LIMIT PROBABILITIES FOR CRITICAL AGE-DEPENDENT BRANCHING PROCESSES WITH IMMIGRATION

BY

HOWARD J. WEINER

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Limit Probabilities for Critical Age-Dependent Branching Processes with Immigration

by

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1. Introduction.

(1.1) Let \( Z(t) \) denote the number of cells alive at time \( t \) in a standard critical age-dependent branching process ([1], Chapter 4) with absolutely continuous cell lifetime distribution function

\[
G(t), \quad G(0^+) = 0
\]

and satisfying

\[
0 < \mu = \int_0^\infty t dG(t).
\]

Let

\[
g(t) = G'(t)
\]

be the density of \( G \). Assume

\[
\int_0^\infty t^{b+4} g(t) \, dt < \infty
\]

with \( b \) given by (1.15).
At the end of each cell life, the original cell disappears, and is replaced by \( k \) new cells with probability \( p_k \geq 0 \) and

\[
(1.6) \quad \sum_{k=0}^{\infty} p_k = 1 ,
\]

satisfying criticality

\[
(1.7) \quad \sum_{k=1}^{\infty} k p_k = 1 .
\]

Let, for \( 0 \leq s \leq 1 \)

\[
(1.8) \quad h(s) = \sum_{k=0}^{\infty} p_k s^k
\]

and assume that, for some \( \varepsilon > 0 \),

\[
(1.9) \quad h(l+\varepsilon) \text{ exists}.
\]

This guarantees, in particular, that for \( n \geq 1 \),

\[
(1.10) \quad \sum_{k=1}^{\infty} k^n p_k \text{ exists}
\]

and that all derivatives of \( h(s) \) for \( 0 \leq s \leq 1 \) exist at \( s = 1 \) and can be evaluated by interchanging derivatives and summation.

Assume in addition that

\[
(1.11) \quad 0 < h''(1) .
\]
(1.12) Let \( N(t) \) denote the total progeny born by time \( t \) in a critical age-dependent process satisfying (1.1)-(1.11).

(1.13) Let \( Z_0(t) \) denote the number of cells alive at \( t \) in a cell immigration process in which new-born cells are introduced at renewal epochs. The (random) time between epochs is governed by a continuous distribution function \( G_0(t), G_0(0+) = 0 \)

with

\[
(1.14) \quad 0 < \mu_0 \equiv \int_0^\infty t dG_0(t)
\]

and for

\[
(1.15) \quad b \equiv \frac{2\mu m_0}{\mu_0 h^n(1)} \quad \text{(with } m_0 \text{ defined below)}
\]

that, as \( t \to \infty, \)

\[
(1.16) \quad t^{b+2}(1-G_0(t)) \to 0.
\]

At each renewal epoch, \( k \) new cells are introduced with probability \( p_{0k} \) and let, for \( 0 \leq s \leq 1+\varepsilon \) for some \( \varepsilon > 0 \)

\[
(1.17) \quad h_0(s) \equiv \sum_{k=0}^{\infty} p_{0k}s^k < \infty
\]

and

\[
(1.18) \quad 0 < m_0 = h'_0(1)
\]

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and

\[ h_0''(l) < \infty, \quad h''(l) < \infty. \]

Each new cell introduced at a renewal epoch now is part of the process and initiates, independent of all other cells and the immigration process, a critical age-dependent branching process satisfying (1.1)-(1.11).

(1.19) Let \( N_0(t) \) denote the total progeny by time \( t \) of the immigration process satisfying (1.1)-(1.18).

