THE EXTINCTION PROBABILITY IN A CRITICAL BRANCHING PROCESS

BY

HOWARD J. WEINER

TECHNICAL REPORT NO. 258
MAY 30, 1978

PREPARED UNDER CONTRACT
N00014-76-C-0475 (NR-042-267)
FOR THE OFFICE OF NAVAL RESEARCH

Reproduction in Whole or in Part is Permitted
for any purpose of the United States Government

Approved for public release; distribution unlimited.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
The Extinction Probability in a Critical Branching Process

By

Howard J. Weiner

TECHNICAL REPORT NO. 258

May 30, 1978

Prepared under Contract
N00014-76-C-0475 (NR-042-267)
For the Office of Naval Research

Herbert Solomon, Project Director

Reproduction in Whole or in Part is Permitted
for any Purpose of the United States Government

Approved for public release; distribution unlimited.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

Partially supported under U.S. Army Research Office Grant DAAG-29-77-G-0031
and issued as Technical Report No. 18.
The Extinction Probability in a
Critical Branching Process

by Howard J. Weiner*

University of California at Davis
and Stanford University

I. Introduction.

Let $Z(t)$ denote the number of cells alive at $t$ in a critical age-
dependent Bellman-Harris branching process with cell lifetime distribution
$G(t)$, $G(0^+) = 0$, non-lattice and assume

\[(1.1) \quad t^2(1-G(t)) \to 0 \text{ as } t \to \infty.\]

\[(1.2) \quad 0 < \mu = \int_0^\infty t dG(t) < \infty.\]

Let the offspring generating function be denoted, for $0 \leq s \leq 1$,

\[(1.3) \quad h(s) = \sum_{k=0}^\infty p_k s^k\]

and

\[(1.4) \quad h'(1) = 1 = \sum_{k=1}^\infty kp_k \quad (\text{criticality})\]

and

\[(1.5) \quad 0 < \sigma^2 = h''(1) = \sum_{k=2}^\infty k(k-1)p_k < \infty.\]

See [1] Chapter 4 for details.

*Partial Support NONR at Stanford University
It is well-known that

\begin{equation}
\lim_{t \to \infty} t P[Z(t) > 0] = \frac{2\mu}{\sigma^2} = b.
\end{equation}

Various proofs of (1.6) and the corresponding result for a critical Galton-Watson or discrete time process have appeared. See [1] Chapters 1, 4 for comments and references. For example, the proof of (1.6) in [2] uses the renewal theorem. The proof in [1], Chapter 4, gives the asymptotic form for a critical generating function, from which (1.6) is a special case, and relates the generating function to that of a critical Galton-Watson process, for which results are obtained. The proof given in this note is elementary and self-contained.

II. Comparisons and Iterations

**Definition.** For \(0 \leq s \leq 1, \ t \geq 0,\)

\begin{equation}
F(s,t) = \sum_{k=0}^{\infty} P[Z(t)=k]s^k.
\end{equation}

It is well-known ([1], Chapter 4), and follows by the law of total probability that

\begin{equation}
F(s,t) = s(1-G(t)) + \int_0^t h(F(s,t-u))dG(u).
\end{equation}

Using the fact that

\begin{equation}
P(t) = P[Z(t) > 0] = 1-F(0,t),
\end{equation}

then from (2.2) it follows that, by a Taylor expansion on \(h(s)\) about \(s=1,\)
\begin{equation}
1 - P(t) = \int_0^t h(1 - P(t-u)) dG(u)
\end{equation}

and

\begin{equation}
1 - P(t) = \int_0^t \left[ 1 - P(t-u) + \frac{\sigma^2}{2} P^2(t-u) + o(P^2(t-u)) \right] dG(u)
\end{equation}

Rewriting (2.5),

\begin{equation}
P(t) = 1 - G(t) + \int_0^t P(t-u) dG(u) - \frac{\sigma^2}{2} \int_0^t P^2(t-u) dG(u) + f(t)
\end{equation}

where \( f(t) \) denotes the remainder.

