ON MINIMIZING AN EXPECTATION SUBJECT TO
CERTAIN SIDE-CONDITIONS

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STANLEY ISAACSON AND HERMAN RUBIN

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1. Introduction.

Chernoff and Reiter [1] have recently considered the problem of choosing a cumulative distribution function $F$ so as to minimize

$$\text{(1.1)} \quad E = \int g(x) \, dF(x)$$

subject to the conditions

$$\text{(1.2)} \quad \int x \, dF(x) = c_1,$$

$$\text{(1.3)} \quad \int x^2 \, dF(x) = c_2,$$

where $c_1$ and $c_2$ are given positive constants $c_2 \geq c_1$ and

$$\text{(1.4)} \quad g(x) = 1 - e^{-\beta x}, \quad (\beta > 0).$$

This problem arose as a result of a bioassay investigation. In this paper we consider a related problem resulting from further consideration of the same bioassay problem. We too are interested in minimizing an expectation involving the function $g$ of (1.4) subject to side-conditions; but in our problem we make certain additional assumptions and hence the lower bound on the expectation which we obtain is greater than that obtained by Chernoff and Reiter.
The problem we consider is the following: We have a countably infinite number of points \( u_1, u_2, \ldots \) which are uniformly and independently distributed over a two-dimensional region \( R \) of infinite area; hence the distribution of the number of points occurring in a subregion of \( R \) with unit area is a Poisson random variable with a parameter we shall call \( \lambda \). The distribution of the number of points occurring in a subregion \( R_j \) of finite area \( A_j \) is therefore also Poisson with parameter \( \lambda A_j \). A certain type of material emanates from each of the points \( u_1 \) and spreads over the region \( R \). We consider two different ways by which the material may spread over \( R \):

Case (i): The concentration at an arbitrary point \( x \) in \( R \) of material emanating from a particular point \( u_1 \) is given by \( f(u_1 - x) \) where \( f \) is a non-negative fixed function and \( u_1 - x \) is the ordinary vector difference between \( u_1 \) and \( x \). Thus the total concentration of material at an arbitrary point \( x \) in \( R \) is

\[
T(x) = \sum_{i=1}^{\infty} f(u_i - x) .
\]

(1.5)

Case (ii): The concentration at an arbitrary point \( x \) in \( R \) of material emanating from a particular point \( u_1 \) is given by \( f_i(u_1 - x) \) where \( f_i \) is now a non-negative random function. We assume that the \( f_i \) are identically and independently distributed. Thus the total concentration of material at an arbitrary point \( x \) in \( R \) is

\[
T_i(x) = \sum_{i=1}^{\infty} f_i(u_1 - x) .
\]

(1.6)
The problem we consider is the following: We have a countably infinite number of points \( u_1, u_2, \ldots \) which are uniformly and independently distributed over a two-dimensional region \( R \) of infinite area; hence the distribution of the number of points occurring in a subregion of \( R \) with unit area is a Poisson random variable with a parameter we shall call \( \lambda \). The distribution of the number of points occurring in a subregion \( R_j \) of finite area \( A_j \) is therefore also Poisson with parameter \( \lambda A_j \). A certain type of material emanates from each of the points \( u_i \) and spreads over the region \( R \). We consider two different ways by which the material may spread over \( R \):

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\[
T(x) = \sum_{i=1}^{\infty} f(u_i - x). \tag{1.5}
\]

Case (ii): The concentration at an arbitrary point \( x \) in \( R \) of material emanating from a particular point \( u_i \) is given by \( f_i(u_i - x) \) where \( f_i \) is now a non-negative random function. We assume that the \( f_i \) are identically and independently distributed. Thus the total concentration of material at an arbitrary point \( x \) in \( R \) is

\[
T(x) = \sum_{i=1}^{\infty} f_i(u_i - x). \tag{1.6}
\]
We note that $T$ in (1.5) is a random function since the $u_i$ are random, while $T_r$ in (1.6) is a random function due to the fact that the $u_i$ and $f_i$ are both random. Thus Case (ii) deals with a more general situation than Case (i).