It is the purpose of this paper to show that for \( k \geq 1 \), as \( t \to \infty \),

(1.20) \[ P_{0k}(t) = P[Z_0(t) = k] \sim \frac{c}{t^b} \]

where

\[ b = \frac{2\mu m_0}{\mu h''(l)} \]

and

where \( c > 0 \) denotes a constant which may depend on \( k \) and under the additional hypotheses that

(1.21) \[ p_{0k} > 0 \quad \text{all} \quad k \geq 0, \]

and that there is a unique \( \alpha > 0 \) defined by

(1.22) \[ P_{00} \int_0^\infty e^{\alpha y} dG_0(y) = 1 \]

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that, as $t \to \infty$, for $k \geq 0$,

$$Q_{0k}(t) = P[N_0(t)=k] \sim ce^{-\alpha t}$$

for $c$ (depending on $k$) some positive constant. A multi-dimensional version and extension are indicated in Section 3.

2. **Integral Equations.**

For reference later, some results about $Z(t)$ are listed. See [1], Chapter 4, for example.

Let, for $0 \leq s \leq 1$

$$E(sZ(t)) = F(s,t).$$

Then, by notation (1.1)-(1.11)

$$F(s,t) = s(1-G(t)) + \int_0^t h(F(s,t-u))dG(u).$$

Under the hypotheses (1.1)-(1.11), denoting

$$P_k(t) = P[Z(t)=k],$$

then [3]

$$P_1(t) = 1 - G(t) + \int_0^t h'(1-P(t-u))P_1(t-u)dG(u)$$

and in general, for $k \geq 2$,

$$P_k(t) = f_k(t) + \int_0^t h'(1-P(t-u))P_k(t-u)dG(u),$$
where

\begin{equation}
(2.6) \quad P(t) = P[Z(t) > 0].
\end{equation}

By [1], [3] respectively,

\begin{equation}
(2.7) \quad P(t) \sim (2\mu)(h''(1)t)^{-1}
\end{equation}

and for \( k \geq 1 \),

\begin{equation}
(2.8) \quad P_k(t) \sim \frac{c_k}{t^k},
\end{equation}

where \( c_k > 0 \) is a constant, possibly depending on \( k \).

Denote, for \( 0 \leq s \leq 1 \),

\begin{equation}
(2.9) \quad F(s,t) = \mathbb{E}s^{Z(t)} = \sum_{k=0}^{\infty} P[Z(t)=k]s^k.
\end{equation}

\begin{equation}
(2.10) \quad F_0(s,t) = \mathbb{E}s^{Z_0(t)} = \sum_{k=0}^{\infty} P[Z_0(t)=k]s^k.
\end{equation}

\begin{equation}
(2.11) \quad H(s,t) = \mathbb{E}s^{N(t)} = \sum_{k=1}^{\infty} P[N(t)=k]s^k.
\end{equation}

\begin{equation}
(2.12) \quad H_0(s,t) = \mathbb{E}s^{N_0(t)} = \sum_{k=0}^{\infty} P[N_0(t)=k]s^k.
\end{equation}

Then the following theorem holds.

**Theorem 1.** Assume (1.1)-(1.18) hold. Then for \( k \geq 0 \), as \( t \to \infty \),

\begin{equation}
(2.13) \quad P[Z_0(t)=k] \sim \frac{c}{t^b}.
\end{equation}
where \( c > 0 \) depends on \( k \).

**Proof.** By [2]

\[
F_0(s,t) = 1 - G_0(t) + \int_0^t h_0(F(s,t-u))F_0(s,t-u)dG_0(u).
\]

For \( \ell \geq 0 \) an integer, denote by

\[
P_{0\ell}(t) \equiv P[Z_0(t) = \ell]
\]

(2.15) \( P_{0\ell}(t) \equiv P[Z_0(t) = \ell] \)

(2.16) \( P_{\ell}(t) \equiv P[z(t) = \ell] \)

and

\[
P(t) = P[Z(t) > 0].
\]

(2.17) \( P(t) = P[Z(t) > 0] \).

From the assumptions we note that

\[
\frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} F_0(s,t) \bigg|_{s=0} = P_{0\ell}(t)
\]

(2.18) \( \frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} F_0(s,t) \bigg|_{s=0} = P_{0\ell}(t) \)

and

\[
\frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} F(s,t) \bigg|_{s=0} = P_{\ell}(t).
\]

(2.19) \( \frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} F(s,t) \bigg|_{s=0} = P_{\ell}(t) \).

