AN OPTIMAL STOPPING PROBLEM FOR SUMS OF 
DICHOTOMOUS RANDOM VARIABLES

BY

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1. Introduction

The limiting behavior of the solution of the following problem can be used to relate the solution of a class of continuous time stopping problems involving a Wiener process to certain discrete time, discrete process, stopping problems. This relationship can be used to estimate the error of a relatively simple computational approximation to the solutions of stopping problems. In the last section we elaborate on this paragraph and relate it to a problem in sequential analysis.

A stochastic process \( \{Y_t, t=-n, -n+1, \ldots, 0\} \) starting at \( Y_{-n} = y \) is observed for at most \( n \) successive times. At each time \( Y_t \) changes by either \( a \) or \( b \) so that the change has mean \( 0 \) and variance \( 1 \). To observe a new value of \( Y_t \) involves a cost of one unit. The observer receives a reward only if he has observed all \( n \) steps and the final value \( Y_0 \) is negative in which case he receives the square of the final value.
Clearly it pays to continue observing if \( Y_t \) is highly negative and to stop if \( Y_t \) is highly positive. In Section 3 we shall prove that an optimal procedure consists of continuing as long as \( Y_t \) stays below \( \tilde{Y}_t \) where \(-t\) is the remaining number of steps to be observed and that \( \tilde{Y}_t \) converges to \( \tilde{y} \) as \( -t \to \infty \). In Section 4 an expression for \( \tilde{y} \) is obtained in terms of a contour integral for the case of rational \( a/b \) and in Section 5 it is shown that \( \tilde{y} \) is continuous in \( a/b \). In Section 6 some related results are presented.

2. Notation

The process \( \{Y_t, t=-n, -n+1, \ldots\} \) starts at \( Y_{-n} = y \) and can be expressed as

\[
Y_{t+1} = Y_t + X_t
\]

where \( \{X_t, t=0, \pm 1, \pm 2, \ldots\} \) are independently and identically distributed according to

\[
P\{X_t = a\} = p
\]

\[
P\{X_t = b\} = 1-p
\]

where \( 0 < p < 1 \) and \( a \) and \( b \) are such that \( E X_t = 0 \) and \( E X_t^2 = 1 \) which is equivalent to

\[
pa + (1-p)b = 0
\]

\[
pa^2 + (1-p)b^2 = 1.
\]
We shall take

\[ a = -\sqrt{(1-p)/p} \quad \text{and} \quad b = \sqrt{p/(1-p)} . \]

Note that \( a/b = -(1-p)/p \) is rational if and only if \( p \) is.

The stopping problem consists of finding a stopping rule which stops after \( N \) steps in order to maximize the expected gain designated by \( v(y,n) \). Here \( N \) is possibly random. Then the gain consists of the payoff \( Y_0^2 \) if \( Y_0 \leq 0 \) and \( N = n \) steps are observed minus a cost of \( N \) if \( N \) steps are observed. In the event that \( N < n \), there is no payoff.

We shall designate the optimal payoff by \( \hat{v}(y,n) \). Associated with an arbitrary stopping rule we are concerned with \( N \), and \( T = -n + N \), the value of the subscript upon termination, and \( \hat{Y} = Y_T \), the value of \( Y_t \) upon termination. The event

\[ F_n = \{N=n, \hat{Y} < 0\} \]

and its complement \( E_n \) are of importance. Note that \( N, T, \hat{Y}, \) and the probability measure depend implicitly upon the initial point \( (y,-n) \) as well as the stopping rule.

3. **Monotonicity and continuity properties and bounds**

In this section we show that an optimal strategy consists of stopping when \( Y_t \geq \hat{Y}_t \) where \( \hat{Y}_n \) is negative and decreases monotonically as \( n \to \infty \) to some number \( \tilde{y} \geq -b/2 \). Also \( \tilde{v}(y,n) \) converges monotonically to \( \tilde{v}(y) \) as \( n \to \infty \) where
\( \tilde{v}(y) \) is continuous and satisfies a simple functional equation.

Comparing the expected gains of taking an observation and stopping, and applying backward induction it is easy to see that

\[
(3.1) \quad \tilde{v}(y,n) = \max\{0, p\tilde{v}(y+a,n-1) + (1-p)\tilde{v}(y+b,n-1) - 1\}, \quad n > 0
\]

with \( \tilde{v}(y,0) = y^2 \) for \( y \leq 0 \) and \( \tilde{v}(y,0) = 0 \) for \( y \geq 0 \).

It is also apparent that an optimal policy consists of stopping after observing \( Y_t = y \), if \( \tilde{v}(y,-t) = 0 \). This describes the optimal policy in terms of a stopping set of points \((y,t)\) at which it pays to stop or the complementary continuation set on which \( \tilde{v}(y,-t) > 0 \). This set does not depend on the initial value \( n \) specified in the problem and thus our solution is simultaneously applicable for all initial points \((y,-n)\), \( n > 0 \).

**Lemma 3.1** \( \tilde{v}(y,n) \) is monotonic decreasing in \( n \).

**Proof:** We observe that

\[
p(y+a)^2 + (1-p)(y+b)^2 - 1 = y^2.
\]

Hence, if \( y + b \leq 0 \),

\[
p\tilde{v}(y+a,0) + (1-p)\tilde{v}(y+b,0) - 1 = y^2
\]

but if \( y + b > 0 \geq y + a \) the left hand side of the above
equality becomes \( p(y+a)^2 - 1 < y^2 \). It easily follows from (3.1) that

\[
\tilde{v}(y,1) = y^2 \quad \text{for } y \leq -b \\
0 < \tilde{v}(y,1) < y^2 \quad \text{for } -b < y < -a - p^{-1/2} < 0 \\
\tilde{v}(y,1) = 0 \quad \text{for } y \geq -a - p^{-1/2}
\]

and hence \( \tilde{v}(y,1) \leq \tilde{v}(y,0) \).