In Case (i) we are interested in choosing $f$ so as to minimize

\begin{equation}
E g(T(x_0))
\end{equation}

for some arbitrary fixed point $x_0$ in $\mathbb{R}$ subject to the conditions that

\begin{equation}
E T(x_0) = \mu (> 0),
\end{equation}

\begin{equation}
\text{Var}(T(x_0)) = \sigma^2,
\end{equation}

where $\mu$ and $\sigma^2$ are given numbers and $g$ is given by (1.4). We solve this problem in section 2 of this paper. It is shown that the minimum value of (1.7) subject to (1.8) and (1.9) is given by

\begin{equation}
E_{\text{min}} = 1 - e^{-\gamma v^2}
\end{equation}

where $v^2 = \frac{\sigma^2}{\mu^2}$ is the squared coefficient of variation and $\gamma = \beta \mu$.

In Case (ii) we are interested in choosing the distribution of the random variables $f_i$ in such a manner as to minimize

\begin{equation}
E g(T_r(x_0))
\end{equation}

subject to the conditions that

\begin{equation}
E T_r(x_0) = \mu (> 0),
\end{equation}
\begin{align}
\text{(1.13)} \quad \mathbb{V} \text{ar}(T_r(x_0)) &= \sigma^2, \\
\end{align}

where \( \mu \) and \( \sigma^2 \) are given numbers and \( g \) is again given by (1.4). This problem is solved in section 3. The answer to Case (ii) is the same as to Case (i).

Chernoff and Reiter [1] obtained

\begin{align}
\text{(1.14)} \quad E_{\min} &= \frac{1 - e^{-\gamma(1 + \nu^2)}}{1 + \nu^2}
\end{align}

as the lower bound of (1.1) subject to (1.2) and (1.3). We show in section 4 that (1.10) for any value of \( \nu^2 \) is never less than (1.14) for the same value of \( \nu^2 \). To compare these lower bounds, the values of \( \gamma \) for which

\begin{align}
\text{(1.15)} \quad 1 - e^{-\gamma} &= .90, .85, .80, \ldots, .05, .01
\end{align}

have been selected. For these values of \( \gamma \) at both (1.10) and (1.14) have been plotted as a function of \( \nu^2 \), for \( 0 \leq \nu^2 \leq 6 \). The graphs are given at the end of this paper.

2. Consideration of Case (i).

In this section we consider the problem of choosing \( f \) so as to minimize (1.7) subject to side-conditions (1.8) and (1.9). Since \( x_0 \) is an arbitrary point in \( \mathbb{R} \), we may take it to be the origin without any loss of generality. Our problem thus is to choose a non-negative function \( f \) so as to minimize

\begin{align}
\text{(2.1)} \quad E g(T(0)) &= E \left( 1 - e^{-\gamma\left( \sum_{i=1}^{\infty} f(u_i) \right)} \right)
\end{align}

subject to
(2.2) \[ E(T(0)) = E\left( \sum_{i=1}^{\infty} f(u_i) \right) = \mu (> 0) , \]

and

(2.3) \[ \text{Var}(T(0)) = \text{Var}\left( \sum_{i=1}^{\infty} f(u_i) \right) = \sigma^2 , \]

where \( \mu \) and \( \sigma^2 \) are given constants.

We will first find the characteristic function of \( T(0) \) and use this to get explicit expressions for (2.1), (2.2), and (2.3). Let \( \varphi(t) \) be the characteristic function of \( T(0) \) and let \( \{ R_j \} \) be a partition of \( \mathbb{R} \) into a denumerable number of mutually exclusive and exhaustive subregions \( R_j \) such that each \( R_j \) has a finite area \( A_j \). Then, using the fact that the \( u_i \) are independently distributed and that the number of points in a subregion \( R_j \) is a Poisson variate with parameter \( \lambda A_j \), we obtain

\[
\varphi(t) = E e^{it T(0)} = E e^{it \sum_{i=1}^{\infty} f(u_i)} = E e^{it \sum_{j=1}^{\infty} \sum_{u_i \in R_j} f(u_i)}
\]