**Lemma 1 (1)** \( P(t) \downarrow 0 \).

**Proof.** A simpler and elementary proof will be given here. Clearly

\begin{equation}
P(t) \downarrow C \geq 0.
\end{equation}

Assume \( C > 0 \). Split the integral on the right side of (2.5) into \( \int_0^t = \int_0^{t/2} + \int_{t/2}^t \). For large \( t \), the integral \( \int_0^{t/2} \) is \( o(t^{-2}) \). Then it is clear that

\begin{equation}
1 - C = 1 - C + \frac{\sigma^2}{2} C + o(1),
\end{equation}

which is a contradiction of \( C > 0 \). This proves the lemma.

\begin{equation}
\text{Let } G^{(n)}(t) = \text{ }^n \text{ iterate of } G \text{ evaluated at } t.
\end{equation}

Define the iterative sequences

\begin{equation}
U_{n+1}(t) = \int_0^t h(U_n(t-u)) dG(u)
\end{equation}

\( U_0(t) = 1. \)
(2.11) \[ I_{n+1}(t) = \int_0^t h(I_n(t-u))dG(u) \]

\[ I_o(t) = \begin{cases} 1, & t \leq T \\ 1 - \frac{b}{t}, & t > T \end{cases} \]

for $T \geq b$.

**Lemma 2** Under assumptions (1.1) - (1.5), for $n \geq 0$, and all $t > T$,

(2.12) \[ 0 \leq U_n(t) - F(0,t) \leq c^{(n)}(t) \]

(2.13) \[ 0 \leq U_n(t) - I_n(t) \leq c^{(n)}(t) \]

(2.14) \[ |I_n(t) - I_o(t)| \leq k(t), \]

where, for $t \to \infty$,

(2.15) \[ tk(t) \to 0. \]

**Proof.** Eq. (212) holds for $n = 0$.

Assume (2.12) for $n$. Then, omitting arguments,

(2.16) \[ 0 \leq U_{n+1} - F = \int_0^t h(U_n) - h(F)dG \leq \int_0^t (U_n - F)dG \]

\[ \leq \int_0^t c^{(n)}(t-u)dG = c^{(n+1)}(t) \]

where the left inequality follows from the induction hypothesis and the monotonicity of $h$, and the right inequalities from the mean value theorem, the fact that $h(1) = 1$, and the induction hypothesis.

To show (2.13), observe that for $t > T$

(2.17) \[ 0 \leq U_o - I_o \leq \frac{b}{t} < 1 = c^{(o)}(t). \]
Assume (2.13) for \( n \) by induction. Then, arguing as in (2.16),

\[
(2.18) \quad 0 \leq U_{n+1} - I_{n+1} = \int_0^t (h(U_n) - h(I_n)) dG \leq \int_0^t (U_n - I_n) dG \\
\leq \int_0^t G^{(n)}(t-u) dG(u) = G^{(n+1)}(t).
\]

To show (2.14) write \( I_1(t) \) as

\[
(2.19) \quad I_1(t) = \int_0^{t/2} + \int_{t/2}^t h(I_0(t-u)) dG(u).
\]

Note that the second integral \( \int_{t/2}^t \) in (2.19) is \( o(t^{-2}) \) by (1.1). The first integral \( \int_0^{t/2} \) may be written by a Taylor expansion of \( h(s) \) about \( s = 1 \) as \( t \gg T \)

\[
(2.20) \quad I_1(t) = o(t^{-2}) + \int_0^{t/2} h(1)\left(1 - \frac{b}{t-u}\right) dG(u) \\
= o(t^{-2}) + \int_0^{t/2} \left\{ h(1) - \frac{b}{t-u} h'(1) + \frac{1}{2} \left(\frac{b}{t-u}\right)^2 h''(1) + o(t^{-2}) \right\} dG(u)
\]

Using \( h(1) = h'(1) = 1 \), and the expansion, \( 0 < u < t/2 \),

\[
(2.21) \quad \frac{1}{t-u} = \frac{1}{t} \left(1 + \frac{u}{t} + o\left(\frac{u}{t}\right)\right)
\]

in the right side of (2.20) yields

\[
(2.22) \quad I_1(t) = o(t^{-2}) + G(t) - \frac{b}{t} \int_0^{t/2} \left(1 + \frac{u}{t} + o\left(\frac{u}{t}\right)\right) dG(u) \\
+ \frac{b^2 g^2}{2t^2} \int_0^{t/2} \left(1 + o\left(\frac{u}{t}\right)\right) dG(u)
\]

as again by (1.1), an integration by parts yields

\[
(2.23) \quad \int_{t}^{\infty} u dG(u) = o(t^{-1}).
\]
It follows that

\[(2.24) \quad I_1(t) = I_0(t) + f(t)\]

where

\[(2.25) \quad 0 \leq |f(t)| \leq K < \infty\]

and as \( t \to \infty \),

\[(2.26) \quad t^2 f(t) \to 0.\]

Then one may write

\[(2.27) \quad I_2(t) = \int_0^t h(I_0(t-u) + f(t-u)) dG(u)\]

\[\quad \geq \int_0^t \{h(I_0(t-u)) + h'(I_0(t-u)) f(t-u)\} dG(u).\]

