FURTHER RESULTS FOR GOODNESS-OF-FIT STATISTICS WITH
UNKNOWN PARAMETERS

by

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TECHNICAL REPORT NO. 180
OCTOBER 4, 1971

PREPARED UNDER CONTRACT N00014-67-A-0112-0053
(NR-042-267)
OFFICE OF NAVAL RESEARCH

Herbert Solomon, Project Director

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Summary. This report revises and extends the report "Asymptotic
results for goodness-of-fit statistics when parameters must be
estimated" (Stephens, 1970); new theoretical results are given, and
then numerical results based on the theory. These lead ultimately
to the provision of percentage points for $W^2$, $U^2$ and $A$ in tests
for normality and exponentiality, when parameters must be estimated.
The previous report gave only means and variances of asymptotic
distributions; some of these are repeated for completeness.

1.1 Introduction. Let $x_1, x_2, \ldots, x_n$ be independent observed random
variables from a continuous distribution $G(x)$, and let $F_n(x)$ be the
empirical distribution function. A well known goodness-of-fit test,
to test the null hypothesis

$$H_0: G(x) = F(x; \theta),$$

where $F(x; \theta)$ contains a parameter $\theta$ (a vector, each of whose
components represents a separate scalar parameter) is based on the
Cramer-von Mises statistic

$$W^2 = n \int (F_n(x) - F(x; \theta))^2 dF(x; \theta).$$

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introduced by Watson (1961) and $A$, introduced by Anderson and Darling (1954); they are defined by
\[
U^2 = n \int \left( F_n(x) - F(x; \theta) \right)^2 \frac{dF(t; \theta)}{dF(x; \theta)} \, dt
\]
and
\[
A = n \int \left( F_n(x) - F(x) \right)^2 \frac{dF(x; \theta)}{F(x; \theta)(1-F(x; \theta))} \, dx.
\]

When $F(x; \theta)$ is completely specified, all three statistics have distributions independent of $F(x; \theta)$; these converge rapidly to the asymptotic distributions, but tables exist giving significance points for $n$ finite and infinite. References are in Stephens (1970b), where also, for the practical user, modified forms of the statistics are given which may be used for goodness-of-fit tests with, for each statistic, only one line of percentage points (those of the asymptotic distributions). Watson (1961) introduces $U^2$ for observations on a circle, since its value does not depend on the choice of origin, but it may be used also for observations on a line. $U^2$ will tend to give significant values, when the sample actually comes from a distribution with a different variance from the null, whereas $W^2$ will be stronger in detecting a shift in mean. The statistic $A$ modifies $W^2$ by giving greater weight to the tails of the distribution; it counteracts the larger variance of the estimate $F_n(x)$ of $F(x; \theta)$ in the tails,
by dividing by the variance. It can be expected to detect discrepancies in the tails better than $\hat{W}^2$.

1.2 Unknown parameters. We have so far considered the case (Case 0) when $F(x; \theta)$ is completely specified, but in many practical situations, the distribution to be tested will be known in form, but $\theta$ will be unknown, and is to be estimated from the sample of $x$-values. The following important examples will be called Cases 1-4.

Case 1. $F(x; \theta)$ is the normal distribution, when $\sigma^2$ is known and $\mu$ is to be estimated;

Case 2. $F(x; \theta)$ is normal, $\mu$ is known, and $\sigma^2$ is to be estimated;

Case 3. $F(x; \theta)$ is normal, and both $\mu$ and $\sigma^2$ are to be estimated;

Case 4. $F(x; \theta)$ is the exponential distribution:

$$F(x) = 1 - e^{-\theta x}, \ x > 0, \ and \ \theta \ is \ to \ be \ estimated.$$ 

Obviously, for the normal distribution, Case 3 is the most likely to occur in practice. In what follows, the estimates of $\mu$ and $\sigma^2$ will always be $\bar{x}$ and $s^2 = \Sigma(x_i - \bar{x})^2/(n-1)$, using obvious notation, and that of $\theta$ will be $1/\bar{x}$.

For these cases the null distributions of $W^2$, $U^2$ and $A$ are drastically changed, even asymptotically, and reliance on the published tables for Case 0 will greatly change the significance level of any test being made.
1.3 Contents of this paper. For the asymptotic distributions, a great deal can be found by extending the results of two pioneering papers by Darling (1955) and Kac, Kiefer, and Wolfowitz (1955, hereafter called KKW). This is done in the present paper. Theoretical results are given in the next section. A particular result is that the asymptotic distributions, in Cases 1-4, as in Case 0, can be represented by a weighted sum of $X_1^2$ variables. In section 3 we give the means and variances of these distributions, and in section 4 the weights in the $X^2$ representation. Finally, these are used to give percentage points of all the asymptotic distributions. These have been used, together with Monte Carlo studies for finite $n$, to give tests for normality and exponentiality based on $W^2$, $U^2$ or $A$. The statistics have good power properties; they are not, for example, very inferior, when used correctly, in tests for normality (Case 3), as previously reported (Shapiro and Wilk, 1965). $A$, in particular, seems a good "omnibus" test statistic against a wide range of alternatives. Practical aspects of these tests (computing formulae, modified test statistics using only the asymptotic points, power studies) are given elsewhere (Stephens, 1969, 1970a).
2. PREVIOUS RESULTS AND EXTENSIONS

2.1 In order to add to Darling and KKW, we start with a summary, somewhat oversimplified, of their results. Extensions are then given as they occur (Theorems 1 to 4 below). The earlier papers deal mostly with $W^2$; in what follows, the results apply equally to $U^2$ and $A$, though independent results will be given only for the numerical work of sections 3, 4, and 5. The notations of the Darling and KKW papers are very different, and we follow Darling's except in §2.6.