\[
= \prod_{j=1}^{\infty} E e^{it \sum_{u_i \in R_j} f(u_i)} = \prod_{j=1}^{\infty} \sum_{n=0}^{\infty} P_j(n) \left( \frac{\lambda A_j}{n!} \right)^n e^{it \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} P_j(n) \left( \frac{\lambda A_j}{n!} \right)^n}
\]

\[
= \prod_{j=1}^{\infty} \sum_{n=0}^{\infty} \frac{\lambda A_j^n}{n!} (e^{it f(u)})^n = \sum_{n=0}^{\infty} \prod_{j=1}^{\infty} e^{it f(u) - 1}
\]

\[
= \lambda \sum_{j=1}^{\infty} (A_j e^{-A_j}) \int_{R_j} (e^{it f(u)} - 1) du = \lambda \int_{R} (e^{it f(u)} - 1) du ,
\]

where \( P_j(n) = \) the probability of \( n \) of the points \( u_i \) being in the region \( R_j \).
and
\[ E_R e^{it} f(u) \equiv E_j(t) = \frac{1}{A_j} \int_R e^{it} f(u) \, du. \]

Thus we see that
\[ \lambda \int_R (e^{it} f(u) - 1) \, du \]
\[ \varphi(t) = E e^{it} T(0) = e^R. \]

In view of this result, it follows from (2.1) that
\[ -\lambda \int_R (1 - e^{-\beta f(u)}) \, du \]
\[ E g(T(0)) = 1 - \varphi(i\beta) = 1 - e^R. \]

In order to get the mean and variance of \( T(0) \), we take \( \psi(t) = \log \varphi(t) \), and consider the first two derivatives of \( \psi(t) \) at \( t = 0 \). Differentiating under the integral sign, we get
\[ (2.6) \ \ E \ T(0) = \frac{\psi'(t)|_{t=0}}{i} = \frac{(\log \varphi(t))'|_{t=0}}{i} \]
\[ = \frac{\lambda \int_R i f(u) e^{it} f(u) \, du|_{t=0}}{i} = \lambda \int_R f(u) \, du; \]

and
\[ (2.7) \ \ \text{Var}(T(0)) = \frac{\psi''(t)|_{t=0}}{i^2} = \frac{\lambda \int_R i^2 f^2(u) e^{it} f(u) \, du|_{t=0}}{i^2} = \lambda \int_R f^2(u) \, du. \]

Substituting (2.6) and (2.7) into (2.2) and (2.3) respectively, we obtain
(2.8) \[ \int_R f(u) \, du = \frac{\mu}{\lambda}, \]

and

(2.9) \[ \int_R f^2(u) \, du = \frac{\sigma^2}{\lambda}; \]

here \( \mu \) and \( \sigma^2 \) are given constants and \( \lambda \) is an unknown constant. Our problem thus is to choose \( f \) so as to minimize (2.5) subject to side-conditions (2.8) and (2.9). We shall see that we need not known \( \lambda \) in order to solve this problem. Now it is obvious that (2.5) is a minimum when

(2.10) \[ \int_R (1 - e^{-\beta f(u)}) \, du \]

is a minimum. Thus our problem is equivalent to minimizing (2.10) as a function of \( f \) subject to (2.8) and (2.9). Since \( f \) is non-negative, this problem is equivalent to finding a measure \( H \) defined for \( a > 0 \) so as to minimize

(2.11) \[ \int (1 - e^{-\beta a}) dH(a) \]

subject to

(2.12) \[ \int a \, dH(a) = \frac{\mu}{\lambda} \]

(2.13) \[ \int a^2 \, dH(a) = \frac{\sigma^2}{\lambda}. \]