For \( n > 0 \), \( \tilde{v}(y,n+1) \) can be considered the optimal payoff of an \( n \) step stopping problem with terminal payoff function \( \tilde{v}(y,1) \leq \tilde{v}(y,0) \). This problem is less favorable than our initial problem and hence \( \tilde{v}(y,n+1) \leq \tilde{v}(y,n) \).

This proof incidentally demonstrates that for \( t = -1 \), the stopping points are \( \{(y,-1):y \geq \hat{v}_1 = -a - p^{-1/2}\} \) where \( \hat{v}_1 < 0 \). Hence applying Lemma 3.1, \( \tilde{v}(y,n) = 0 \) for \( y \geq \hat{v}_1 \).

**Lemma 3.2** \( \tilde{v}(y,n) \) is monotonic decreasing in \( y \).

**Proof:** For a given initial point \((y,-n)\), the optimal procedure can be described in terms of the \( X_{-n}, X_{-n+1}, \ldots \), which lead to stopping. Apply this same rule for the initial point \((y-\varepsilon,-n)\) with \( \varepsilon > 0 \). (Here this rule is possibly suboptimal).

Then

\[
\tilde{v}(y-\varepsilon,n) - \tilde{v}(y,n) \geq \int_{F_n} [(\hat{X}-\varepsilon)^2 - \hat{X}^2]dP \geq 0
\]

where we recall that \( F_n = \{N=n, \hat{Y}<0\} \) and \( P \) is the probability distribution induced by the optimal procedure with initial point \((y,-n)\).
Theorem 3.1 follows immediately from Lemmas 3.1 and 3.2.

**Theorem 3.1** The optimal stopping set can be described as

\[(y, n) : y \geq \tilde{y}_n, n \geq 1\]

where \{\tilde{y}_n\} is a monotonic decreasing negative sequence.

Furthermore \(\tilde{v}(y, n) = 0\) for \(y > \tilde{y}_n\).

We may now define

\[(3.2) \quad \tilde{y} = \lim_{n \to \infty} \tilde{y}_n\]

and

\[(3.3) \quad \tilde{v}(y) = \lim_{n \to \infty} \tilde{v}(y, n).\]

**Lemma 3.3** \(y^2 \geq \tilde{v}(y, n) > y^2 - b^2/4\) for \(y \leq 0\),

and

\(\tilde{y}_n > -b/2\).

**Proof:** Since \(\{W_i = y_{n+i} - n, i = 0, 1, 2, \ldots\}\) is a martingale the optional stopping theorem [5, p. 300] yields

\[(3.4) \quad y^2 = E(\hat{y}^2 - N)\]

for any stopping procedure. Then

\[(3.5) \quad v(y, n) = E(\hat{y}^2 - N) - \int_{E_n} \hat{y}^2 dP\]

where \(E_n = \{N < n \text{ or } N = n, \hat{y} > 0\}\).
Let \( y < 0 \) and consider the special possibly sub-optimal procedure which consists of continuing as long as \( Y_t < c \leq 0 \). Then

\[
v(y,n) \geq y^2 - \max[c^2, (c+b)^2].
\]

Taking \( c = -b/2 \) we have \( \tilde{v}(y,n) \geq v(y,n) \geq y^2 - b^2/4 \). Thus \( \tilde{v}(y,n) > 0 \) for \( y < -b/2 \) and \( \tilde{y}_n \geq -b/2 \).

Some useful continuity properties are included in

**Theorem 3.2.** \( \tilde{v}(y,n) \) and \( \tilde{v}(y) \) are continuous and for \( y < 0 \), their derivative numbers are bounded between \( 2(y-b) \) and \( \min[0, 2(y-\tilde{y})] \).

**Proof:** First we note that \( \{ Y_t, -\infty < t < 0 \} \) is a martingale and hence the optional stopping theorem yields

\[
y = E\tilde{y} = \int_{E_n} \hat{y} dP + \int_{F_n} \hat{y} dP
\]

for any stopping procedure. Let us restrict ourselves to procedures which stop if \( Y_t > 0 \) and do not stop before \( t = 0 \) if \( Y_t < \tilde{y} \). These include all optimal procedures. Then if \( y < 0 \),

\[
y - b \leq \int_{F_n} \hat{y} dP \leq y - \tilde{y}.
\]

Applying the argument of Lemma 3.2, we have, for \( \varepsilon > 0 \),
\[ \tilde{v}(y-\varepsilon, n) - \tilde{v}(y, n) \geq -2\varepsilon \int F_n \hat{y} dP + \varepsilon^2 \int dP \geq -2\varepsilon (y - \tilde{y}). \] The same argument yields
\[ \tilde{v}(y+\varepsilon, n) - \tilde{v}(y, n) \geq \int F_n [(\hat{y}+\varepsilon)^2 - \hat{y}^2] dP - \int F_n \{y+\varepsilon > 0\} (\hat{y}+\varepsilon)^2 dP \geq 2\varepsilon (y - \tilde{y}) - \varepsilon^2. \]

These bounds which are independent of \( n \) and the previous results imply our Theorem 3.2. □

Let \( n \to \infty \) in Equation 3.1. With the help of the previous results it is easy to see that