2.2 Darling deals with the asymptotic distribution of $W^2$ when one parameter in $F(x; \theta)$ must be estimated. He later particularizes to tests for normality and exponentiality. KKW treat normality only, but when both parameters must be estimated.

2.3 An important result (which makes it worthwhile, from the practical standpoint, to consider the problems in Cases 1 to 4) is the following: provided, in $F(x; \theta)$, $\theta$ contains location and or scale parameters only, and provided the estimator satisfies certain conditions, the asymptotic distribution of $W^2$ (add also $U^2$, $A$) depend only on the family of distributions $F(x; \cdot)$ and not on the true value of $\theta$

Maximum likelihood estimators satisfy the required conditions.

Darling investigates this more thoroughly, for one location or scale parameter; KKW give the result for the particular case of the normal distribution.
2.4 The asymptotic distribution of the statistic considered is then that of \( \int_0^1 (y(t))^2 \, dt \), where \( Y(t) \) is a Gaussian process with covariance function \( \rho(s,t) \), dependent on \( F(x,\theta) \) and on the parameter to be estimated. Given \( \rho(s,t) \), the characteristic function of the distribution (the word asymptotic will be dropped) is given by 

\[
\phi(x) = \lambda \int_0^1 \rho(x,y)f(y) \, dy 
\]  

(1)

The distribution is the same as that of the sum

\[
S = \sum_{i=1}^{\infty} \frac{z_i}{\lambda_i},
\]

(2)

where \( \lambda_i \) are the solutions of \( D(\lambda) = 0 \), and the \( z_i \) are independent \( \chi^2_1 \) variables. Further, the cumulants of the distribution are given by

\[
K_j = 2^{j-1}(j-1)! \int_0^1 \rho_j(s,s) \, ds
\]

(3)

where \( \rho_j(s,t) \) is defined by

\[
\rho_j(s,t) = \int_0^1 \rho_{j-1}(s,u) \rho(u,t) \, du, \quad j \geq 2;
\]

\[
\rho_1(s,t) = \rho(s,t).
\]

From the representation (2) we have also the cumulants in the form

\[
K_j = 2^{j-1}(j-1)! \sum_{i=1}^{\infty} \frac{1}{\lambda_i^j}
\]

(4)
2.5 The covariance function.

In order to use the results of §2.4, one must find \( \rho(s,t) \). KKW gives it for the Case 3 situation. Darling shows that when one location or scale parameter is estimated, and subject to regularity conditions, \( \rho(s,t) \) takes the form

\[
\rho(s,t) = \rho_0(s,t) - \phi(s)\phi(t)
\]

(5)

where \( \rho_0(s,t) \) is the covariance of the relevant statistic in Case 0, i.e., where \( F(x;\theta) \) is completely specified. The function \( \phi(s) \), as expected, depends on \( F(x;\theta) \) and on which parameter is to be estimated, but not on the true value. Darling shows how to find \( \phi(s) \), and how to get \( D(\lambda) \) in this Case. We use his techniques later on for Cases 1, 2 and 4, but first we give an extension to cover Case 3.

2.6 Covariance when location and scale parameters are estimated.

Theorem 1. Let \( \theta \) in \( F(x;\theta) \) consist of location and scale parameters, to be estimated by maximum likelihood, and let the conditions of Darling (1955), Lemmas 3.1, 3.2 be satisfied; when only the location parameter is estimated, let \( \phi(s) \) in (5) above be the function \( \phi_1(s) \), and when only the scale parameter is estimated, let it be \( \phi_2(s) \); then, when both parameters are estimated, the covariance of \( Y(t) \) is

\[
\rho(s,t) = \rho_0(s,t) - \phi_1(s)\phi_1(t) - \phi_2(s)\phi_2(t).
\]

(6)

The proof will be omitted; it follows by taking Darling's equation (3.1), expanding for two parameters instead of one, and following the
argument on the next several pages till his Theorem 4.1. Note that Case 3 is a special case of Theorem 1 above, and \( \rho(s,t) \), given by KKW, is an example of (6).

2.7 Solution for \( D(\lambda) \). The problem is now to solve the integral equation (1) for a kernel of type (6). We follow closely Darling, Theorem 6.2. Suppose first that \( \rho_0(s,t) \) has Fredholm determinant \( d_1(\lambda) \), roots \( 0 \leq \lambda_1 \leq \lambda_2 \ldots \), and corresponding eigenfunctions \( f_1(x), f_2(x), \ldots \). Define

\[
a_j = \int_0^1 f_j(x) \phi_1(x) dx; \quad b_j = \int_0^1 f_j(x) \phi_2(x) dx;
\]

\[
a_0 = (1 + \lambda \sum_{i=1}^{\infty} a_i^2/(1-\lambda/\lambda_i)); \quad b_0 = (1 + \lambda \sum_{i=1}^{\infty} b_i^2/(1-\lambda/\lambda_i));
\]

\[
a_{ab} = (\lambda \sum_{i=1}^{\infty} a_i b_i/(1-\lambda/\lambda_i)).
\]

Further, let \( c_i(g) = \int_0^1 g(x) \phi_1(x) dx \), \( i=1,2 \). When \( \rho(s,t) \) takes the form \( \rho_0(s,t)\phi_1(s)\phi_1(t) \), i.e., when a location parameter only is estimated, Darling shows that the Fredholm determinant is

\[
D_1(\lambda) = d_1(\lambda) S_a(\lambda); \quad (7)
\]

when a scale parameter is estimated, \( \phi_1(s) \) is replaced by \( \phi_2(s) \), and the new determinant is

\[
D_2(\lambda) = d_1(\lambda) S_b(\lambda). \quad (8)
\]