In order to solve this problem, we make use of the Hahn-Banach Theorem (see Munroe [2], pp. 56-57) together with certain unpublished results due to Rubin.
The Hahn-Banach theorem may be stated as follows: If $\mathcal{H}$ is a linear subspace of the linear space $\mathcal{M}$, if $\lambda$ is subadditive functional on $\mathcal{M}$ such that $\lambda(a \varphi) = a \lambda(\varphi)$ for $a \geq 0$, and if $L$ is an additive, homogeneous functional on $\mathcal{H}$ such that $L(\varphi) \leq \lambda(\varphi)$ for every $\varphi \in \mathcal{M}$, then there is an additive, homogeneous extension $\overline{L}$ of $L$ to the whole of $\mathcal{M}$, such that

$$\overline{L}(\varphi) \leq \lambda(\varphi)$$

for every $\varphi \in \mathcal{M}$; furthermore, for some given $\varphi_0 \in \mathcal{M} - \mathcal{H}$, the extension $\overline{L}$ of $L$ may be chosen so to minimize $\overline{L}(\varphi_0)$; this may be done by defining

$$\overline{L}(\varphi_0) = \sup_{\psi \in \mathcal{H}} \left[ -\lambda(\psi - \varphi_0) - L(\psi) \right]. \tag{2.14}$$

To apply the Hahn-Banach theorem to our problem, let $\mathcal{H}$ be the totality of functions of the form $\alpha a + \delta a^2$ where $\alpha$ and $\delta$ are given constants and $a \geq 0$; then let $\mathcal{M}$ consists of all continuous functions which are defined for $a \geq 0$ and which are bounded by elements of $\mathcal{H}$. Thus $\mathcal{M}$ is a linear space and $\mathcal{H}$ is a linear subspace of $\mathcal{M}$. Let $\varphi_0 = 1 - e^{-\beta a}$; we note that $\varphi_0 \in \mathcal{M}$. Now we define

$$L(\alpha a + \delta a^2) = \frac{\alpha \mu}{\lambda} + \frac{\delta \sigma^2}{\lambda}. \tag{2.15}$$

Thus $L$ is defined for all elements of $\mathcal{H}$ and is obviously an additive, homogeneous functional on $\mathcal{H}$; furthermore if $\alpha a + \delta a^2$ is a positive function, then $L$ is positive. Now if we define for all $\varphi \in \mathcal{M}$,

$$\lambda(\varphi) = \inf_{\psi \in \mathcal{H}} \left\{ L(\psi) \right\}, \tag{2.16}$$

$$\inf_{\psi \leq \varphi} \psi \in \mathcal{H},$$
then \( \lambda \) as can be readily verified satisfies the hypothesis of the
Hahn-Banach theorem. Thus an extension \( \overline{L} \) of \( L \) to the whole of \( \mathcal{H} \) such
that \( \overline{L}(\varphi_0) = \overline{L}(1 - e^{-\beta a}) \) is a minimum exists, and by (2.14)

\[
(2.17) \quad \overline{L}(\varphi_0) = \sup_{\psi \in \mathcal{H}} \left[ -\lambda(-\psi - \varphi_0) - L(\psi) \right].
\]

Now, we can simplify (2.17) as follows:

\[
-\lambda(-\psi - \varphi_0) = \inf_{\xi \in \mathcal{H}} L(\xi) \leq \sup_{\xi \in \mathcal{H}} L(\xi) = \sup_{\xi \leq \varphi_0} L(\xi) = \sup_{\xi \leq \varphi_0} L(\psi + \varphi_0) \leq \sup_{\varphi \leq \varphi_0} \{ L(\psi) + L(\varphi) \},
\]

\[
= \sup_{\varphi \leq \varphi_0} L(\psi + \varphi) = \sup_{\varphi \leq \varphi_0} \{ L(\psi) + L(\varphi) \} = L(\psi) + \sup_{\varphi \leq \varphi_0} L(\varphi) = L(\psi) - \lambda(-\varphi_0).
\]

Hence in (2.17) we have

\[
\overline{L}(\varphi_0) = \sup_{\psi \in \mathcal{H}} \{-\lambda(-\varphi_0)\} = -\lambda(-\varphi_0),
\]

so that

\[
(2.18) \quad \overline{L}(\varphi_0) = -\lambda(-\varphi_0) = \sup_{\varphi \leq \varphi_0} L(\varphi) .
\]