By (2.18) applied to (2.14) for \( \ell = 0 \)

\[
P_{00}(t) = 1 - G_0(t) + \int_0^t h_0(1-F(t-u))P_{00}(t-u)dG_0(u).
\]

(2.20) \( P_{00}(t) = 1 - G_0(t) + \int_0^t h_0(1-F(t-u))P_{00}(t-u)dG_0(u) \).

Define
(2.21) \[ R(t) = 1 - G_0(t) + \frac{1}{\mu_0} \int_0^t h_0(1-P(t-u))R(t-u)e^{-\frac{(t-u)}{\mu_0}} du \]

or equivalently,

\[ R(t) = 1 - G_0(t) + \frac{e}{\mu_0} \int_0^t h_0(1-P(u))R(u)e^{-\frac{u}{\mu_0}} du \]

Taking the derivative w.r.t. \( t \) in (2.21) and simplifying leads to the differential equation

(2.22) \[ R'(t) + \frac{\left(1-h_0(1-P(t))\right)}{\mu_0} R(t) = f(t) \]

where

(2.23) \[ f(t) = o(t^{-b-2}) \]

Expanding \( 1-h_0(1-P(t)) \) in a Taylor series, using (2.7) and the idea of the proof of Claim IV of ([3] pp 480-481), one may solve for \( R(t) \) asymptotically to get

(2.24) \[ R(t) \sim ct^{-b}, \text{ where } c > 0 \]

is a constant whose value may change from equation to equation. From (2.20), (2.21),

(2.25) \[ P_{00}(t) - R(t) = \int_0^t h_0(1-P(t-u))(P_{00}(t-u) - R(t-u))dG_0(u) \]

\[ + \int_0^t h_0(1-P(t-u))R(t-u)(dG_0(u) - dE(u)) \]
where

\[(2.26) \quad E(t) = 1 - e^{-\mu_0 t}.\]

Define

\[(2.27) \quad \Delta(t) = |P_{00}(t) - R(t)|.\]

Then, iterating (2.25) repeatedly, one obtains

\[(2.28) \quad \Delta(t) \leq \Delta \cdot G_{0n}(t) + R \cdot |G-E| \cdot U_0(t)\]

for all \(n,t\), and the dots denote convolution integral, where \(G_{0n}(t)\) is the \(n^{th}\) convolution of \(G_0\) with itself, and

\[U_0(t) = \sum_{\ell=0}^{\infty} G_{\ell}(t) \sim \frac{t}{\mu_0}.\]

Let \(n \to \infty\), then \(t \to \infty\), and the law of large numbers and the properties of \(R, G, E, U_0\) yield that

\[(2.29) \quad t^{\frac{b}{\Delta(t)}} \to 0 \text{ as } t \to \infty.\]

This yields the result of Theorem 1 for \(P_{00}(t)\).

The argument for \(P_{01}(t)\) is similar and uses the result for \(P_{00}(t)\).

The general result for \(P_{0n}(t)\) follows by induction using Leibniz' rule for successive differentiation, and is omitted.

Remark: The proof of Theorem 1 of [3] on pp. 482-483 is incompletely justified and would go through by an argument as above.
Theorem 2. Assume (1.1)-(1.22) to hold. Then, for \( k \geq 0 \) an integer

\[
Q_{0n}(t) = P[N_0(t)=k] \sim c e^{-\alpha t}
\]

for some \( c > 0 \) depending on \( k \), where \( \alpha \) is as given in (1.22).

