = |x-\mu|, \text{ if } |x-\mu| \leq 1, f(x) = 0 \text{ otherwise },$$

then $E|X|^r$ is an increasing function of $|\mu|$ and $E(e^{-X^2})$ is a decreasing function of $|\mu|$. However, interesting applications of the theorem concern structurally rich $\mathbb{R}^n$, that is, they concern random vectors with invariant probability density functions. Some of the groups of transformations of special interest to statisticians are the orthogonal group, the group of reflections in the origin, various groups
of reflections in coordinate planes, and, perhaps most interesting, the full group of permutations. We now discuss some of these particular cases.

3. Some Particular Cases.

(i) If $G$ is the group of reflections in the origin, then any element in the convex-hull of the $G$-orbit of any $\varphi \in \mathbb{R}^n$ is of the form $k\varphi$, $-1 \leq k \leq 1$, and $\mathcal{C}$ is the closed convex cone generated by compact, convex sets symmetric about the origin. In particular, all functions $h \geq 0$ on $\mathbb{R}^n$, symmetric about the origin, decreasing along any ray through the origin and having convex contours, as well as convolutions of such functions, belong to $\mathcal{C}$. So the following is a corollary to the theorems in the previous section.

**Corollary 1.** If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a random vector with the probability density function $h(x; \varphi)$, where $h(x)$ belongs to the closed convex cone generated by compact, convex sets symmetric about the origin, and if $T(\mathbf{x})$ is a statistic in the same cone, then

$$E_{k\varphi}(T) \geq E_{\varphi}(T), \quad -1 \leq k \leq 1.$$  

If $T(\mathbf{x})$ is a symmetric statistic such that $\{x : T(x) \geq t\}$ is convex for each $t$ (and therefore in $\mathcal{C}$), then (10) is a fortiori true because of the inequality

$$E_{k\varphi}^m(t) \leq E_{\varphi}^m(t), \quad -1 \leq k \leq 1, \quad -\infty < t < \infty.$$
(ii) If \( G \) is the orthogonal group, then the \( G \)-orbit of any \( \theta = (\theta_1, \theta_2, \ldots, \theta_n) \) is the hypersphere centered at the origin and with the radius \( (\Sigma \theta_i^2)^{1/2} \), any element \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \) in the convex-hull of the \( G \)-orbit of \( \theta \) is of the form \( kA\theta \), \( 0 \leq k \leq 1 \), for some orthogonal matrix \( A \), and satisfies \( \Sigma \omega_i^2 \leq \Sigma \theta_i^2 \). The cone \( \mathcal{C} \) in this case is generated by the family of hyperspheres centered at the origin and coincides with the family of spherically symmetric functions which are unimodal in Anderson's sense. So we have

**Corollary 2.** Let \( X \) be a random vector with the probability density function \( h(x - \theta) \), such that \( h(Ax) = h(x) \) for each \( x \in \mathbb{R}^n \) and each orthogonal matrix \( A \), and \( \{ x | h(x) \geq u \} \) is convex for each \( u \), \( 0 \leq u \leq \infty \). Then for any spherically symmetric statistic \( T(X) \) such that \( \{ X | T(X) \geq t \} \) is convex for each \( t \), we have

\[
\frac{F_{k\theta}^T(t)}{F_{\theta}^T(t)} \leq \frac{F_{k\theta}^T(t)}{F_{\theta}^T(t)},
\]

and

\[
F_{k\theta}(T) \geq F_{\theta}(T),
\]

\( 0 \leq k \leq 1 \), and each orthogonal matrix \( A \). The inequalities are reversed if for the spherically symmetric statistic \( T \), \( \{ X | T(X) \leq t \} \) is convex for each \( t \), \( -\infty < t < \infty \).