Now we have seen that \( L \) is positive on \( \mathcal{H} \) (i.e., \( L \) is positive for any
positive function on \( \mathcal{H} \)); hence in order to calculate \( \overline{L}(\varphi_0) \), we must
take $\mathcal{S} \in \mathcal{N}$ so that $L(\mathcal{S})$ is as large as possible subject to the constraint that $\mathcal{S} \leq \mathcal{S}_0$. But the condition $\mathcal{S} \leq \mathcal{S}_0$ is in our case the conditions

$$
(2.19) \quad \alpha a + \mathcal{S} a^2 \leq 1 - e^{-\beta a}.
$$

We must therefore choose $\alpha$ and $\mathcal{S}$ so that $L(\alpha a + \mathcal{S} a^2)$ is as large as possible subject to the constraint that $\alpha a + \mathcal{S} a^2$ is always less than $1 - e^{-\beta a}$, since $a > 0$, (2.19) is equivalent to

$$
(2.20) \quad \alpha + \mathcal{S} a \leq \frac{1 - e^{-\beta a}}{a}.
$$

But now we are in a position to choose $\alpha$ and $\mathcal{S}$ so that $L(\mathcal{S})$ will be a maximum subject to the constraint (2.20). On the left-hand side we have a linear function of $a$; on the right-hand side, a monotone-decreasing concave function of $a$. Hence we choose $\alpha$ and $\mathcal{S}$ so that the straight line will be tangent to the concave function at some point $a_0$; and then choose $a_0$ so that $L(\mathcal{S})$ will be a maximum. Now if $\alpha + \mathcal{S} a$ is to be tangent to $\frac{1 - e^{-\beta a}}{a}$ at $a_0$, these functions and their first derivations must be equal at $a_0$. Hence we have

$$
(2.21) \quad \alpha + \mathcal{S} a_0 = \frac{1 - e^{-\beta a_0}}{a_0}
$$

and

$$
(2.22) \quad \mathcal{S} = -\frac{1 - e^{-\beta a_0}}{a_0^2} \left(1 + a_0 \beta \right)
$$

consequently

$$
(2.23) \quad \alpha = \frac{1 - e^{-\beta a_0}}{a_0} - \mathcal{S} a_0 = \frac{2 - e^{-\beta a_0}}{a_0} \left(2 + \beta a_0 \right).
$$
Using these values of $\alpha$ and $\delta$ and (2.23) and being the definition of $L$ in (2.15), we obtain

\begin{equation}
\bar{L}(\eta_0) = \bar{L}(1 - e^{-\beta a})
\end{equation}

\begin{equation}
= \left(\frac{2 - e^{-\beta a}}{a_0} \left(1 + \beta a_0\right)\right) \frac{\mu^2}{\lambda} - \frac{\sigma^2}{\lambda} \left(1 - e^{-\beta a_0} \left(1 + a_0\beta\right)\right).
\end{equation}

Now we must choose $a_0$ so to make (2.24) as large as possible. It is easy to verify that (2.24) is maximized by taking

\begin{equation}
a_0 = \frac{\sigma^2}{\mu}.
\end{equation}

For this choice of $a_0$, we obtain from (2.24),

\begin{equation}
\bar{L}(\eta_0) = \frac{\mu^2}{\lambda \sigma^2} \left(1 - e^{-\beta \frac{a_0^2}{\mu}}\right).
\end{equation}

Thus we have used the Hahn-Banach to extend the linear functional given in (2.15) from $\gamma$ to $\gamma$ in such a manner as to obtain the minimum value for $\bar{L}(\eta_0)$ which is given in (2.26).

The question remains whether the extended linear functional constitutes a measure; that it does follows from results of an unpublished paper by Rubin which is applicable to this situation. Furthermore, it follows from Rubin's results that this measure concentrates all its weight on the single point $a_0$.