Theorem 3.3 \( 0 \geq \tilde{y} \geq -b/2 \)

\[ y^2 \geq \tilde{v}(y) \geq y^2 - b^2/4, \quad \text{for } y \leq 0 \]

and

\[ \tilde{v}(y) = p\tilde{v}(y+a) + (1-p)\tilde{v}(y+b) - 1 \quad \text{for } y \leq \tilde{y} \]
\[ \tilde{v}(y) = 0 \quad \text{for } y \geq \tilde{y}. \]

Note that the two equations of (3.7) imply \( p\tilde{v}(\tilde{y}+a) + (1-p)\tilde{v}(\tilde{y}+b) - 1 = 0. \)

4. The case of rational \( p \)

Suppose \( p = r/m \) where \( r \) and \( m \) are relatively prime integers with \( 0 < r < m \). Let \( s = m-r > 0 \). Then \( a = -sh \) and \( b = rh \) where \( h = (rs)^{-1/2} \). In this case the possible values of \( X_t \) are commensurate, i.e., integral multiples of \( h \).
and the possible values of \( y_t \) are restricted to a lattice of values \( y + ih, i = 0, \pm 1, \pm 2, \cdots \). For convenience we represent this set as \{c+ih:i=0,\pm 1,\pm 2,\cdots\} where \( c \) is selected so that \( \tilde{y} - h < c < \tilde{y} \).

For each value of \( c \), Equation (3.7) becomes a classical \( m \)-th order linear difference equation and \( \tilde{v}(y) \) is a solution of

\[
(4.1) \quad v(y) = \frac{1}{m} v(y-sh) + \frac{s}{m} v(y+rh) - 1 \quad y \leq \tilde{y}
\]

\[
(4.2) \quad v(y) = 0 \quad y \geq \tilde{y}
\]

A particular solution of (4.1) is given by \( v(y) = y^2 \).

The general solution of (4.1) can be expressed in terms of the roots \( x_i \) of the algebraic equation

\[
(4.3) \quad m x^s = x + sx^m
\]

which is easily seen to have a double root at \( x = 1 \) and no other double root. Then the general solution of (4.1) is

\[
(4.4) \quad v(y) = y^2 + d_0 + d'_0 y + \sum_{i=1}^{m-2} d_i x_i^{(y-c)/h}.
\]

Then the bound on \( \tilde{v}(y) \) in Theorem 3.3 implies that for \( \tilde{v}(y) \), \( d'_0 = 0 \) and \( d_i = 0 \) for all \( i \) for which \( |x_i| < 1 \). The fact that \( \tilde{v}(y) = 0 \) for \( y \geq \tilde{y} \) implies
(4.5) \( \tilde{v}(c+ih) = 0 \) for \( i = 1,2,\ldots,r \)

which imposes \( r \) conditions on the remaining \( d_i \). (The fact that \( \tilde{v}(y) = 0 \) for \( y > c+rh \) is not useful since Equation (4.1) applies only for \( y \leq \tilde{y} \) or equivalently \( y + rh \leq \tilde{y} + rh < c + (r+1)h \).) Thus to determine the coefficients \( d_i \) it is desirable to have \( r - 1 \) roots \( x_i \) for which \( |x_i| > 1 \). In the following lemma we establish this property and use it to derive an expression for \( \tilde{y} \) in terms of a contour integral.

Lemma 4.1 The equation \( mx^s = r + sx^m \) has one double root at \( x = 1 \), \( r - 1 \) distinct roots for which \( |x_i| > 1 \), and \( s - 1 \) distinct roots for which \( |x_i| < 1 \).

Proof: Since the \( x^{-1} \) satisfy the same equation with \( r \) and \( s \) interchanged, it suffices to show that there are \( s - 1 \) roots inside the unit circle. Let

(4.6) \[ A(x) = x^{-S}B(x) = \frac{r}{m} x^{-s} + \frac{s}{m} x^r - 1 \]

Consider the path \( C \) which follows the unit circle counterclockwise in the complex plane except for a short vertical line near \( x = 1 \) from \( (1-\epsilon) - i\eta \) to \( (1-\epsilon) + i\eta \). The number of times \( A(x) \) circles the origin as \( x \) goes around \( C \) is \( s \) less than the number of \( x_i \) such that \( |x_i| < 1 \). Our proof will be complete if we show that this is \( -1 \), i.e., if there is one clockwise circuit.
Near $x = 1$, $A(x) \approx rs(x-1)^2/2$ and along the line segment, $A(x)$ moves from a point in the second quadrant clockwise about 0 to a point in the third quadrant. Along the circular part of $C$, $A(x)$ is a weighted average of points on the unit circle $-1$, and is confined to the half plane where the real part is negative. The lemma follows.

Let $x_0 = 1, x_1, x_2, \ldots, x_{r-1}$ be the $r - 1$ distinct roots of (4.3) outside the unit circle and $x_r, x_{r+1}, \ldots, x_m$ be the distinct roots inside the unit circle. Let

\[
D = \begin{pmatrix}
1 & x_1 & \cdots & x_{r-1} \\
1 & x_1^2 & \cdots & x_1^{r-1} \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \\
1 & x_r & \cdots & x_r^{r-1}
\end{pmatrix}
\]

(4.7)

which is non-singular and let

\[d' = (d_0, d_1, \ldots, d_{r-1})\]

and let $i^j$ be the column vector whose $i^{th}$ element is $i^j$, $1 \leq i \leq r$, with $1 = i^0$. Then condition (4.5) translates to

\[(4.8) \quad Dd = -(c^2_1 + 2c_1i + h^2 i^2).\]
Hence,

\[(4.9) \quad \tilde{v}(c) = \frac{1}{2} d + c^2 = c^2 (1 - \frac{1}{2} D^{-1} \mathbf{1}) - 2 c \mathbf{1}' D^{-1} \mathbf{1} - h^2 \mathbf{1}' D^{-1} \mathbf{1}^2.\]

The following Theorem "evaluates" \( \tilde{v}(c) \) and \( \tilde{y} \).