(iii) The full permutation group is among the more interesting groups from the point of view of statistical applications. Invariance under such a group of transformations has a role in the analysis of random vectors having symmetric distributions, in particular, the random
vectors with independently, identically distributed components generated, for example, by simple random sampling. The members of the permutation group can be represented in terms of the $N = n!$ permutation matrices $P_i$, $i=1,2,...,N$. It is well known that for any $\theta \in \mathbb{R}^n$, any element in the convex-hull of the $N$ transforms $P_i \theta$, $i=1,2,...,N$, is of the form $S\theta$, where $S$ is a doubly stochastic matrix. Furthermore, for a given pair $S, \omega \in \mathbb{R}^n$, if $\omega$ belongs to the convex-hull of the permutations of $S$, that is, there exists a doubly stochastic matrix $\tilde{S}$ such that $\omega = \tilde{S}\theta$, if, and only if,

$$\sum_{j=1}^{k} \omega(j) \leq \sum_{j=1}^{k} \theta(j), \quad k = 1,2,...,n-1,$$

and

$$\sum_{j=1}^{n} \omega(j) = \sum_{j=1}^{n} \theta(j),$$

where $\theta(1) \geq \theta(2) \geq \cdots \geq \theta(n)$ and $\omega(1) \geq \omega(2) \geq \cdots \geq \omega(n)$ are the ordered values of the coordinates $\theta_1, \theta_2, \ldots, \theta_n$ of $\theta$ and $\omega_1, \omega_2, \ldots, \omega_n$ of $\omega$, respectively. The closed cone $\mathcal{C}$, in this case is generated by compact, convex sets symmetric with respect to permutations and contains all nonnegative, permutation symmetric, unimodal (in Anderson's sense) functions and their convolutions. Now let $X$ be a random vector with the probability density function $h(x|\theta)$ such that $h \in \mathcal{C}$. Then for any compact, convex, permutation symmetric set $E$, any $\theta \in \mathbb{R}^n$ and any doubly stochastic matrix $S$

$$\int_{E+S\theta} h(x) dx \geq \int_{E+\theta} h(x) dx,$$
By suitably choosing statistics $T$ and using for $E$ sets of the form $\{x|T(x) \leq t\}$ or of the form $\{x|T(x) \geq t\}$, we get the following

**Corollary 3.** Let $X$ be a random vector with the probability density function $\, h(x-q), \,$ such that $he\mathcal{C}$, the closed convex cone generated by compact, convex, permutation symmetric sets. If $T(x) \geq 0$ is a function in $\mathcal{C}$, then

\[(15) \quad E_{\mathcal{E}_T}(T) \geq E_{\mathcal{E}_T}(T) .\]

If instead a permutation symmetric statistic $T(X)$ is either a monotonically increasing function of a real-valued concave function of $X$ or a monotonically decreasing function of a real-valued convex function of $X$, then (15) a forteriori is true because of the relation

\[(16) \quad F_{\mathcal{E}_T}(t) \leq F_{\mathcal{E}_T}(t) ,\]

for each $t$. The inequalities in (15) and (16) are reversed if the symmetric statistic $T(X)$ is either a monotonically decreasing function of a real-valued concave function of $X$ or a monotonically increasing function of a real-valued concave function of $X$.

We now proceed to discuss some examples of relevant probability density functions and statistics.

4. **Examples of Populations and Statistics.**

The inequalities in the foregoing theorems and corollaries are applicable to monotone functions of real-valued convex and concave functions of observations with joint probability density functions.
belonging to certain closed convex cones of invariant functions. As has been noted earlier, the invariant unimodal functions as well as their convolutions belong to these cones. If the observations are independently distributed, then the following theorems give sufficient conditions for the invariant joint probability density function to be a member of the cone.

**Theorem 7.** Let $X_1, X_2, \ldots, X_n$ be $n$ independently distributed random variables with the probability density functions $f_{X_1}(x_1)$, which are concave functions of $x_1$, $i=1,2,\ldots,n$. Then the joint probability density function of $X_1, X_2, \ldots, X_n$ is unimodal.