Thus we have seen that there exists a measure which subject to the side-conditions (2.12) and (2.13) assigns the minimum value

\begin{equation}
\frac{\mu^2}{\lambda \sigma^2} \left(1 - e^{-\beta \frac{a_0^2}{\mu}}\right).
\end{equation}
to (2.11). We are really interested in the minimum of (2.5) subject to (2.8) and (2.9); we now see immediately that this minimum is given by

\[ -\frac{\sigma^2}{\mu^2} (1 - e^{-\beta \frac{\sigma^2}{\mu}}) - \frac{1}{\nu^2} (1 - e^{-\gamma \nu^2}) \]

\[ 1 - e^{-\frac{\sigma^2}{\mu^2}} = 1 - e^{-\nu^2}, \]

(2.28)

where \( \nu^2 \) is the squared coefficient of variation defined by

\[ \nu^2 = \frac{\sigma^2}{\mu^2}, \]

(2.29)

and

\[ \gamma = \frac{\beta}{\mu}. \]

(2.30)

In passing, we note that an argument similar to the one above will show that no measure exists which maximizes (2.5) subject to (2.8) and (2.9); but that a least upper bound for (2.5) subject to (2.8) and (2.9) is given by \( (1 - e^{-\gamma}) \).

3. Consideration of Case (ii).

In this section we consider the problem of choosing a distribution for the independently and identically distributed random functions \( f_i \) in such a manner as to minimize (1.11) subject to (1.12) and (1.13). Much of the argument is the same as given in section 2 for Case (i); we shall only sketch these portions of the argument in this section.

We again take \( x_0 \) as the origin. We then have

\[ T(0) = \sum_{i=1}^{\infty} f_i(u_i). \]

(3.1)
By an argument exactly like the one employed to obtain (2.4), we can show that the characteristic function \( \zeta_T(t) \) of \( T_r(0) \) is given by

\[
\zeta_T(t) = E e^{\lambda \int_{T_r(0)} e^{itf(u)} - 1} du
\]

(3.2)

where by \( E_f \) we mean the expectation taken with respect to the distribution of the random function \( f \). Using (3.2) we obtain,

\[
E(g(T_r(0))) = 1 - \zeta(i\beta) = 1 - e^{-\lambda \int_{T_r(0)} (1 - E_f e^{-\beta f(u)}) du}
\]

(3.3)

\[
\int_{T_r(0)} (E_f f(u)) du = \frac{\mu}{\lambda};
\]

(3.4)

\[
\int_{T_r(0)} (E_f f^2(u)) du = \frac{\sigma^2}{\lambda},
\]

(3.5)

where \( \mu \) and \( \sigma^2 \) are given numbers.

Our problem is to choose the distribution of \( f \) so as to minimize (3.3) subject to the side-conditions (3.4) and (3.5). This is equivalent to minimizing

\[
\int_{T_r(0)} E_f (1 - e^{-\beta f(u)}) du
\]

(3.6)

subject to (3.4) and (3.5) since \( f \) is non-negative, we can solve this problem in two stages. First holding \( u \) fixed, we will choose a distribution function for \( f(u) \) minimizing

\[
E_f (1 - e^{-\beta f(u)})
\]

(3.7)
subject to

(3.8) \[ E_f f(u) = m_1(u) \]

and

(3.9) \[ E_f f^2(u) = m_2(u) \]

where \( m_1 \) and \( m_2 \) are for the present considered as given functions \( (m_2 \geq m_1^2) \). Then, having solved this problem, we will choose \( m_1(u) \) and \( m_2(u) \) so as to minimize the expression obtained by substituting the minimum value of (3.7) into (3.6) subject to the conditions

(3.10) \[ \int_R m_1(u) du = \frac{\mu}{\lambda} \]

and

(3.11) \[ \int_R m_2(u) du = \frac{\sigma^2}{\lambda} \]

The problem of minimizing (3.7) subject to (3.8) and (3.9) is the one solved by Chernoff and Reiter [1]. Their solution is to take the c.d.f. \( \tilde{G}(u) \) of \( f(u) \) to be

(3.12) \[ \tilde{G}(u) = \begin{cases} 
0 \text{ for } f(u) < 0 \\
1 - \frac{m_1^2(u)}{m_2(u)} \text{ for } 0 \leq f(u) < \frac{m_2(u)}{m_1(u)} \\
1 \text{ for } f(u) \geq \frac{m_2(u)}{m_1(u)}
\end{cases} \]

Using the solution given by (3.12), the minimum value of (3.7) is given by
Thus our problem now becomes that of choosing the functions \( m_1 \) and \( m_2 \) (\( m_2 \geq m_1^2 \)) so as to minimize

\[
\int \left( 1 - e^{-\frac{\beta}{m_1(u)}} \right) \frac{m_1^2(u)}{m_2(u)} \, du
\]

subject to (3.10) and (3.11).