Proof. By arguments similar to those used to establish (2.14) by the law of total probability,

\[
H_0(s,t) = 1 - G_0(t) + \int_0^t H_0(H(s,t-u))H_0(s,t-u) dG_0(u).
\]

The assumptions of the theorem allow derivatives with respect to \( s \) to be taken under the summation sign in (2.11)-(2.12) and that for \( \ell \geq 0 \),

\[
\frac{1}{\ell!} \left. \frac{\partial^\ell}{\partial s^\ell} H(s,t) \right|_{s=0} = P[N(t)=\ell] = Q_\ell(t)
\]

and

\[
\frac{1}{\ell!} \left. \frac{\partial^\ell}{\partial s^\ell} H_0(s,t) \right|_{s=0} = P[N_0(t)=\ell] = Q_\ell(t),
\]

and note that

\[
Q_0(t) = P[N(t)=0] = 0.
\]

Applying (2.32)-(2.34) to (2.31) for \( \ell = 0 \) yields

\[
Q_{00}(t) = 1 - G_0(t) + P_{00} \int_0^t Q_{00}(t-u) dG_0(u).
\]
But (2.35) is in the standard form of the integral equation for the mean number of cells at time \( t \) in a Bellman-Harris age-dependent branching process with cell lifetime distribution function \( q_0(t) \) and mean number of progeny per parent of \( 0 < p_{00} < 1 \), the subcritical case. (See [1] pp 162-168). Hence [1] as \( t \to \infty \),

\[
q_{00}(t) \sim ce^{-\alpha t},
\]

where \( c > 0 \) may be explicitly evaluated [1], but since no general tractable expression for corresponding constants in the asymptotic form for \( q_{0\ell}(t) \) seems obtainable, such constants will not be evaluated explicitly, although this proof indicates how they may be obtained recursively.

Applying (2.32)-(2.34) to (2.31) for \( \ell = 1 \) yields

\[
q_{01}(t) = p_{01} \int_0^t q_1(t-u)q_{00}(t-u)dg_0(u) + p_{00} \int_0^t q_{01}(t-u)dg_0(u),
\]

which can be expressed in the form

\[
q_{01}(t) = f(t) + p_{00} \int_0^t q_{01}(t-u)dg_0(u),
\]

where, from [1] and (2.37), it follows that, as \( t \to \infty \),

\[
f(t) \sim ce^{-\alpha t}.
\]

By Theorem 1 (i) of ([1] p. 145) and the argument of equation (9)-(11) on page 146 of [1], one then obtains
\[ Q_{01}(t) \sim c e^{-\lambda t}, \]

for a \( c > 0 \) which may be evaluated, as indicated in the remark following (2.37).

The rest of the argument proceeds by induction analogous to that used in Theorem 1.

3. **Multidimensional Case.**

Let

\[ Z_{ij}(t) = \text{the number of cells of type } j \text{ at time } t \]

starting with one new-born cell of type \( i \) at \( t = 0 \) with \( 1 \leq i \leq m \) in an \( m \)-type critical age-dependent branching process described as follows. At time \( t = 0 \), one newly born cell of type \( i \) starts the process, for some \( 1 \leq i \leq m \). The cell lives a random time described by a continuous distribution function

\[ G_i(t), \quad G_i(0^+) = 0. \]

At the end of its life, cell \( i \) is replaced by \( j_1 \) new daughter cells of type \( 1 \), \( j_2 \) new cells of type \( 2, \ldots, j_m \) cells of type \( m \) with probability \( \mathcal{P}_{ij_1j_2j_3\cdots j_m} \).

Define the generating functions, for \( s = (s_1, \ldots, s_m), \quad j = (j_1, \ldots, j_m), \)

\[ s^j = \left(j_1, \ldots, j_m\right). \]

\[ h_i(s_1, \ldots, s_m) = h_i(s) = \sum_{(j_1, \ldots, j_m)} \mathcal{P}_{ij_1j_2j_3\cdots j_m} j_1^{s_1} \cdots j_m^{s_m} = \sum_i p_i s^i. \]
Each daughter cell proceeds independently of the state of the system, with each cell type \( j \) governed by \( G_j(t) \) and \( h_j(s) \).