We now give the solution for the kernel of equation (6).
Theorem 2. For the conditions of Darling, Theorem 6.2, the Fredholm determinant for the kernel (6) is

\[ D_{3}(\lambda) = d_{1}(\lambda)(s_{a}(\lambda)s_{b}(\lambda) - (s_{ab}(\lambda))^{2}) \]  

(9)

Corollary. If the functions \( \phi_{1}(x), \phi_{2}(x) \) are such that either \( a_{1} \) or \( b_{1} \) is zero for each \( i \), then \( S_{ab}(\lambda) = 0 \) and

\[ d_{1}(\lambda)D_{3}(\lambda) = D_{1}(\lambda)D_{2}(\lambda). \]

We shall show later that for Cases 1 and 2, the above corollary becomes true for \( W^{2}, U^{2} \) or \( A \). Then, for any of these three:

Theorem 3. When Theorem 2, corollary, holds, let the characteristic function for Case i be \( C_{i}(t) \). Then

\[ C_{0}(t)C_{3}(t) = C_{1}(t)C_{2}(t). \]

If \( K_{ij} \) is the \( j^{th} \) cumulant, Case i:

\[ K_{0j} + K_{3j} = K_{1j} + K_{2j}, \]

equivalent to

\[ K_{0j} - K_{3j} = (K_{0j} - K_{1j}) + (K_{0j} - K_{2j}). \]

Thus the decrease in value of a cumulant in proceeding form Case 0 to Case 3 is the sum of the decreases for Cases 1 and 2. This result is very valuable in the numerical calculations to follow.

Proof of Theorem 2. Darling's equation (6.4), with \( \rho(s,t) \) given by (6), and with \( \rho_{0}(s,t) \) for Darling's \( k_{1}(s,t) \), becomes

\[ g(x) = \lambda \int_{0}^{1} \rho_{0}(x,y)g(y)dy - \lambda \phi_{1}(x)c_{2}(s) - \lambda \phi_{2}(x)c_{1}(s). \]  

(10)
sides. Multiply (11) by \( \phi_1(x) \) and integrate, and similarly by \( \phi_2(x) \) and integrate, and get
\[
c_1(g) = -c_1(g)(S_a(\lambda)-1) - c_2(g)S_{ab}(\lambda)
\]
\[
c_2(g) = -c_2(g)S_{ab}(\lambda) - c_2(g)(S_b(\lambda)-1)
\]

The various possible solutions of these equations are:

(i) both \( c_1(g), c_2(g) \) are zero. Then (Darling), \( \lambda \) satisfies
\[
d_1(\lambda) = 0;
\]

(ii) one only (say \( c_2(g) \)) is zero; then the problem reduces to
Darling's, and \( \lambda \) is a root of \( S_a(\lambda) = 0 \). The condition \( c_2(g) = 0 \)
implies \( S_{ab}(\lambda) = 0 \), and \( \lambda \) is a root of
\[
P_3(\lambda) = S_a(\lambda)S_b(\lambda) - (S_{ab}(\lambda))^2 = 0;
\]

(iii) if neither is zero, \( \lambda \) is a root of \( P_3(\lambda) = 0 \).

All three contingencies are covered by \( \lambda \) a solution of \( D_3(\lambda) = 0 \)
as given in (9). Theorem 2, Corollary, follows at once.

For Theorem 3, recall that \( C_0(t) = d_1(2it)^{-1/2} \); for the other
cases \( C_i(t) = (D_i(2it)^{-1/2}, i=1,2, \text{or } 3 \). Then Theorem 3 at once
follows from Theorem 2, Corollary, and well-known properties of
characteristic functions and cumulants.
2.8 Related KKW results. In this paragraph we use KKW notation. They solve (8) above by solving the related differential equation, in their §2.6. They point out that the Fredholm determinant for Case 3, there called $D(\mu)$, can be factored: $D(\mu) = -2 D_1(\mu) D_2(\mu)$, where $D_1(\mu), D_2(\mu)$ are given by their (2.18) and (2.19), and where $\mu$ is related to the $\lambda$ of this paper by $\mu^2 = \lambda$. Suppose now we solve Case 1 and Case 2 integral equations by the differential equation method, to give Fredholm determinants $D_1^*(\mu), D_2^*(\mu)$. The solution follows KKW, §2.5, and the long algebraic details will be omitted. The results are $D_1^*(\mu) = 2 \sin \frac{1}{2} \mu D_1(\mu)/\mu$ and $D_2^*(\mu) = -2 \cos \frac{1}{2} \mu D_2(\mu)$. Thus

$$D_1^*(\mu) D_2^*(\mu) = -2 \sin \mu D_1(\mu) D_2(\mu)/\mu = \sin \mu D(\mu)/\mu.$$ 

Apart from notational changes, this is the result of Theorem 2, Corollary. The KKW factorisation $D(\mu) = -2 D_1(\mu) D_2(\mu)$ is essentially expressing this result, for Case 3, as a particular example.