Since \( m_2 \geq m_1^2 \geq 0 \), if we define

\[
x = m_1 \\
y = m_2 - m_1^2
\]

then we see that our problem is that of finding a measure \( F(x,y) \) defined for \( x > 0, y > 0 \) so as to minimize

\[
\int \left( 1 - e^{-\beta \left( x + \frac{y}{x} \right)} \right) \frac{x^2}{x^2 + y} \, d F(x,y)
\]

subject to

\[
\int x \, d F(x,y) = \frac{\mu}{\lambda},
\]

and

\[
\int (x^2 + y) \, d F(x,y) = \frac{\sigma^2}{\lambda}.
\]

We can again apply the Hahn–Banach theorem and Rubin's results. \( \mathcal{N} \) will be the totality of functions of the form \( \propto x + \delta(x^2 + y) \) where
\( \alpha \) and \( \delta \) are constants and \( x > 0, y > 0 \); \( \mathcal{N} \) will consist of all continuous functions defined for \( x > 0, y > 0 \) which are bounded by elements of \( \mathcal{N} \).

We let

\[
(3.19) \quad \varphi_0 = \frac{x^2}{x^2 + y} \left( 1 - \beta \frac{x + y}{x} \right).
\]

We then define

\[
(3.20) \quad L(\alpha x + \delta (x^2 + y)) = \frac{\alpha}{\lambda} + \frac{\delta \sigma^2}{\lambda},
\]

and

\[
(3.21) \quad \lambda(\varphi) = \inf_{\psi \in \mathcal{N}} L(\psi).
\]

Then with these definitions the minimum value of \( \bar{L}(\varphi_0) \) subject to

\( (3.17) \) and \( (3.18) \) is again given by

\[
(3.22) \quad \bar{L}(\varphi_0) = \sup_{\varphi \leq \varphi_0} L(\varphi).
\]

The condition \( \varphi \leq \varphi_0 \) is in our case

\[
(3.23) \quad \alpha x + \delta (x^2 + y) \leq \frac{x^2}{x^2 + y} \left( 1 - \beta \frac{x + y}{x} \right),
\]

or, since \( x > 0 \),

\[
(3.24) \quad \alpha + \delta \frac{x + y}{x} \leq \frac{x}{x^2 + y} \left( 1 - \beta \frac{x + y}{x} \right).
\]
Now if we let

\[(3.25) \quad a = x + \frac{y}{x},\]

then \(a > 0\), and (3.24) becomes

\[(3.26) \quad \alpha + \delta a \leq \frac{1 - e^{-\beta a}}{a}.\]

But (3.26) is exactly the same condition as (2.20). Hence \(\alpha\) and \(\delta\) are again given by (2.22) and (2.23), \(a_0\) is given by (2.25), and \(L(\varphi_0)\) by (2.26). Once again Rubin's results are applicable and tell us that the extension of \(L\) which we have obtained actually defines a measure. Thus we see that the minimum of (3.3) subject to (3.4) and (3.5) is given by

\[(3.27) \quad l - e^{-\frac{\sigma^2}{2}(1 - e^{-\beta \frac{\sigma^2}{\mu}})} = l - e^{-\frac{1}{v^2}(1 - e^{-\beta v^2})},\]

which, of course, is the same expression obtained for the lower bound in Case (i). Thus allowing \(f\) to be random does not make matters worse and the lower bound for the expectation remains the same.


Chernoff and Reiter [1] obtained the lower bound

\[(4.1) \quad G(y^2, \gamma) = \frac{1 - e^{-\gamma(1 + v^2)}}{1 + v^2}.\]

We obtain the lower bound,

\[(4.2) \quad H(y^2, \gamma) = 1 - e^{-\frac{1}{v^2}(1 - e^{-\beta v^2})}.\]
In both (4.1) and (4.2), $\gamma > 0$.