**Proof.** We have to show that $h_X(x) = \prod_{i=1}^{n} f_{X_i}(x_i)$ is such that $\{x \mid f_X(x) \geq u\}$ is convex for each $u$. If $\alpha \geq 0$, $\bar{\alpha} = 1-\alpha > 0$ and $x, y \in \mathbb{R}^n$, then

$$\left\{ h_X(\alpha x + \bar{\alpha} y) \right\}^{1/n} = \left\{ \prod_{i=1}^{n} f_{X_i}(\alpha x_i + \bar{\alpha} y_i) \right\}^{1/n}$$

$$\geq \left\{ \prod_{i=1}^{n} [\alpha f_{X_i}(x_i) + \bar{\alpha} f_{X_i}(y_i)] \right\}^{1/n}$$

$$\geq \alpha \prod_{i=1}^{n} f_{X_i}(x_i) + \bar{\alpha} \prod_{i=1}^{n} f_{X_i}(y_i),$$

because of the concave nature of the geometric mean $\left( \prod_{i=1}^{n} t_i \right)^{1/n}$ of any $n$ nonnegative numbers $t_1, t_2, \ldots, t_n$. Thus $\{h_X(x)\}^{1/n}$ is a concave function of $x$, which implies the unimodality of $h_X(x)$. 

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Theorem 8. Let \( X_1, X_2, \ldots, X_n \) be independently distributed random variables such that \( \log f^\prime_{X_i}(x_i) \), where \( f^\prime_{X_i}(x_i) \) is the probability density function of \( X_i \), is a concave function of \( X_i \), \( i=1,2,\ldots,n \).
Then the joint probability density function \( h_x(x) = \frac{1}{c} \prod_{i=1}^{n} f^\prime_{X_i}(x_i) \) is unimodal.

Proof. If \( h^\prime_x(x) > c \) and \( h^\prime_x(y) > c \), then for \( \alpha > 0, \bar{\alpha} = 1-\alpha > 0, \)

\[
\log h^\prime_x(\alpha x + \bar{\alpha} y) = \sum_{i=1}^{n} \log f^\prime_{X_i}(\alpha x_i + \bar{\alpha} y_i) \\
\geq \alpha \sum_{i=1}^{n} \log f^\prime_{X_i}(x_i) + \bar{\alpha} \sum_{i=1}^{n} \log f^\prime_{X_i}(y_i) \\
\geq \alpha \log c + \bar{\alpha} \log c = \log c ,
\]

which implies the convexity of \( \{x|h^\prime_x(x) > c \} \) and consequently the unimodality of \( h^\prime_x(x) \).

Corollary 4. If \( X_1, X_2, \ldots, X_n \) are independently distributed random variables with PF2 probability density functions, then the joint probability density function of \( X_1, X_2, \ldots, X_n \) is unimodal.

If \( X_1, X_2, \ldots, X_n \) are independently distributed random variables with, for instance, any of the following examples as their probability density functions, then the unimodality of the joint probability density function of the \( X_i \)'s may be verified either by checking the second derivative of \( \log f \), wherever possible, and concluding its concavity, or by appealing to the PF2 character of the individual probability density functions.
(i) Normal populations. \( f(x) = \left(\sigma \sqrt{2\pi}\right)^{-1} \exp\left[-(x-\mu)^2/(2\sigma^2)\right] \)
\[-\infty \leq x \leq \infty ,\]

(ii) Gamma populations. \( f(x) = \alpha^{p} [\Gamma(p)]^{-1} e^{-x} x^{p-1} \),
\[0 \leq x \leq \infty ,\]
\[= 0 , \text{ otherwise},\]
for \( p \geq 1 \). This includes the exponential population.

(iii) Doubly exponential population. \( f(x) = \frac{1}{2} e^{-|x|} \),
\[-\infty \leq x \leq \infty .\]

(iv) Beta population. \( f(x) = [\beta(p, q)]^{-1} x^{p-1} (1-x)^{q-1} \),
\[0 \leq x \leq 1 ,\]
\[= 0 , \text{ otherwise},\]
for \( p, q \geq 1 \). This includes the rectangular population over \((0, 1)\).