**Theorem 4.1** For some positive constant \( k \) (independent of \( c \))

\[(4.10) \quad \tilde{v}(c) = -2 k h \left\{ c + h \left[ \frac{1}{2} \sum \left| x_i \right| (1-x_i)^{-1} \right] \right\} \quad \text{for} \quad \tilde{y} - h < c < \tilde{y} \]

and

\[(4.11) \quad \tilde{y} = - \left\{ \frac{1}{2} + \sum \left(1-x_i\right)^{-1} \right\} h \]

**Proof:** Let \( E(x) = \frac{1}{2} D^{-1} x - 1 \) where \( x' = (x, x^2, \ldots, x^r) \). Since \( D^{-1} x_{i-1} \) is the \( i \)-th column of the identity matrix \( E(x_{i-1}) = 0 \) for \( i = 1, 2, \ldots, r \). Since \( E(0) \neq 0 \), \( E \) is a nondegenerate \( r \)-th degree polynomial in \( x \) and it follows that

\[ E(x) = k_1 \prod_{i=0}^{r-1} (x-x_i) \]

where \( k_1 \neq 0 \). Moreover,

\[ E(1) = \frac{1}{2} D^{-1} \mathbf{1} - 1 = 0, \]

\[ E'(1) = \frac{1}{2} D^{-1} \mathbf{1} = k_1 \prod_{i=1}^{r-1} (1-x_i) \neq 0, \]

\[ E''(1) = \frac{1}{2} D^{-1} (\mathbf{1}^2 - \mathbf{1}) = 2 E'(1) \sum_{i=1}^{r-1} (1-x_i)^{-1} \]
and applying (4.9)

\[ \tilde{v}(c) = -2chE'(1) - h^2[E''(1) + E'(1)]. \]

Equation (4.10) follows with \( k = E'(1) \). Since \( \tilde{v}(c) \geq 0 \), and is decreasing in \( c \) for \( c < \tilde{y} \), it follows that \( E'(1) > 0 \). Equation (4.11) is an immediate consequence of the fact that \( \tilde{v}(\tilde{y}) = 0 \).

To obtain an expression for \( \tilde{y} \) in terms of a contour integral, we establish two lemmas. The second expresses \( \sum_{|x_i| < 1} (1-x_i)^{-1} \) in terms of a contour integral and the first obtains a simple expression for \( \sum_{x_i \neq 1} (1-x_i)^{-1} \).

**Lemma 4.2** The expansion of \( B'(x)/B(x) \) about \( x = 1 \) is of the form

\[ B'(x)/B(x) = 2(x-1)^{-1} + \frac{r+2s-3}{3} + \frac{(x-1)}{18}[s^2+rs+r^2-12s-6r+15] + \cdots \]

and

\[ \sum_{x_i \neq 1} (1-x_i)^{-1} = \frac{r+2s-3}{3}. \]

**Proof:** Since \( B(x) = \prod_\{m\} (x-x_i) \cdot (x-1)^2 \), it is easy to see that \( B'(x)/B(x) = d[\log B(x)]/dx = 2(x-1)^{-1} + \sum_{x_i \neq 1} (x-x_i)^{-1} \), and the constant in the expansion of \( B'(x)/B(x) \) about \( x = 1 \)
is \( \sum_{x_i \neq 1} (1-x_i)^{-1} \). Expanding \( B'(x) \) and \( B(x) \) about \( x = 1 \) and dividing yields the expansion above. \( \square \)

**Lemma 4.3**

\[
\sum_{|x_i| < 1} (1-x_i)^{-1} = \frac{1}{2\pi} \int_0^{2\pi} \frac{ms(e^{i\theta} - 1)e^{i(s-1)\theta}}{e^{i\theta} - 1 - r+2s-3} \frac{e^{i\theta} d\theta}{1-e^{i\theta}}.
\]

**Proof:** By virtue of the expansion of Lemma 4.2,

\[
\left[ \frac{B'(x)}{B(x)} - 2(x-1)^{-1} - \frac{r+2s-3}{3} \right] \frac{1}{1-x} = \frac{\sum_{|x_i| < 1} (1-x_i)^{-1}}{1-x}.
\]

which has poles of magnitude \( (1-x_i)^{-1} \) at each \( x_i \neq 1 \), is regular at \( x = 1 \). Our result follows by taking the contour integral about the unit circle. \( \square \)

Combining Theorem 4.1 with the last two lemmas we can write

\[
(4.12) \quad \tilde{y} = [S + \frac{1}{2} - \frac{r+2s}{3}]h
\]

where \( S \) is the right hand side of the statement of Lemma 4.3.

It should be remarked that the cases where \( p = l/m \) and \( p = (m-1)/m \) are simple to handle without using the contour integral. When \( p = l/m, r = 1 \) and there are no roots of (4.3) outside the unit circle. Then \( \tilde{y} = -h/2 = -b/2 = -(m-1)^{1/2}/2 \).

When \( p = (m-1)/m, r = m-1, s = 1 \) and except for the double root at \( x = 1 \), all the roots are outside the unit circle. Then
\[ \sum \frac{1-|x_i|}{|x_i|} = (r+2s-3)/3 = (m-2)/3 \] and \( \bar{y} = -h(2m-1)/6 \), where \( h = (m-1)^{-1/2} = -a \). Since \( b = (m-1)h \), \( \bar{y} \approx -b/3 \) in that case.