In this section we will show that for a fixed value of $\gamma^2$, (4.2) is never less than (4.1).

First we note that

\[(4.3) \quad G(\gamma^2, 0) = H(\gamma^2, 0) = 0.\]

Now let $G_{\gamma}$ and $H_{\gamma}$ denote the partial derivatives of $G$ and $H$ with respect to $\gamma$. Then

\[(4.4) \quad G_{\gamma} = e^{-\gamma(1 + v^2)}\]

and

\[-\gamma v^2 - \frac{1}{v^2} (1 - e^{-\gamma v^2})\]

\[(4.5) \quad H_{\gamma} = e.\]

If for fixed $\gamma^2$, we can show that $H_{\gamma} \geq G_{\gamma}$ for all values of $\gamma$, then we have, of course, completed our proof. But $H_{\gamma} \geq G_{\gamma}$ is equivalent to

\[(4.6) \quad \gamma v^2 + \frac{1}{v^2} (1 - e^{-\gamma v^2}) \leq \gamma (1 + v^2)\]

or

\[(4.7) \quad \frac{1}{v^2} (1 - e^{-\gamma v^2}) \leq \gamma.\]

But for $\gamma > 0$, (4.7) is a well-known inequality. Hence our result is established.
References


- $e^y = .01$

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Minimizing Distribution with Random Distribution of Centers

Minimizing Distribution

For this case, the two curves are indistinguishable.

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Expected Value of $g_2$

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$v^2$: Coefficient of Variation
\[ 1 - e^{v^2} = .05 \]

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Minimizing Distribution

For this case the two curves are indistinguishable.
\[ 1 - e^{-y} = 10 \]

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Minimizing Distribution with Random Distribution of Centers

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Minimizing Distribution

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For this case the two curves are indistinguishable.
\[ I - e^Y = 15 \]

Minimizing Distribution with Random Distribution of Centers

Minimizing Distribution

Expected Value of \( q \)

\( v^2 \): Coefficient of Variation
$1 - e^{-y} \approx 20$

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Minimizing Distribution with Random Distribution of Centers

Minimizing Distribution

Expected Value of $q$

$v^2$: Coefficient of Variation
$1 - e^{-\gamma} = .25$

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--- Minimizing Distribution

Expected Value of $g_1$

$v^2$: Coefficient of Variation
\[ 1 - e^{-x} = 0.30 \]

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Expected Value of \( g_i \)

\( v^2 \): Coefficient of Variation
\[ 1 - e^{-Y} = 35 \]

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- Minimizing Distribution

Expected Value of \( \theta \)

\[ v^2: \text{Coefficient of Variation} \]
\[ f(x) = e^{-x^2/4} \]

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--- Minimizing Distribution

**Expected Value of g:**

\[ v^2: \text{Coefficient of Variation} \]
\(-e^{-\frac{1}{2}} = .50\)

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Minimizing Distribution

Expected Value of \(q\)

\(v^2: \text{Coefficient of Variation}\)
1.8^Y = 0.60

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Minimizing Distribution

Expected Value of g₁

v²: Coefficient of Variation
$e^{-\gamma} = 0.86$

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Minimizing Distribution with Random Distribution of Centers
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Minimizing Distribution

Expected Value of $g$

$v^2$: Coefficient of Variation
Minimizing Distribution with Random Distribution of Centers

Minimizing Distribution

Expected Value of $g$

$v^2$: Coefficient of Variation
$1 - e^{-\gamma} = \delta$

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Minimizing Distribution with Random Distribution of Centers

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Minimizing Distribution
\[ t - e^{-r/\sigma} = .85 \]

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Minimizing Distribution with Random Distribution of Centers

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Minimizing Distribution

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Expected Value of \( q \)

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\( v^2: \text{Coefficient of Variation} \)
$1 - e^{-v} = .9$

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Minimizing Distribution with Random Distribution of Centers

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Minimizing Distribution

**Expected Value of $g_1$**

$v^2$: Coefficient of Variation