Assume, for \( \underline{1} + \varepsilon = (1+\varepsilon, \ldots, 1+\varepsilon) \) and \( \underline{1} = (1, \ldots, 1) \), m-vectors,

\[
\text{(3.4)} \quad h_i(\underline{1}+\varepsilon) < \infty \text{ for } 1 \leq i \leq m.
\]

This insures that all moments of \( h_i(s) \) evaluated at \( s = \underline{1} \) may be computed by partial differentiations under the summation sign.

Define, for \( 1 \leq i, j \leq m \),

\[
\text{(3.5)} \quad m_{ij} = \frac{\partial h_i(s)}{\partial s_j} \bigg|_{s=\underline{1}} = h_{ij}(\underline{1})
\]

and assume

\[
\text{(3.6)} \quad m_{ij} > 0 \text{ all } 1 \leq i, j \leq m,
\]

and let the first moment \( m \times m \) matrix be

\[
\text{(3.7)} \quad M = (m_{ij}).
\]

By standard Frobenius theory ([1], p. 185), there is a largest eigenvalue in absolute value, denoted \( \rho \), which is positive.

The basic assumption of criticality is that

\[
\text{(3.7)(i)} \quad \rho = 1.
\]

It follows that there are strictly positive eigenvectors \( u > 0 \), \( v > 0 \) such that (see [4]),
(3.7)(ii) \[ M u = u, \quad vM = v, \]

\[ \sum_{i=1}^{m} u_i = 1 = u \cdot 1, \]

and

\[ u \cdot u = \sum_{\ell=1}^{m} u_{\ell} v_{\ell} = 1. \]

Assume

(3.7)(iii) \[ \infty > \frac{\partial^2 h_1(l)}{\partial s_j \partial s_k} > 0, \quad l \leq j, k \leq m. \]

Denote

(3.7)(iv) \[ Q(u) = \frac{1}{2} \sum_{i=1}^{m} \sum_{\ell=1}^{m} \sum_{r=1}^{m} \frac{\partial^2 h_1(l)}{\partial s_i \partial s_r} u_{\ell} u_{\ell} v_{r} < \infty, \]

where, for \( l \leq i \leq m \), for \( a > 0 \) (3.9)

(3.8)(i) \[ \int_{0}^{\infty} t^{a+b} dG_i(t) < \infty, \]

and denote, \( 0 \leq i \leq m \)

(3.8)(ii) \[ 0 < u_i = \int_{0}^{\infty} t dG_i(t), \]

where \( a > 0 \) is given by

\[ a = \frac{\left( \sum_{\ell=1}^{m} h_{\ell} \left( \frac{l}{h}\right) u_{\ell} \right)}{\mu_0 Q(u)}, \]

\( 14 \)
with \( h_{02}(\mathbf{1}) = \frac{3}{3!} h_{0}(\mathbf{1}) \), assumed to exist.

Let

\begin{equation}
Z_4(t) = (Z_{i1}(t), Z_{i2}(t), \ldots, Z_{im}(t)) .
\end{equation}

Let

\begin{equation}
N_i(t) = (N_{i1}(t), N_{i2}(t), \ldots, N_{im}(t))
\end{equation}

denote the \( m \)-vector with entries

\begin{equation}
N_{ij}(t) = \text{total progeny of type } j \text{ born by } t \text{ in the above critical } m \text{-type process starting with one new cell of type } i.
\end{equation}