Incidentally, in the case \( p = 1/m \), application of (4.4) and (4.8) yields

\[ \tilde{v}(y) = y^2 - (c+b)^2 \quad \text{for } y \leq \bar{y} = -b/2 \]

where \( c \) is the remainder, \(-3b/2 < c \leq -b/2\), after a suitable multiple of \( b \) is added to \( y \). The term \( (c+b)^2 \) can also be expressed as

\[ \inf_{j} (y+jb)^2 , \]

i.e., the square of the distance from \( y \) to the closest multiple of \( b \), and varies from 0 to \( b^2/4 \).

5. Continuity in \( p \)

In Section 4, the functional equation (3.7) reduced to a difference equation which was used to derive expressions for \( \tilde{y} \) and \( \tilde{v}(y) \) for the case where \( p \) is rational. In the irrational case that technique is not directly applicable. However, we shall show that \( \tilde{y} \) and \( \tilde{v}(y) \) are continuous in \( p \) and thus their values in the irrational case can be approximated by replacing \( p \) by a nearby rational.

We shall find it expedient to introduce another stopping problem which is equivalent to the original problem. That
consists of minimizing

\[ u(y, n) = \int_{E_n} \tilde{y}^2 \, dP \]

and has minimizing value \( \tilde{u}(y, n) \). To compare the cases for two values \( p \) and \( p_o \), we shall use the subscript \( o \) to represent the case of \( p_o \). Thus \( \tilde{u}_o(y, n), \tilde{y}_o, \tilde{y}_o, \tilde{y}_o \) etc. all correspond to the case of \( p_o \).

Lemma 5.1 implies the equivalence of the original problem and the minimization problem and the fact that as \( n \to \infty \),

\[ \tilde{u}(y, n) \to \tilde{u}(y) = y^2 - \tilde{v}(y). \]

**Lemma 5.1** For any stopping rule, \( v(y, n) = y^2 - u(y, n) \).

**Proof:** This result is an immediate consequence of Equations (3.4) and (3.5). \( \square \)

It should be remarked that there are some conveniences to be gained from the equivalence of these two problems. The original problem made certain monotonicity properties easy to obtain. The new problem permits us to deal with \( \hat{y} \) on \( E_n \), where \( \hat{y} \) is bounded between \(-b/2\) and \( b \).

Our overall plan consists of using a bound on \( P(E_n) \) to bound \( \tilde{v}(\hat{y}) - \tilde{v}(\tilde{y}_o, n) \). This with the bound on the derivative numbers of \( \tilde{v}(y, n) \) (of Theorem 3.2) leads to a bound on \( \tilde{y} - \tilde{y}_n \). Finally a bound on \( \tilde{v}(y, n) - \tilde{v}_o(y, n) \) leads to a bound on \( \tilde{y}_n - \tilde{y}_o \) which combines with two applications of the previous result to bound \( \tilde{y} - \tilde{y}_o \).
The desired bound on $P(E_n)$ follows from the following more general theorem on the distribution of the time a sum of i.i.d. random variables first exceeds a value $c > 0$.

**Theorem 5.1** (First passage time bound)

\[
\text{If } Z_1, Z_2, \cdots \text{ are i.i.d. random variables with mean } \\
\mu_Z > 0 \text{ and } M_Z(\lambda) = E(e^{\lambda Z}) < \infty \text{ for some positive } \lambda, \text{ then for } \\
c > 0
\]

\[
P\left\{ \left( \max_{1 \leq m < n} \sum_{i=1}^{m} Z_i \right) < c \right\} \leq \frac{1-e^{-\lambda c}}{1-e^{-\lambda c}} \frac{M_Z(\lambda)}{\log M_Z(\lambda)}.
\]

**Proof:** Let $S_n = Z_1 + Z_2 + \cdots + Z_n$. We desire to bound $P(N \geq n)$ where $N$ is the first time $S_n > c$. The sequence $\exp[\lambda S_n - n \log M_Z(\lambda)]$ is a martingale with mean 1. Because $\mu_Z > 0$, $u = -\log M_Z(\lambda) < 0$. By optional stopping

\[
1 = E(e^{\lambda S_n + Nu}) \leq e^{\lambda c M_Z(\lambda) M_N(u)}
\]

where $M_N$ is the moment generating function (m.g.f.) of $N$. But

\[
M_N(u) \leq [1 - P(N \geq n)] + e^{nu} P(N \geq n)
\]

The theorem follows from the above two inequalities. \[ \]

**Lemma 5.2** If $y$ is in a bounded interval and $p$ is in
a closed subinterval of \((0,1)\), there is a constant \(K\) such that
\[
P(F_n) \leq Kn^{-1/2}
\]
when the optimal procedure is applied.

**Proof:** The case \(y > 0\) is trivial. Note that \(P(F_n) \leq \)
\[
P\{\max_{1 \leq m < n} \sum_{i=1}^{m} X_i < -y\}.
\]
Apply Theorem 5.1 with \(\lambda = n^{-1/2}\)
and \(c = -y\). Then \(M_2(\lambda) = 1 + \frac{1}{2} \lambda^2 + O(\lambda^3)\) and it is easy
to see that the right hand side of (5.1) is asymptotically
equivalent to \((-y)(1-e^{-1/2})^{-1}n^{-1/2}\). A more detailed calculation yields our desired result. □

**Lemma 5.3** If \(y\) is in a bounded interval and \(p\) is in a
closed subinterval of \((0,1)\) there is a constant \(K\) such that
\[
\tilde{v}(y,n) \geq \tilde{v}(y) \geq \tilde{v}(y,n) - b^2 Kn^{-1/2}
\]
and
\[
\tilde{y}_n - \tilde{y} \leq bk^{1/2}n^{-1/4}.
\]