2.9 If, in the notation of §2.5, $S_{ab}(\lambda)$ is not zero, a result weaker than Theorem 3 may still hold.

**Theorem 4.** If the functions $\phi_1(x), \phi_2(x)$ are orthogonal, i.e.,

$$\int_0^1 \phi_1(x) \phi_2(x) dx = 0,$$

the means and variances of Cases 1-4 satisfy

$$K_{0j} + K_{3j} = K_{1j} + K_{2j}, \quad j = 1, 2.$$ 

**Proof.** Follows from the cumulant formulae in (3). The proof is obvious for the mean; for the more difficult variance,
\[ K_{02} + K_{22} = \int_0^1 \int_0^1 \left( (\rho_0(s,t))^2 + (\rho_0(s,t) - \phi_1(s)\phi_1(t) - \phi_2(s)\phi_2(t))^2 \right) ds \, dt \]
\[ = \int_0^1 \int_0^1 (\rho_0(s,t) - \phi_1(s)\phi_1(t))^2 ds \, dt + \int_0^1 \int_0^1 (\rho_0(s,t) - \phi_2(s)\phi_2(t))^2 ds \, dt \]
\[ = K_{12} + K_{22} , \]

if \( \int_0^1 \phi_1(s)\phi_2(s) ds = 0 \). Note that the orthogonality condition implies only \( \sum_{i=1}^n a_i b_i = 0 \), not \( a_i b_i = 0 \) for all \( i \).

2.9 Calculation of the weights \( \lambda_i \).

KKW continued to calculate \( D_1(\mu) \) and \( D_2(\mu) \) and hence to find the zeros \( \mu_i \) of these functions (\( \lambda_i \) in our notation is then \( \sqrt{\mu_i} \)), but the techniques were computationally difficult and only the first eight \( \lambda \)'s were found. These they used to approximate the \( W^2 \) distribution by \( S^* = \sum_{i=1}^n z_i / \lambda_i \); this approximation is examined in §6.2.

With Theorem 2 we can calculate the \( \lambda_i \) to a large order very quickly; the values of \( a_i, b_i \) involve only single integrals and the solutions of \( S_a(\lambda) = 0, S_b(\lambda) = 0 \) are straightforward. This will be done in §4. First we find the functions \( \phi(x) \) in all the Cases, and then get exact values of means and variances of the various distributions, using (3).
3. COVARIANCE FUNCTIONS: TEST FOR NORMALITY

3.1 Preliminaries. This section introduces notations and definitions of functions which will be used throughout the paper. Let \( d = (2\pi)^{-1/2}; \)
\( n(x) = \hat{d} \exp(-x^2/2); \)
\( N(x) = \int^x n(t)dt. \) When \( s = N(x), \) let \( x = J(s), \)
i.e., \( J(\cdot) \) is the inverse of \( N(\cdot). \)

Define functions:

\[
\begin{align*}
  r(s,t) &= \min(s,t)-st \\
  B(s) &= \hat{d} \exp(-J^2(s)/2) \\
  b(s,t) &= -B(s)B(t) \\
  c(s) &= d(J(s)/\sqrt{2}) \exp(-J^2(s)/2) \\
  c(s,t) &= -c(s)c(t) \\
  w(s,t) &= ((s-s^2)(t-t^2))^{-1/2} \\
  a(s,t) &= 1/12 - ((s-s^2) + (t-t^2))/2 \\
  E(s) &= B(s) - 1/(2 \sqrt{\pi}) \\
  e(s,t) &= -E(s)E(t)
\end{align*}
\]

Notation for double integrals. It will be convenient to use a notation for double integrals, e.g.

\[
R^2 = \int_0^1 \int_0^1 r^2(s,t)ds \, dt; \quad RB = \int_0^1 \int_0^1 r(s,t)b(s,t)ds \, dt.
\]

The capital letters indicate the functions in the integrand; the limits are all \( 0, 1. \)
Useful integrals. The integrals below arise frequently in subsequent calculations of means and variances. They might be useful also in other contexts (very similar integrals arise in the theory of normal order statistics).

(i) \( A_1(k) = \int n^k(x)dx = d^{k-1}/\sqrt{k} \).

(ii) \( A_2(k) = \int n^k(x)N(x)dx = d^{k-1/2}/\sqrt{k} \).

(iii) \( A_3(a,b) = \int n^b(x)N(x)N(ax)dx = d^{b+1}(\tan^{-1}(a/\sqrt{b(1+b+a^2)^{1/2}})/\sqrt{b}) + d^{b-1/4}/\sqrt{b} \).

Proof. Differentiate with respect to \( a \), and prove

\[
\frac{dA_2}{da} = \frac{d^{b+1}}{(b+a^2)(b+1+a^2)^{1/2}}.
\]

Integration then gives \( A_3(a,b) \) as above.

(iv) \( A_4(k,y) = \int x n^k(x)N(x)dx = -d^kN(y)(\exp(-ky^2/2))k
\]

\[
+ \frac{d^k}{k} \int \exp(-x^2(1+k)/2)dx
\]

\[
= -d^kN(y)\exp(-ky^2/2))k + d^k(N((1+k)^{1/2}y))/k(1+k)^{1/2};
\]

then

\( A_4(k,\infty) = d^k/k(1+k)^{1/2} \).

(v) \( A_5(k,b) = \int x^b n^b(x)dx = \frac{(k-1)}{b} A_5(k-2,b) \).
then

\[ A_5(0,b) = \frac{\mathbf{b}^{b-1}}{\sqrt{b}} ; \quad A_5(1,b) = \frac{\mathbf{b}^{b-1}}{b^{3/2}} ; \quad A_5(2,b) = \frac{\mathbf{b}^{b-1}}{b^{5/2}} . \]

(vi) \[ A_6(k;x) = \int_x^{\infty} n_k(t) dt = (a_k^{-1/\sqrt{k}})(1-N(\sqrt{k} x)) . \]

3.2 Calculation of covariance functions. For the covariance of the type in equation (5), Darling shows how \( \phi(s) \) is found. Recall that \( \theta \) contains only one unknown, called \( \theta \).