Note that for \( t \gg T \),

\[(2.28) \quad I_1(T) = \int_0^T h(I_0(T-u)) dG(u)\]

and

\[(2.29) \quad h(1-\varepsilon)G(T) \leq I_1(T) \leq h(1 - \frac{a}{T})G(T),\]

for some \( a > 0, \varepsilon > 0, \)

\[(2.30) \quad I_2(t) \leq \int_0^t \{h(I_0(t-u)) + h'(\max\{h(1 - \frac{a}{T})G(T), 1 - \frac{\gamma}{T}\}) f(t-u)\} dG(u)\]

for some \( a > 0, \gamma > 0.\)
For all $t \gg T$, (2.27) - (2.30) yield that

\[(2.31) \quad |I_2(t) - I_1(t)| \leq (1 - \frac{\alpha}{t})^2 \int_0^t f(t-u)dG(u).\]

Hence, repeating the argument of (2.27) - (2.31),

\[(2.32) \quad |I_3(t) - I_2(t)| \leq (1 - \frac{\alpha}{t})^2 \int_0^t f(t-u)dG^{(2)}(u),\]

where

\[(2.33) \quad 0 < \alpha \text{ is a constant}.\]

An induction yields that

\[(2.34) \quad |I_n(t) - I_0(t)| \leq \sum_{\ell=0}^\infty (1 - \frac{\alpha}{t})^\ell \int f \ast G^{(\ell)}(t) = k(t),\]

where "\ast" denotes the usual convolution integral.

The right side of (2.34) is now broken up into a number of parts, and upper bounds for each part is obtained.

\[(2.35) \quad k(t) = \sum_{\ell=0}^{t^2} (1 - \frac{\alpha}{t})^\ell \int f \ast G^{(\ell)}(t).\]

Since

\[(2.36) \quad |f| \ast G^{(\ell)}(t) \leq K\]

the second term of (2.35) is dominated by

\[(2.37) \quad K \sum_{\ell=t^2}^\infty (1 - \frac{\alpha}{t})^\ell \leq \frac{Kte^{-\alpha t}}{\alpha}.\]
The first term on the right of (2.35) is now written as

\[ t^2 \sum_{\ell=0} \left( 1 - \frac{\alpha}{t} \right)^{\ell} f(x)_{\ell}(t) \]

\[ = \frac{t^2}{\alpha} \left( \sum_{\ell=0} \left( 1 - \frac{\alpha}{t} \right)^{\ell} \right) \left[ \int_0^{t-t/\ln t} f(t-u)dG(u) + \int_{t-t/\ln t}^t f(t-u)dG(u) \right]. \]

The first term on the right side of (2.38) is bounded above by

\[ \frac{t}{\alpha} \left( \frac{(\ln t)^2}{t^2} \right) = o(1/t), \]

by the asymptotic behavior of \( f \) and summing the geometric series.

The second term on the right side of (2.38) is split into the three terms, ignoring distinctions between \([\ell]\) and \(\ell\),

\[ \sum_{\ell=0}^{\sqrt{t/\ln t}} t \ln t + \sum_{\ell=\sqrt{t/\ln t}}^t \frac{t^2}{\ell \ln t} \int_{\ell-t/\ln t}^t f(t-u)dG(u). \]

Since \(|f| \leq 1\), Chebyshev's inequality yields

\[ \int_{t-t/\ln t}^t |f(t-u)|dG(u) \leq \frac{a\ell}{(t-t/\ln t - \ell\mu)^2}, \]

where

\[ 0 < a^2 = \int_0^{\infty} (t-\mu)^2 dG(u). \]

An application of (2.41) to the first term of (2.40) yields

\[ \sum_{\ell=0}^{\sqrt{t/\ln t}} \int_{t-t/\ln t}^t f(t-u)dG(u) \leq \frac{\beta}{t(\ln t)^2} \]

for some \( \beta > 0 \).
The Central Limit theorem may be applied to the second and third terms of (2.40) since the summation index \( \ell \) is large.

For \( \ell \geq \sqrt{t/\ln t} \),

\[
G^{(\ell)}(t) - G^{(\ell)}(t-t/\ln t) \sim \frac{x}{a \sqrt{\ell}} - \frac{x}{a \sqrt{\ell}}. 
\]

(2.44)

Applying (2.44) to the second term of (2.40) and using the mean value theorem and

\[
\varphi(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} 
\]

yields

\[
(2.45)
\]

\[
\int_{t-t/\ln t}^{t} f(t-u) dG^{(\ell)}(u) 
\]

\[
\leq \sum_{\ell = \sqrt{t/\ln t}}^{t \ln t} \varphi \left( \frac{t - \ell \mu}{a \sqrt{\ell}} \right) \left( \frac{t}{a (\ell \ln t) \ell} \right) 
\]

\[
\leq C e^{-t/2} \ln t \frac{t^{3/2}}{a (\ln t)} 
\]

where \( C \) is a positive constant.