(v) Logistic population. \( f(x) = e^{-x/\alpha}/[\alpha(1+e^{-x/\alpha})^2] \),
\[-\infty < x < \infty , \alpha > 0.\]

Remark. Clearly in many cases the unimodality of the joint probability density functions could be directly concluded. For example, if the joint probability density function is of the form \( \exp -U(x) \), where \( U \) is a convex function on \( \mathbb{R}^n \), then it is unimodal. Also, in the case of independently distributed components, the verification of the invariance could be done in terms of the properties of the component
random variables. For example, if the probability density functions of the components are symmetric about the origin, then so is their joint probability density function; and if the components are independently identically distributed, then their joint probability density function is invariant with respect to the permutation group.

Both examples and properties of convex functions can be found in many places, e.g., [2], [5]. Convex functions and concave functions on \( \mathbb{R}^n \) may be constructed from similar functions on the real line. For example, if \( T_j(x_j), j=1,2,\ldots,n, \) are convex on \( \mathbb{R} \), then
\[
T(x_1,x_2,\ldots,x_n) = \sum_{j=1}^{n} c_j T_j(x_j), \quad c_j \geq 0, \quad j=1,\ldots,n,
\]
is convex on \( \mathbb{R}^n \) and so is \( T(x_1,x_2,\ldots,x_n) = \max_j T_j(x_j) \); if \( T_j(x_j), j=1,2,\ldots,n, \) are concave on \( \mathbb{R}^n \), then \( \sum_{j=1}^{n} c_j T_j(x_j), \quad c_j \geq 0, \) as well as \( \min_j T_j(x_j), \) is concave on \( \mathbb{R}^n \). The following are some well known examples of convex functions and concave functions on \( \mathbb{R}^n \).

(i) Linear functions of order statistics. If \( X(1) \geq X(2) \geq \cdots \geq X(n) \) are the order statistics for the sample \( (X_1,X_2,\ldots,X_n) \), then for \( c_1 \geq c_2 \geq \cdots \geq c_n \geq 0, \) \( T_1(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j \) is a convex function and \( T_2(\mathbf{x}) = \sum_{j=1}^{n} c_j x_{(n-j)} \) is a concave function of \( \mathbf{x} \). Note that this example includes the extreme order statistics and the arithmetic mean of the random sample \( \mathbf{x} \). The convexity of \( T_1 \) (concavity of \( T_2 \)) may be established by first proving convexity of (concavity of) functions
\[
\sum_{i=1}^{k} X(i), \quad i=1,2,\ldots,n; \sum_{n=1+1}^{n} X(i), \quad k=1,2,\ldots,n
\]
and then noting that \( T_1 \) can be (or \( T_2 \) can be) expressed as a linear function, with nonnegative coefficients, of these simpler functions.
(ii) Elementary symmetric functions. If \( E_k(t_1, t_2, \ldots, t_n) \) denotes the \( k \)th symmetric function of \((t_1, t_2, \ldots, t_n)\), then \( T_3(\overline{x}) = E_k^{1/k}(x_1, x_2, \ldots, x_n) \), and \( T_4(\overline{x}) = E_k(x_1, x_2, \ldots, x_n)/E_{k-1}(x_1, x_2, \ldots, x_n) \), 1 \( \leq \) \( k \) \( \leq \) \( n \), are concave functions of \( \overline{x} \), provided that \( \overline{x} \geq 0 \). This example includes arithmetic and geometric means of \( \overline{x} \).

(iii) Completely symmetric functions. If \( H_k^{1/k}(t_1, t_2, \ldots, t_n) \) denotes the \( k \)th completely symmetric function, that is, the coefficient of \( a^k \) in the formal expansion of \( \prod_{i=1}^{n} (1-at_i)^{-1} \), then \( T_5(\overline{x}) = H_k^{1/k}(x_1, x_2, \ldots, x_n) \), \( 1 \leq k \leq n \), is a convex function of \( \overline{x} \). Completely symmetric functions are not very common in statistics. It may be observed that examples (ii) and (iii) may be covered by the following example.