Let \( s = F(x;\theta) \) define \( x \) implicitly in terms of \( s \). Let

\[ f(x;\theta) = \frac{\partial}{\partial x} F(x;\theta) ; \quad g(s) = \frac{\partial}{\partial \theta} F(x;\theta) ; \quad \text{and let} \quad k^2 \quad \text{be defined by} \]

\[ \frac{1}{k^2} = \int (\frac{\partial}{\partial \theta} \ln f(x;\theta))^2 f(x;\theta) dx ; \]

then

\[ \phi(s) = k^2 \cdot g(s) \]

and

\[ \phi'(s) = k^2 \frac{\partial}{\partial \theta} \ln f(x;\theta) . \]

Note that the presence of \( k \) in the definition of \( \phi(s) \) ensures that

\[ \int_0^1 (\phi'(s))^2 ds = 1 . \]

3.3 Covariance functions for \( W^2 \). We now apply the methods of the previous paragraph to \( W^2 \). In Case 1, we may take the known variance to be 1 (if it were \( \sigma^2 \), values of \( x/\sigma \) would be tested to come from a normal distribution with variance 1). The unknown \( \theta \) is the mean \( \mu \),
and it is easily shown, following §3.2, that $\phi_1(s) = -B(s)$, where $B(s)$ is defined in §3.1. Similarly, for Case 2 we take $\mu$ as zero; the unknown $\theta$ is the variance $\sigma^2$, and $\phi_2(s)$ becomes $-C(s)$. The covariances are then given by equation (4), with $\rho_0(s,t) = \min(s,t)-st = r(s,t)$. For Case 3, apply Theorem 1:

$$\rho(s,t) = r(s,t) - \phi_1(s)\phi_1(t) - \phi_2(s)\phi_2(t)$$
$$= r(s,t) + B(s,t) + C(s,t).$$

If arguments are omitted when there is no ambiguity, the covariance functions $\rho$, for $W^2$, can be summarized as follows:

<table>
<thead>
<tr>
<th>Case</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>$r$</td>
<td>$r+b$</td>
<td>$r+c$</td>
<td>$r+b+c$</td>
</tr>
</tbody>
</table>

Equation (3) can now be used to find cumulants of $W^2$; in practice, the calculations become extremely long and only the means and variances have been calculated, as follows.

**Means and Variances: $W^2$. Case 1.** Use of (3) with $\rho$ given by (5), gives, for Case 1,

$$\mu = K_1 = \mu_0 - a^2 \int_0^1 \exp(-\gamma^2(s))ds$$

where $\mu_0$ is the mean for Case 0; this is known to be $1/6$. Let $x = \gamma(s)$; then
\[ \mu = \frac{1}{6} - \int n^3(x) \, dx = \frac{1}{6} - A_1(3) = 0.074779. \]

The variance is

\[ \sigma^2 = K_2 = 2 \int \int (r+b)^2 \, ds \, dt = 2(R^2 + 2RB + B^2), \]

where (from Case 0, known result), \( 2R^2 = 1/45 \) and \( 2B^2 = 2(\mu - \frac{1}{6})^2 \); \( 4RB \) must be found. The substitution \( x = J(s), y = J(t) \) gives

\[ RB = 2 \int_0^1 \int_0^t s(1-t)b(s,t) \, ds \, dt. \]

\[ = -2 \int (1-N(y)) n^2(y) \int N(x)n^2(x) \, dx \, dy \]

\[ = -2 \int n^2(y)N(x)n^2(x) \, dx \, dy + 2 \int \int N(y)N(x)n^2(y)n^2(x) \, dx \, dy \]

\[ = I_1 + I_2, \] say.

In \( I_1 \), change the order of integration:

\[ I_1 = -2 \int n^2(x)N(x) \int n^2(y) dy \]

\[ = -\sqrt{2} \, d \int n^2(x)N(x)(1-N(\sqrt{2} \, x)) \, dx, \text{ using } A_6(2,x); \]

\[ = -\sqrt{2} \, d(A_2(2) - A_2(\sqrt{2}, 2)); \]

\[ = -\frac{d^2}{2} + d^4 \tan^{-1}(\frac{1}{\sqrt{5}}) + \frac{d^2}{4} = -\frac{d^2}{4} + d^4 \tan^{-1} \frac{1}{\sqrt{5}}. \]

Also
\[ I_2 = \iint N(y)n^2(y)N(x)n^2(x)\,dx\,dy \]
\[ = (A_2(2))^2 = d^2/8. \]

Then \[ 4R_B = 4(I_1 + I_2) = -0.036970; \] this gives \[ \sigma^2 = 0.002159. \]

**Case 2.** The mean is
\[ \mu^2 = \frac{1}{6} - \frac{d^2}{2} \int_0^1 J^2(s)\exp(-J^2(s))\,ds \]
\[ = \frac{1}{6} - \frac{1}{12 \sqrt{5} \pi} = 0.15135. \]

The variance is
\[ \sigma^2 = 2(R^2 + 2RC + C^2); \]
\[ 2R^2 = \frac{1}{45}, \] and \[ C^2 = (\mu - \frac{1}{6})^2 = 1/(432 \pi^2); \] \[ 4RC \] is needed.

\[ RC = -d^2 \int_0^1 \int_0^1 (1-t)sJ(s)J(t)\exp(-J^2(s)+J^2(t))/2)\,ds\,dt. \]