The Central Limit theorem is applied to the third term of (2.40), and since the arguments are large, use will be made of the standard approximation

\[
1 - \varphi(x) \sim \frac{\beta}{x} \varphi(x) 
\]

(2.47)

for some \( \beta > 0 \), as \( x \to \infty \).
Applying (2.44) and (2.47) to the third term of (2.40) yields

\[
(2.48) \quad \left| \frac{t^2}{\sigma \ln t} \int_{t-\mu \ln t}^{t} f(t-u) dG(u) \right| \\
\leq \beta \frac{t^2}{\sigma \ln t} \varphi(t-\mu) \left[ \frac{a}{t-t/\ln t} - \frac{a \sqrt{t}}{t-\mu} \right] \\
\leq \frac{\beta t^2 \varphi(-\mu \sqrt{t/\ln t})}{a t^{1/2}} \left( a t/\ln t \right) \leq \frac{ye^{-\delta t \ln t}}{\ln t}.
\]

Now (2.37), (2.39), (2.43), (2.46), (2.48) applied to (2.34) yield that for all sufficiently large \( t \), equation (2.14) holds.

**Theorem 1.** Under assumptions (1.1) to (1.5)

\[
(1.6) \quad \lim_{t \to \infty} tP[Z(t) > 0] = b.
\]

**Proof.** Combine (2.12) - (2.15) to yield

\[
(2.49) \quad |P[Z(t) > 0] - \frac{b}{t}| \leq G^{(n)}(t) + o(t^{-1}).
\]

Let \( n \to \infty \). The weak law of large numbers yields the result.

**III. Extension.** The result of Theorem 1 can be strengthened by the method.

**Theorem 2.** Under the assumptions (1.2) - (1.5) and in addition, for \( t \to \infty \),

\[
(3.1) \quad t^3(1-G(t)) \to 0
\]
and

\[ h^{(3)}(1) < \infty, \]

then for \( t \) large,

\[ p[Z(t) > 0] \sim \frac{b}{t} + \frac{c \ln t}{t^2} \]

for some unspecified constant \( c \).

**Remark.** Conditions (3.1) and (3.2) are not as stringent as in [2], where more terms in the asymptotic expansion of \( P(t) \) are given.

**Outline of Proof.**

Define

\[ J_{n+1}(t) = \int_{0}^{t} h(J_n(t-u))dG(u) \]

\[ J_0(t) = \begin{cases} 1, & t \leq T \\ 1 - \frac{b}{t} - \frac{c \ln t}{t^2}, & t > T \end{cases} \]

for \( T >> b \).

As in the proof of (2.13) one obtains

\[ 0 \leq U_n(t) - J_n(t) \leq G^{(n)}(t). \]

An expansion of \( h \) about 1 to four terms in the Taylor expansion yields that

\[ J_1(t) = J_0(t) + o(t^{-3}). \]
From this and a similar tedious sequence of estimations as in Theorem 1, one obtains

\[ |J_n(t) - J_0(t)| \leq C^{(n)}(t) + o(t^{-2}) \]

and again one lets \( n \to \infty \) and applies the weak law of large numbers to complete the argument.
REFERENCES


REPORT DOCUMENTATION PAGE

1. REPORT NUMBER  258

4. TITLE (and Subtitle)
The Extinction Probability in a Critical Branching Process

7. AUTHOR(s)
Howard J. Weiner

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Department of Statistics
Stanford University
Stanford, CA 94305

11. CONTROLLING OFFICE NAME AND ADDRESS
Office of Naval Research
Statistics and Probability Program Code 436
Arlington, VA 22217

16. DISTRIBUTION STATEMENT (of this Report)
Approved for Public Release; Distribution is Unlimited.

18. SUPPLEMENTARY NOTES
This report partially supported under U.S. Army Research Office Grant DAAG29-77-G-0031 and issued as Technical Report No. 18.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Extinction Probability, critical branching process, integral equations
Iterations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
Please see reverse side.
Let $Z(t)$ be the number of cells alive at $t$ in a critical age-dependent Bellman-Harris process with lifetime distribution $G(t)$ and offspring generating function $h(s)$. A simplified proof that $tP(Z(t) > 0) \rightarrow b > 0$, a specified constant, is given.