**Proof:** For the stopping problem with initial point \((y,-n_1)\),
with \(n_1 > n\), apply the optimal procedure for initial point
\((y,-n)\). More precisely stop if \(Y_{t} \geq \tilde{y}_{n-n_1-t}\) for \(-n_1 \leq t < n-n_1\).
For \(t \geq n-n_1\) stop if \(Y_t \geq -b/2\) or when \(t = 0\). This sub-
optimal procedure leads to \(u(y,n_1)\) where
\[
\tilde{u}(y,n_1) \leq u(y,n_1) \leq \tilde{u}(y,n) + b^2 P(F_n)
\]
Let \(n_1 \to \infty\) and apply Lemmas 5.1 and 5.2 and the first part of the
result follows. Now let \( y = \tilde{y} \geq -b/2 \) and hence
\[
\tilde{v}(\tilde{y}, n) - \tilde{v}(\tilde{y}_n, n) = \tilde{v}(\tilde{y}, n) \leq b^2 p(F_n)
\]

However, Theorem 3.2 implies that \( \tilde{v}(\tilde{y}, n) - \tilde{v}(\tilde{y}_n, n) \geq (\tilde{y}_n - \tilde{y})^2 \).

**Lemma 5.4** If \( y \leq 0 \)
\[
\tilde{v}_o(y, n) \geq \tilde{v}(y, n) - (\varepsilon_{1n} b^2 + 2b\varepsilon_{2n} + \varepsilon_{2n}^2)
\]

where \( \varepsilon_{1n} = n|p - p_o| \) and \( \varepsilon_{2n} = n \max(|a - a_o|, |b - b_o|) \).

**Proof:** We shall apply the optimal stopping rule for \((y, n, p)\) to \((y, n, p_o)\) where that rule will be interpreted in terms of the "history" of positive and negative values of \(X_t\) which lead to stopping. It is convenient to think of \(X_t\) and \(X_{ot}\) as being formed by generating a random variable \(Z_t\) uniform on \((0,1)\) and letting \(X_t = a\) if \(Z_t \leq p\) and \(b\) otherwise. The same \(Z_t\) can serve to generate \(X_t\) and also \(X_{ot}\) corresponding to \(p_o\). Thus we see that in \(n\) steps, the "histories" of positive and negative steps for \(X_t\) and \(X_{ot}\) will differ only if some \(Z_t\) is between \(p\) and \(p_o\), an event with probability no larger than \(\varepsilon_{1n}\). To help define our stopping rule, let \(X_t^* = a\) when \(X_{ot} = a_o\) (i.e., when \(Z_t \leq p_o\)) and let \(X_t^* = b\) when \(X_{ot} = b_o\). Our rule consists of stopping the \(Y_{ot}\) process when \(Y_t^* > \tilde{y}_t\).

If the histories don't differ, then \(\hat{Y}_t = \tilde{y}\) and \(|\hat{y} - \tilde{y}_o| \leq \varepsilon_{2n}\).
Let $H_n^+$ be the event of common histories and $H_n^-$ be its complement. Then

$$u_o(y,n) = \int_{H_n^+E_{n\mathbb{E}}} \hat{\gamma}_o^2 \, dp + \int_{H_n^+E_{n\mathbb{E}}} \hat{\gamma}_o^2 \, dp + \int_{H_n^-E_{n\mathbb{E}}} \hat{\gamma}_o^2 \, dp$$

and

$$\tilde{u}(y,n) = \int_{\hat{\gamma}^2} \hat{\gamma}^2 \, dp \geq \int_{H_n^+E_{n\mathbb{E}}} \hat{\gamma}^2 \, dp$$

On $H_n^+E_{n\mathbb{E}}$, $\hat{\gamma}_o^2 \leq \hat{\gamma}^2 + 2b\varepsilon_2n + \varepsilon^2_2n$. On $H_n^+E_{n\mathbb{E}}$, $N = N_o = n$ and $\hat{\gamma} < 0$ but $\hat{\gamma}_o \geq 0$ and hence $\hat{\gamma}_o^2 \leq \varepsilon^2_2n$. Finally on $E_{\mathbb{E}}$, $|\hat{\gamma}_o - \hat{\gamma}^*| \leq \varepsilon_2n$ and $|\hat{\gamma}^*| \leq b$ and hence $\hat{\gamma}_o^2 \leq (b + \varepsilon^2_2n)^2$. Hence

$$\tilde{u}_o(y,n) \leq u_o(y,n) \leq \tilde{u}(y,n) + 2b\varepsilon_2n + \varepsilon^2_2n + \varepsilon_1b^2. \square$$

Lemma 5.5 If $\tilde{\gamma}_n > \tilde{\gamma}_{\mathbb{E}}$, then

$$\tilde{\gamma}_n - \tilde{\gamma}_{\mathbb{E}} \leq (\varepsilon_1b^2 + 2b\varepsilon_2n + \varepsilon^2_2n)^{1/2}$$

Proof:

$$0 = \tilde{v}_o(\tilde{\gamma}_{\mathbb{E}}, n) \geq \tilde{v}(\tilde{\gamma}_{\mathbb{E}}, n) - (\varepsilon_1b^2 + 2b\varepsilon_2n + \varepsilon^2_2n)$$

By Theorem 3.2

$$\tilde{v}(\tilde{\gamma}_{\mathbb{E}}, n) = \tilde{v}(\tilde{\gamma}_{\mathbb{E}}, n) - \tilde{v}(\tilde{\gamma}_n, n) \geq (\tilde{\gamma}_n - \tilde{\gamma}_{\mathbb{E}})^2. \square$$
Theorem 5.2 \( y \) is continuous in \( p \).