(iv) Let \( T_{r,k}(\overline{x}) = \sum_{i_1 + i_2 + \cdots + i_n = r} \frac{n!}{i_1! i_2! \cdots i_n!} \overline{x}_{i_1}^{i_1} \overline{x}_{i_2}^{i_2} \cdots \overline{x}_{i_n}^{i_n} \), where \( i_j \) are nonnegative integers, \( j=1,2,\ldots,n \); \( k \) is any number, provided that if \( k > 0 \) and not an integer, then \( k > (n-1) \), and

\[
\delta_i = \begin{cases} 
\binom{k}{i}, & \text{if } k > 0 \\
(-1)^i \binom{k}{i}, & \text{if } k < 0 
\end{cases}
\]

If \( \overline{x} > 0 \), then \( T_{r,k}(\overline{x}) \) is a convex function of \( \overline{x} \) if \( k < 0 \) and a concave function of \( \overline{x} \) if \( k > 0 \). Notice that \( T_{r,1}(\overline{x}) = E_r(\overline{x}) \) and \( T_{r,-1}(\overline{x}) = H_r(\overline{x}) \).
(v) Sums of powers. $T_6(\bar{X}) = \left( \sum_{i=1}^{n} x_i^k \right)^{1/k}$, $k \leq 1$, is a concave function of $\bar{X}$, provided that $\bar{X} > \emptyset$, and $T_7(\bar{X}) = \left( \sum_{i=1}^{n} x_i^k \right)^{1/k}$, $k \geq 1$, is a convex function of $\bar{X}$. As $k \to -\infty$, $T_6(\bar{X}) \to \min(X_1, X_2, \ldots, X_n)$, $\lim_{k \to -\infty} T_7(\bar{X}) = \max(\vert X_1 \vert, \vert X_2 \vert, \ldots, \vert X_n \vert)$, and $\lim_{k \to 0} \frac{1}{n} T_6(\bar{X}) = \left( \prod_{i=1}^{n} x_i \right)^{1/n}$. Also for $k = -1$, $T_6(\bar{X})$ is a harmonic mean and for $k = 1$, $T_6(\bar{X})$ is the arithmetic mean of $\bar{X}$.

5. Remarks.

(i) Generalized Monotonicity. The inequalities (6), (7), and (8) of section 2 can be considered as meaningful generalizations of the monotonicity properties of the expected values and distribution functions mentioned in the introduction. If $\omega$ and $\phi$ are in the same G-orbit, that is, if $\phi = g \omega$, for some $g \in G$, let us denote the equivalence by $\phi \sim \omega$. Also if $\omega$ is in the convex-hull of the G-orbit of some $\phi$, let us write $\omega \ll \phi$. The relation $\ll$ is then a partial order and the inequalities (14), (15), and (16) may be regarded as the monotonicity properties of $F_\phi^T(t)$ and $E_\phi(T)$ with respect to this partial order. Thus in the particular case of the permutation group, we say $\phi \sim \omega$ if $\phi = \pi \omega$ for some permutation matrix $\pi$, and $\omega \ll \phi$ if $\omega = \phi \phi$ for some doubly stochastic matrix $\phi$. Both the distribution functions and the expected values of the particular case (iii) of section 3 are invariant with respect to the permutation group and are monotone functions of the parameter vector with respect to the partial order $\ll$. 

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(ii) Schur Functions. A real-valued function \( I(\theta) \) on \( \mathbb{R}^n \) is said to be S-concave (Schur-concave) \([9]\) if \( I(S\theta) \geq I(\theta) \), for every \( \theta \) and every doubly stochastic matrix \( S \). The function \( I(\theta) \) is said to be S-convex (Schur-convex) if \( I(S\theta) \leq I(\theta) \). A necessary and sufficient condition for a symmetric function \( I(\theta) \) to be S-concave is

\[
(\theta_i - \theta_j) \left( \frac{\partial I}{\partial \theta_i} - \frac{\partial I}{\partial \theta_j} \right) \leq 0 ,
\]

for all \( \theta_i, \theta_j \), where \( \theta_i, \theta_j \) are coordinates of the vector \( \theta \).