Using \( x = J(s), y = J(t), \)
\[ RC = -\iint xy^2(x)n^2(y)N(x)dxdy + \iint xyn^2(x)n^2(y)N(y)N(x)dxdy. \]
\[ = -I_3 + I_4. \]

Each integral separately gives
\[ I_3 = \int y n^2(y) \int x n^2(x)N(x)dxdy \]
\[ = \int y n^2(y) A_4(2,y)dy \]
\[ = 1/(32 \pi^2 \sqrt{5}), \]
using integration by parts.

\[ I_4 = \frac{1}{2} \iint xy n_1^2(x)n_2^2(y)N(x)N(y) \, dx \, dy \]

\[ = \frac{1}{2} (A_4(2))^2 = 1/(96 \cdot x^2) . \]

Then \( 4 \sigma^2 = 0.00144 \) and \( \sigma^2 = 0.02125 \).

**Case 3.** Using Theorem 3, we have for Case 3

\[ \mu = 0.07478 + 0.15135 - 0.16667 = 0.05946 \cdot \]

\[ \mu^2 = 0.00214 + 0.02125 - 0.02222 = 0.00117 \cdot \]

All these means and variances are recorded in Table 1.

3.4 **Covariance functions for \( U^2 \).** Watson (1961) has shown that the limiting distribution of \( U^2 \) is that of \( \int_0^1 Q(t) \, dt \), where \( Q(t) \) is the Gaussian process \( Y(t) = \int_0^1 Y(u) \, du \), and where \( Y(t) \) is the same Gaussian process as used in the limiting distribution of \( W^2 \), for the corresponding Case 0, 1, 2 or 3. Let the covariance function of \( Q(t) \) then be \( \rho^*(s,t) \) and that of \( Y(t) \) be \( \rho(s,t) \); then

\[ \rho^*(s,t) = \rho(s,t) + \int_0^1 \int_0^1 \rho(s,t) \, ds \, dt - \int_0^1 \rho(s,t) \, ds - \int_0^1 \rho(s,t) \, dt \]

\[ = \rho(s,t) + E_1 + E_2 + E_3, \text{ say.} \]

For each case it is necessary to find the integrals \( E_1, E_2 \) and \( E_3 \), corresponding to the \( \rho(s,t) \) given in §3.3. This again needs only straightforward algebra, and, with the notation \( a(s,t) \) and \( e(s,t) \)
introduced in §3.1, the covariance function \( \rho^*(s,t) \) for \( U^2 \) can be summarized as follows.

Case : 0 : 1 : 2 : 3
\[ r + a : r + a + e : r + a + c : r + a + c + e \]

Note that the Case 3 covariance is of type (6).

When these are used in (1) extensive algebra (details in Stephens, 1970c), gives the moments recorded in Table 1.

3.5 Covariance functions for A. For the Anderson-Darling statistic \( A \), the covariance functions for the asymptotic distribution Gaussian process are respectively those of \( W^2 \), given in §3.3, multiplied by \( v(s,t) = ((s^2 - s^2)(t^2 - t^2))^{-1/2} \). The resulting integrals do not seem tractable by analytic methods, but since this particular statistic has some special advantages for goodness-of-fit testing (Stephens 1969), the means have been calculated numerically and are given in Table 1. The variances are calculated in §4.
4. CALCULATION OF WEIGHTS

4.1 In this section the results of Theorems 2 and 3 are used to find the weights \( \lambda_i \) in the representation (2). For each case, these are the solutions of the relevant \( D_1(\lambda) = 0 \). For each statistic, Case 0, let the weights \( \lambda_i \); solutions of \( d_1(\lambda) = 0 \), be called the standard weights; for all other cases the weights consist of a subset of the standards, plus a new set \( \lambda_i^* \) labelled with an asterisk. The standards are quoted below and the values of \( 1/\lambda_i^* \) are given in Table 2, for \( i \) from 1 to 10. 

\( W^2 \): Case 1. For \( W^2 \), \( d_1(\lambda) = \sin \sqrt{\lambda}/\sqrt{\lambda} \), and the standards are \( \lambda_i = \pi^2 i^2 \), with \( f_1(x) = \sin \pi x \). For Case 1, \( D_1(\lambda) = d_1(\lambda)S_a(\lambda) \); the zeros \( \lambda_i \) of \( d_1(\lambda) \) are simple zeros, and will not be zeros of \( D_1(\lambda) \) unless, in \( S_a(\lambda) \), the corresponding \( a_i \) is zero. For Case 1, \( \phi_1(x) = -B(x) \), from §3.3, and is easily shown to be symmetric around \( x = 0.5 \); then the coefficients \( a_i \) are zero for \( i \) even, and the subset of the standards given by \( \lambda_i = 4\pi^2 i^2 \), \( i = 1, 2, \ldots \) are then zeros of \( D_1(\lambda) \). The other zeros \( \lambda_i^* \) are solutions of \( S_a(\lambda) = 0; \) the first twenty have been calculated, and the reciprocals of the first ten are tabulated in Table 2. A plot of \( 1/\lambda_i^* \) against \( i^{-2} \) is a smooth curve asymptotic to the line through the origin with slope \( 1/(4\pi^2) \). Values of \( 1/\lambda_i^* \) for \( i > 20 \) have been found from this curve. The values of \( \sum 1/\lambda_i + \sum 1/\lambda_i^* \) converge only slowly and after 400 values the value of the mean is 0.0737 to compare with the exact value of 0.0748 in Table 1. However, higher cumulants, calculated from (2), converge much faster, and the variance, given in Table 3, converges exactly to the
value in Table 1. This provides a check on the values of \(1/\lambda_1^*\). The third and fourth cumulants and values of \(\beta_1 = k_3^2/k_2^3\) and \(\beta_2 = k_4/k_2^2\) are also given in Table 3.