Proof: From Lemma 5.3 we have bounds for \( \tilde{y}_n - \tilde{y} \) and \( \tilde{y}_{on} - \tilde{y}_o \). From Lemma 5.5 we have a bound on \( \tilde{y}_n - \tilde{y}_{on} \) if \( \tilde{y}_n > \tilde{y}_{on} \) and a similar bound if \( \tilde{y}_n < \tilde{y}_{on} \). These combine to give a bound on \( |\tilde{y} - \tilde{y}_o| \). Given an interval containing \( p \) and \( p_o \), then \( a, b, a_o, \) and \( b_o \) are bounded, and a constant \( K \) (from Lemma 5.3) is determined. For \( n \) sufficiently large the bounds for \( \tilde{y}_n - \tilde{y} \) and \( \tilde{y}_{on} - \tilde{y}_o \) derived from Lemma 5.3 can be made arbitrarily small. Given that value of \( n, \varepsilon_{ln} \) and \( \varepsilon_{2n} \) can be made sufficiently small by taking \( p - p_o \) small enough so that the bound on \( |\tilde{y}_n - \tilde{y}_{on}| \) is arbitrarily small. Hence \( |\tilde{y} - \tilde{y}_o| \) can be made arbitrarily small. \( \square \)

Theorem 5.2 permits us to approximate \( \tilde{y}_o \) for irrational \( p_o \) by computing \( \tilde{y} \) for nearby rational \( p \). Moreover the derivation carries implicitly in it a method of estimating the error of this approximation.

In Figure 1, we present the values of \( \tilde{y} \) determined from (4.12) by numerical integration for a finite sequence of rational values of \( p \). This figure suggests the conjecture that \( \tilde{y} \) is not differentiable in \( p \).

6. Related results

An alternative proof was developed for the continuity of \( \tilde{y} \) as a function of \( p \). The proof in Section 5 has the advantage that it provides a means of computing bounds on
\[
\tilde{y}(1-p) = \tilde{y}(p) - \sqrt{(1-p)/p} - \sqrt{p/(1-p)}
\]
The alternative proof involves several results of intrinsic interest in themselves and we shall present these here.

Where Section 5 used Theorem 5.1 to bound the probability distribution of the time of first passage above a constant of a sum of i.i.d. random variables, the alternative uses

\begin{equation}
(6.1) \quad w_c(y) = E(\hat{\gamma}_c^2)
\end{equation}

where \( \hat{\gamma}_c \) is the value of \( Y_t \) at the first time where \( Y_t > c \). We also define

\begin{equation}
(6.2) \quad w^*_c(y) = E(\hat{\gamma}^*_c^2)
\end{equation}

where \( \hat{\gamma}^*_c \) is the value of \( Y_t \) at the first time \( Y_t \geq c \). We omit the subscript \( c \) when \( c = \tilde{\gamma} \). Note that these definitions (6.1) and (6.2) are independent of the \( t \) coordinate of the initial point \( (y,t) \).

In this section we shall prove that \( \hat{\gamma}_n > \tilde{\gamma} \) for all \( n \) and use this to prove that

\[ w(y) = \tilde{u}(y) = w^*(y) \]

A corollary is that \( c = \tilde{\gamma} \) minimizes \( w_c(y) \) and \( w^*_c(y) \). The result that \( w_c(y) = w^*_c(y) \) for \( c = \tilde{\gamma} \) is not trivial. It is not true for general \( c \) and its proof which invokes the fact that \( \hat{\gamma}_n > \tilde{\gamma} \), seems to be tied to the smoothness imposed by the fact that \( c = \tilde{\gamma} \) is the solution of an
optimization problem.

Lemma 6.1 \( \tilde{y}_n > \tilde{y} \).

Proof: Direct computation in the proof of Lemma 3.1 shows that \( \tilde{v}(y,1) < \tilde{v}(y,0) \), for \(-b < y < 0\). If \((y,-n-1)\) is an initial point such that there exists a possible path \( Y_t \) which leads to \((y_1,-1)\) with \(-b < y_1 < 0\) without stopping under the optimal procedure, then

\[
(6.3) \quad \tilde{v}(y,n+1) < \tilde{v}(y,n)
\]

since \( \tilde{v}(y,n+1) \) can be considered as the optimal payoff of an \( n \)-step problem with terminal payoff \( \tilde{v}(y,1) \). In particular if \( y = \tilde{y} \), \( \tilde{y} > -b/2 \) by Theorem 3.3) and \( n_1 \) is designated, such a path is easily constructed for some \( y_1 \) between \(-b\) and \( \tilde{y} \) and for some \( n > n_1 \).

Hence, for arbitrary \( n_1 > 0 \), there is an \( n > n_1 \), so that

\[
\tilde{v}(\tilde{y},n_1) > \tilde{v}(\tilde{y},n) > \tilde{v}(\tilde{y},n+1) > 0
\]

using monotonicity, (6.3), and the non-negativity of \( \tilde{v} \), and thus

\[ \tilde{y} < \tilde{y}_{n_1} \]

Lemma 6.2 \( w(y) = \tilde{u}(y) \).
Proof: Let \( \hat{Y}, T, \) and \( N \) correspond to the suboptimal rule of stopping when \( Y_t > \hat{Y} \), and let \( \tilde{Y}, \tilde{N}, \) and \( \tilde{T} \) correspond to the optimal rule. Then by Lemma 6.1, \( N < \tilde{N} \) and

\[
(6.4) \quad \tilde{u}(y, n) = \int_{\{N=N<n/2\}} \hat{Y}^2 dP + \int_{C} \tilde{u}(\hat{Y}, -T) dP
\]

where \( C \) is the complement of \( \{N=N<n/2\} \). Thus on \( C \) either \( N \geq n/2 \) or \( N < n/2 \) and \( N < \tilde{N} \).