The function \( I(\theta) \) is S-convex if and only if

\[
(\theta_i - \theta_j) \left( \frac{\partial I}{\partial \theta_i} - \frac{\partial I}{\partial \theta_j} \right) \geq 0 ,
\]

for all \( \theta_i, \theta_j \). The inequalities (15) and (16) thus express, respectively, the S-concave and S-convex nature of the distribution functions and the expected values. We can therefore infer gradient properties similar to (17) and (18) for these functions.

(iii) G-convexity and G-concavity. The concepts of S-convexity and S-concavity can be generalized. Corresponding to a group \( G \) of linear Lebesgue measure-preserving transformations, a function \( I(\theta) \) may be defined as G-convex, if for each \( \theta \) and each \( \varphi \) in the convex-hull of the G-orbit of \( \theta \), \( I(\varphi) \leq I(\theta) \). The function \( I(\theta) \) is G-concave if \( I(\varphi) \geq I(\theta) \). Theorem 6 of section 2 gives some examples of G-concave and G-convex functions.
(iv) **Monotonicity of Moments.** Let \( X_1, X_2, \ldots, X_n \) be a sample from any of the populations mentioned in section 3 and let \( T(\lambda) \) be one of the statistics catalogued there. By particularizing Corollary 3 we can obtain some monotonicity properties for the moments of \( T \).

Now \( T^r \) is an increasing function of the real-valued statistic \( T \geq 0 \) if \( r > 0 \) and a decreasing function of \( T \) if \( r < 0 \). So if \( T \) is a symmetric convex (concave) statistic of a random sample from a population yielding unimodal samples, then the d.f. \( F_\theta^T(t) \), for each \( t \), is an \( S \)-concave (\( S \)-convex), that is, in the generalized sense, a monotonically decreasing (increasing) function of \( \theta \). Furthermore, a moment of positive order, \( E(T^r) \), \( r > 0 \) of \( T \) is an \( S \)-convex (\( S \)-concave), that is, in the generalized sense, a monotonically increasing (decreasing) function of \( \theta \). A moment of negative order, \( E(T^r) \), \( r < 0 \) of \( T \) is an \( S \)-concave (\( S \)-convex) function of \( \theta \).

(v) **Optimization.** Obviously, the monotonicity properties mentioned here can be used for optimization of various statistics. Suppose that an available resource \( \theta_1 \) is to be allocated as \( \theta_1, \theta_2, \ldots, \theta_n \) to \( n \) components of a system with random outputs \( X_1, X_2, \ldots, X_n \). Suppose that the final output is a real-valued function \( T = T(X_1, X_2, \ldots, X_n) \) of the component outputs \( X_i \), \( i = 1, 2, \ldots, n \). If the allocations \( \theta_i \) enter as translation parameters in the distributions of \( X_i \) and if the joint distribution of \( X_i \) and the final output satisfy the relevant conditions of Corollary 3, then the corollary gives the behavior of the distribution function and expected value of \( T \). Also if \( I(\theta) \) is \( S \)-convex, then \( I(\lambda \theta) = I(\bar{\theta}, \bar{\theta}, \ldots, \bar{\theta}) \leq I(\theta) \), where \( \bar{\theta} \) is the doubly
stochastic matrix with all elements equal and \( \bar{\theta} = \frac{1}{n} \sum \theta_i \). On the other hand, if \( I(\theta) \) is \( S \)-concave then \( I(\bar{\theta}, \bar{\theta}, \ldots, \bar{\theta}) \geq I(\theta) \). These inequalities, together with the gradient properties (\( \_ \_ \_ \) ), indicate the optimization possibilities.
References


ON A MONOTONE CHARACTER OF SOME INVARIANT STATISTICS

TECHNICAL REPORT

MUDHOLKAR, GOVIND S.

November 3, 1969

Technical Report No. 149

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Some results on variations of distribution functions and moments of convex or concave statistics from certain populations with respect to population translation parameters are derived and their implications and applications are discussed. The results generalize similar results in Mudholkar [7].
**Invariance convexity or concavity statistics, Distribution functions, Moments, G-convexity, G-concavity Unimodality Optimization, Schur-functions, Generalized monotonicity.**

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