\(\hat{u}_2^2\): Case 2. For Case 2, with \(\phi_2(x) = -C(x)\), the \(b_i\) are zero for odd \(i\); then in the solution of \(D_2(\lambda) = 0\), the subset of \(\lambda_1\) given by \(\lambda_1 = \pi^2(2i - 1)^2\), \(i=1,2,\ldots\) are included; values of \(1/\lambda_1^*\), calculated from \(S_b(\lambda_1^*) = 0\) are given in Table 2. For large \(i\), they are asymptotic to \(1/(\pi^2(2i - 1)^2)\). The cumulants are given in Table 3; the variances check perfectly with the exact calculation on Table 1.

\(\hat{u}_3^2\): Case 3. For Case 3, \(D_3(\lambda) = d_1(\lambda) S_a(\lambda) S_b(\lambda)\), and none of the standard \(\lambda_1\) can be a solution, but only the two sets of \(\lambda_1^*\) given by Cases 1 and 2. The values in this set of weights are those given in KKW, with a very slight numerical discrepancy in the fifth decimal place, for the largest values.

\(\hat{u}_5^2\): Cases 1 and 2. The discussion of \(\hat{u}_5^2\) is slightly more complicated. The roots \(\lambda_1\) of \(d_1(\lambda) = 0\) are double roots, given by \(\lambda_1 = 4\pi^2\), and the corresponding eigenfunctions are \(f_1^*(x) = \sin 2\pi ix\) and \(f_1^*(x) = \cos 2\pi ix\). Suppose \(a_1\) and \(a_1^*\), \(b_1\) and \(b_1^*\) are the coefficients obtained using \(f_1(x)\) and \(f_1^*(x)\) respectively. Then \(S_a(\lambda)\) becomes

\[
1 + \lambda \left( \sum_i \frac{a_i^2}{1-\lambda/\lambda_i} + \sum_i \frac{a_i^*}{1-\lambda/\lambda_i} \right),
\]

and

\[
S_{ab}(\lambda) = \lambda \left( \sum_i \frac{a_i b_i}{1-\lambda/\lambda_i} + \sum_i \frac{a_i^* b_i^*}{1-\lambda/\lambda_i} \right).
\]

When the coefficients are calculated, it is found that either \(a_1\) or
$a^*_i$ is zero, but not both, and $b^*_i$ or $b^*_i$ is zero, but not both, such
that $a^*_i b^*_i = a^*_i b^*_i = 0$ for all $i$. Thus $S_{ab}(\lambda) = 0$ and Theorem 3
still applies. In Case 1, $D_1(\lambda) = d_1(\lambda)S_a(\lambda) = 0$ gives
$\lambda^*_1 = \frac{4\pi}{12}$. A solution, but not repeated; similarly when $D_2(\lambda) = 0$. In each case
another set $\lambda^*_{2i}$ is found from $S_{a}(\lambda^*) = 0$, or $S_{b}(\lambda^*) = 0$. Values of
$1/\lambda^*_{2i}$ from these two sets are given in Table 2, Columns 3 and 4. For
both cases, when $i$ is large, $\lambda_{2i}^* \rightarrow (2i+1)^2$.

U^2: Case 3. For Case 3, the standard set $\lambda^*_1 = \frac{4\pi}{12}$ cannot be
included, and the zeros of $D_3(\lambda)$ are those of $S_{a}(\lambda)$, $S_{b}(\lambda)$, given
in Table 2, Columns 3 and 4. The cumulants, calculated from (2), are
in Table 3; the variances agree exactly with those in Table 1.

A: Cases 1 and 2. For A, the standard $\lambda^*_1$ are $i(i+1)$, $i=1,2,\ldots$;
the functions $f_1(x)$ are $P_1^1(2x-1)$, where $P_1^1(t)$ are Ferrer
associated Legendre functions (Anderson and Darling (1952), though the
$\lambda^*_1$ and $f_1(x)$ are both misprinted). For Cases 1 and 2, $a^*_i = 0$ for
$i$ even, and $b^*_i = 0$ for $i$ odd, and the solution is similar to that
of $W^2$. Thus in Case 1, the standard $\lambda^*_1 = 2i(2i+1)$, $i=1,2,\ldots$ are
solutions, and values of $1/\lambda^*_1$, where $S_{a}^*(\lambda^*_1) = 0$, are given in
Table 2, Column 5. For Case 2, $\lambda^*_1 = 2i(2i-1)$ are solutions, and the
values of $1/\lambda^*_1$, where $S_{b}^*(\lambda^*_1) = 0$, are in Table 2, Column 6. The
$\lambda^*_1$ in Case 1 converge, for large $i$, on $(2i+1)(2i+2)$; those in Case
2 on $2i(2i+1)$.
A: Case 3. A with $W^2$ and $U^2$, the weights for Case 3 are the $\lambda^*_i$ of Cases 1 and 2. The cumulants, calculated from the weights, are given in Table 3. The variances cannot now be checked against exact calculations; estimates of the variance, involving numerical integration of double integrals, were found from (1), and differed only in the third decimal place. The values in Table 3 are considered more reliable since they involve only single integrals calculated numerically (the $a_i, b_i$).
5. TESTS FOR EXPONENTIALITY

5.1 For Case 4, the null hypothesis is $H_0$: a sample $x_1, x_2, \ldots, x_n$ comes from $F(x) = 1 - \exp(-\theta x)$, $x \geq 0$; $\theta$ is estimated by $1/x$. Then (Darling, 1955) $\psi(s) = s \ln s$ for $W^2$, and the covariance function is $r(s,t) - st \ln s \ln t$. Following section 3 and using notation of §3.1 we have the covariance function $\rho^*(s,t)$ for $U^2$:

$$\rho^*(s,t) = r(s,t) + a(s,t) + m(s,t)$$

where $m(s,t) = -(s \ln s + 0.25)(t \ln t + 0.25)$. Note that this is of the form (5).