As in Theorem 5.1 it follows that as \( n \to \infty \), \( P\{N\geq n/2\} \to 0 \). On the set where \( N < n/2 \) and \( N < \tilde{N} \), \( \tilde{Y} < \hat{Y} < \tilde{Y}_{[n/2]} \to \tilde{Y} \) and

\[|\tilde{u}(\hat{Y}, -T) - \hat{Y}^2| = \tilde{V}(\hat{Y}, -T) < 3b |\hat{Y} - \tilde{Y}_{[n/2]}| \to 0 \]

where the last inequality derives from Theorem 3.2 and the fact that \( \tilde{Y} > -b/2 \). Hence

\[\tilde{u}(y, n) - \int_{\{N<n/2\}} \hat{Y}^2 dP \to 0.\]

The first term converges to \( \tilde{u}(y) \). The second term converges to \( w(y) \).

Theorem 6.1 \( w(y) = \tilde{u}(y) = w^*(y) \).

Proof: Let \( \hat{Y}^*, T^*, \) and \( N^* \) correspond to the rule of stopping when \( Y_t \geq \hat{Y} \). While the proof of Lemma 6.2 required Lemma 6.1 to infer \( N \leq \tilde{N} \), the fact that \( N^* \leq \tilde{N} \) follows from the definition. The remainder of the proof of Lemma 6.2 applies to \( w^*(y) \) directly.

Note that if \( c \) is a possible value of \( Y_t \), the distribution of \( \hat{Y}_c \) and \( \hat{Y}^*_c \) are quite different and \( w_c(y) \) is
not in general equal to $w_c^*(y)$.

**Corollary 1.** $w_c(y) \geq w(y)$ and $w_c^*(y) \geq w^*(y)$.

**Proof:** The suboptimal strategy of stopping when $Y_t > c$ leads to

$$u(y,n) = \int_{Y < n} \hat{Y}^2 dP + \int_{N=n,Y > 0} \hat{Y}^2 dP \geq \tilde{u}(y,n).$$

As $n \to \infty$ the second term in the sum approaches 0 and the first approaches $w_c(y)$. At the same time $\tilde{u}(y,n) + \tilde{u}(y) = w(y)$. The same proof applies for $w_c^*(y)$. \[ ]

7. **Background**

In a series of papers [1, 2, 3, 4], the sequential problem of testing whether the mean of a normal distribution with known variance is positive or negative was approximated by the continuous time problem of deciding the sign of the drift of a Wiener process. The latter problem reduces to a stopping problem involving a zero drift standard Wiener process $Z(t)$. If stopping takes place at $(\hat{Z}, T)$, there is a payoff $g(\hat{Z}, T)$. The continuous time problem has the advantage that its solution is related to a problem in analysis, a free boundary problem involving the heat equation. Moreover a numerical solution of the continuous time problem can be approximated by applying backward induction on a truncated version of the original discrete time problem.

This apparent circularity seems more embarrassing than is
the case. First, the continuous time problem allows us to derive valuable characteristics of the solution including asymptotic approximations. Moreover there is an excellent and simple approximate relation between the solution of the discrete and continuous time solutions which allow us to use a single backward induction to approximate the solution of the continuous time problem and the solution of an entire class of discrete time problems.

More specifically, a discrete time version of the above stopping problem is obtained when one is permitted to stop only on a discrete set of possible values of \( t \), say \( \{ t_0 + n\delta, n=1,2,\ldots \} \). Then, between successive possible stopping times \( Z \) changes by a normal deviate with mean 0 and variance \( \delta \). It is shown in [4] that the difference between the optimal boundaries \( \bar{z}(t) \) and \( \bar{z}_\delta(t) \) of the two problems is approximately given by

\[
\bar{z}_\delta = \bar{z} \pm 0.5824 \delta^{1/2}
\]

(the sign is determined so as to make the continuation region smaller). The number 0.5824 comes from the limiting value of \( \tilde{y} \) in the solution of an associated problem. That problem is the same as the one originally posed in this paper except that the \( X_t \) are normally distributed with mean 0 and variance 1.

The programming of the backward induction for the numerical calculation is easier if the Wiener process is approximated, not by the sum of independent normal random variables, but by the sum of dichotomous variables which take on the values \( \pm \delta^{1/2} \).
with probability $1/2$. In this case the above approximation is replaced by

$$\tilde{z}_\delta = \tilde{z} \pm 0.5\delta^{1/2}$$

where .5 is the limiting value $\tilde{y}$ in the problem of this paper when $p = 1/2$.

The last result would be adequate to approximate the solution of the continuous time stopping problems. However in a recent case [6], the continuous time problem was used to approximate a discrete time dichotomous problem derived from an application to clinical trials. There, the parameter of concern was the probability of success which could be far from $1/2$. Thus we can use the special $p = 1/2$ problem to correct the numerical approximation to the solution of the continuous time problem. But now we need $\tilde{y}$ of the problem for general $p$ for the approximation $\tilde{z}_\delta = \tilde{z} \pm \tilde{y}\delta^{1/2}$ to relate the solution of the continuous time problem to the dichotomous variable, discrete time, clinical trials problem.
Bibliography


**An Optimal Stopping Problem for Sums of Dichotomous Random Variables**

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**ABSTRACT**

A stopping problem for sums of dichotomous random variables is defined. The optimal procedure is determined and the limiting behavior of this procedure is examined. This limiting behavior can be used to relate the solution of a class of continuous time stopping problems involving a Wiener process to the solution of certain discrete time, discrete process, stopping problems. These relations are useful in calculating numerical approximations to the solutions of various stopping problems.