An \( m \)-type branching process with immigration is defined as follows. At renewal epochs with inter-arrival time continuous distribution

\begin{equation}
G_0(t) ,
\end{equation}

\begin{equation}
G_0(0^+) = 0, G_0(t) < 1 \text{ for all } t > 0 ,
\end{equation}
satisfying

\begin{equation}
t^{4+a}(1-G_0(t)) \to 0 \text{ as } t \to \infty
\end{equation}

\( m \)-types of new cells are introduced such that there are \( i_1 \) new cells of type 1, \( i_2 \) new cells of type 2, \ldots, \( i_m \) cells of type \( m \) introduced with probability \( p_{0i}, \ldots, i_m \). Denote
(3.16) \[ h_0(s) = \sum_{i_1, \ldots, i_m = 0}^{\infty} P_{i_1} \cdots P_{i_m} s_{i_1} \cdots s_{i_m} = \sum_{l=0}^{\infty} P_0 s^l, \]

and assume

(3.17) \[ h_0(1+\epsilon) \text{ exists} \]

for some \( \epsilon > 0 \).

Each new cell of type \( i \) initiates an \( m \)-type critical age-dependent branching process [1] independent of all other cells and of the renewal process, satisfying (3.1)-(3.12).

Define, for \( 1 \leq i \leq m \),

(3.18) \[ Z_{0i}(t) \text{ and } N_{0i}(t) \]

to be the number of cells of type \( i \) alive at \( t \) and the total progeny born by \( t \), respectively, in the \( m \)-type branching process satisfying (3.1)-(3.17), called an \( m \)-type critical age-dependent branching process with immigration.

Denote

(3.19) \[ Z_0(t) = (Z_{01}(t), Z_{02}(t), \ldots, Z_{0m}(t)) \]

(3.20) \[ N_0(t) = (N_{01}(t), N_{02}(t), \ldots, N_{0m}(t)). \]

**Theorem 3.** Under assumptions (3.1)-(3.12), for \( k = (k_1, \ldots, k_m) \) a vector of non-negative integers, at least one of which is strictly positive,
\begin{align}
(3.21) \quad \lim_{t \to \infty} t^2 P[Z_1(t)=k] &= c > 0 \\
(3.22) \quad \lim_{t \to \infty} P[N_i(t)=k] &= d > 0
\end{align}

where $c, d$ are constants which may depend on $i, k$.

**Proof.** The proof follows the one-dimensional case using [4] and is omitted.

**Theorem 4.** Under assumptions (3.11)-(3.20), for $\mathbf{l} = (l_1, \ldots, l_m)$ a vector of non-negative integers,

\begin{align}
(3.23) \quad \lim_{t \to \infty} t^a P[Z_0(t)=\mathbf{l}] &= c > 0
\end{align}

for some constants $c$.

If

\begin{align}
(3.24) \quad p_{0\mathbf{l}} > 0
\end{align}

and there is a unique $\alpha > 0$ defined by

\begin{align}
(3.25) \quad h_0(0) \int_0^\infty e^{\alpha u} d \zeta_0(u) = 1,
\end{align}

then

\begin{align}
(3.26) \quad \lim_{t \to \infty} e^{\alpha t} P[N_0(t)=\mathbf{l}] &= c > 0.
\end{align}

**Proof.** Theorem 4 follows from Theorem 3 in a proof similar to Theorems 1 and 2, respectively.
Remark: If the quantities $Z_1(t)$, $N_1(t)$, $Z_0(t)$, $N_0(t)$, $k$, $l$ in Theorems 3 and 4 are replaced by corresponding marginal vectors of dimension $1 \leq d < m$, the corresponding results of Theorems 3 and 4 hold and are of the same form, since the method of proof is the same, with expressions of the same form.

References


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20. **ABSTRACT (CONTINUE ON REVERSE SIDE IF NECESSARY AND IDENTIFY BY BLOCK NUMBER)**
    PLEASE SEE REVERSE SIDE
Let $Z_0(t), N_0(t)$ denote, respectively, the number of cells alive at $t$ and the total progeny born by $t$ in a process with a random number of new cells introduced at renewal epochs, each new cell initiating a critical age-dependent branching process. As $t \to \infty$, the forms of $P[Z_0(t) = k]$ and $P[N_0(t) = k]$ are obtained for $k = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$, respectively. A multi-dimensional version and extension are indicated.