Finally, the covariance function for $A$ is $(r(s,t)-st \ln s \ln t)(w(s,t))$. Straightforward calculations give the results for $W^2$ (first given by Darling) and for $U^2$ in Table 1.

5.2 Calculations for $A$. These require more complicated integrals. For the mean,

$$\mu = \int_0^1 (s(1-s)-s^2 \ln s)ds/s(1-s)$$

$$= 1 - \int_0^1 s \ln^2 s \frac{ds}{1-s}$$

$$= 1 - \sum_{k=1}^{\infty} \int_0^1 s^k \ln^2 s \, ds$$

$$= 1 - 2 \sum_{k=1}^{\infty} (k+1)^{-3} = 1 -.404114 = .595886$$
For the variance,

$$\sigma^2 = 4 \int_0^1 \int_0^t \frac{(s(1-t) - st \ln s \ln t)^2}{s(1-s)t(1-t)} \, ds \, dt$$

$$= \sigma_o^2 - 8 \int_0^1 \int_0^t \frac{s \ln s \ln t}{(1-s)} \, ds \, dt + 4 \int_0^1 \int_0^t \frac{st \ln^2 s \ln t}{(1-s)(1-t)} \, ds \, dt$$

where $\sigma_o^2$ is the variance in Case 0 (.579786). The final term is

$$2[\int_0^1 \frac{s \ln^2 s}{1-s} \, ds]^2 = 2 (0.4041)^2$$

from the calculations for the mean; the remaining integral is, changing the order,

$$-8 \int_0^1 \int_s^1 \ln t \, dt \, s(1-s)^{-1} \ln s \, ds$$

$$= 8 \int_0^1 (s^2(1-s) \ln^2 s - s \ln s) \, ds$$

$$= 15 \sum_{k=1}^{\infty} \frac{1}{(k+2)^3} - 2$$

by methods similar to those for the mean. These calculations give

$$\sigma^2 = .139303.$$

5.3 **Calculation of weights.** The parameter $\theta$ is a scale parameter, and Case 4 is similar to Case 2. We therefore call the Fredholm determinant $D_4(\lambda)$; it equals $d_1(\lambda) S_d(\lambda)$, using the notation of Theorem 2, where the $b_i$ are calculated with the appropriate $\phi'$s for Case 4. These functions are such that no $b_i$ is now zero. For $W^2$ and $A$, this means that no standard weight is a zero of $D_4(\lambda);$
6. CALCULATION OF PERCENTAGE POINTS

6.1 Where possible, the first four cumulants have been used to fit Pearson curves to the distributions, and to give the uppertail percentage points. The points are found from the tables of Johnson, Nixon, Amos and Pearson (1963). For smooth curves such as these, with long tails, the points can be expected, from past experience, to be very accurate. A check is provided by plotting points for finite n, found from Monte Carlo studies, and plotting against $1/n$ (or $1/\sqrt{n}$) and extrapolating to $1/n = 0$. This was done before the Pearson curve fits were made, and the results agree extremely well. Where the values of $\beta_1$, $\beta_2$ are beyond the range of the Pearson curve tables, (for $W^2$ and $A$, Cases 2 and 4) an approximation of the form $a + b \chi^2_p$ has been used; the values of $a$, $b$, $p$ are chosen to match the first three cumulants. These are expected also to be quite accurate. The upper 10, 5, $2^{1/2}$%, and 1% points for all statistics, all Cases, are given in Table 4.

6.2 The KKW approximation for Case 3. KKW used only the first four $\lambda^*_i$ in the approximation $S^* = \sum_{i=1}^{4} z_i / \lambda^*_i$ for the asymptotic distribution for $W^2$, Case 3. This gives a mean 0.0415 and a variance 0.00113; both are too low, as expected, and will give significance points which are too low. The $S^*$ approximation gives roughly 0.085, 0.109 and 0.153 for the 10, 5, and 1% points. These differ from the values in Table 4 by roughly the difference in the $S^*$ mean and the true mean.
6.3 Comments on Table 4: The points for Case 0, calculated from known asymptotic distributions, are included in Table 4 for comparison. It is clear that when one is allowed to improve the fit by estimating one or more parameters, the values of the goodness-of-fit statistics, even asymptotically, become stochastically much smaller. In testing for normality, being allowed to estimate the mean makes a much greater improvement to the fit, in general, than the variance, particularly as measured by the statistics $\bar{W}$ and $A$. Fuller details of the associated tests are in Stephens (1970a,b).

Acknowledgment. This work was partly supported by the National Research Council of Canada, and by the U.S. Office of Naval Research Contract N00014-67-A-0112-0053. The author thanks both these agencies for their support. He also acknowledges with thanks conversations with Dr. J. Farina in connection with Theorem 2.
## Asymptotic Means and Variances of $W^2$, $U^2$ and $A$

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<th></th>
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REFERENCES


