DETECTING CHANGE POINTS IN SAMPLING FROM
MULTINOMIAL DISTRIBUTIONS

By

ALAN E. GELFAND

TECHNICAL REPORT NO. 282
March 4, 1980

Prepared under Contract
NO0014-76-C-0475 (NR-042-267)
For the Office of Naval Research

Herbert Solomon, Project Director

Reproduction in Whole or in Part is Permitted
for any Purpose of the United States Government

Approved for public release; distribution unlimited.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
Detecting Change Points in Sampling
from Multinomial Distributions

Alan E. Gelfand

1. Introduction

Suppose that we are observing a sequence of categorical variables. Our problem addresses the matter of if and when a change in the underlying cell probabilities has occurred. In other words, we are trying to detect, along the sequence a shift from one multinomial distribution to another.

The literature to date has discussed the case of shifts for one-dimensional observations (i.e., binomial shifts in our setting). Page [3] studies a cumulative sum over time. Chernoff and Zacks [1] showed that a particular weighted sum weighing recent observations more heavily arises from a Bayesian approach. Kander and Zacks [2] generalized this result to random variables whose distributions belong to the one parameter exponential family. The latter two articles develop the problem from a hypothesis testing point of view. Sclove [4] examines these results in the binomial case. We shall see that the multidimensional problem is a bit less tractable. Sclove also suggests two possible applications. One application is in quality
control where one might wish to detect whether a production process has shifted from being "in control" to being "out of control" (i.e., from a low probability of producing a defective item to a higher one). A second application provides an epidemiology context where we would be concerned with whether the probability of contracting a disease has changed.

A broader application is the detection problem. As a given field is scanned there is a certain probability of detecting an object therein. If at some unknown time point something occurs in the field resulting in a change in the probability of detection then we are precisely in the situation described. The first example is easily extended to a multinomial framework if the production process yields items manufactured according to specification limits. We then have three natural categories: (i) the item is below the lower specification limit, (ii) the item is within the specification limits and (iii) the item is above the upper specification limit. The third example is easily extended by further elaborating the detection and no detection classifications. Yet another example involves attempting to detect shifts in political, sociological or psychological response over time with respect to categorical questions.

In certain applications the initial cell probabilities may be known, i.e., the probability of a defective
when the process is in control or the probability of contracting a disease under nonepidemic conditions. In other applications no probabilities will be known. We confine ourselves to this latter circumstance.

In all of our examples it may be that change occurs gradually over time. However we will presume that each change is somewhat sharp and that the changed distribution persists for a long period of time relative to the frequency of observation before a next change occurs. We consider exclusively the probability of detecting the first distributional change. If several distributional changes may be expected then subsequent changes would be discovered by continuous monitoring of the process using the procedures developed in the subsequent sections.

The format of the paper, then, is the following. In section 2 we formalize the problem and developed weighted and unweighted cumulative statistics and their properties. In the third section we create one-dimensional decision functions of these statistics according to three different motivations which lead to the most effective procedure we have obtained thus far. Finally in section 4 we examine a portion of the results of a large simulation study.

2. Preliminaries

Formally our problem is the following. Vector valued observations $X_i$ are taken with components $X_{ij}$,
\[ j=1,\ldots,r \text{ such that } X_{i,j} \text{ is distributed as a generalized } \]
binomial random variable, i.e., one multinomial trial with
associated probability vector \( p \) with components \( p_1, p_2, \ldots, p_r \)
for \( i=1,\ldots,k \) while \( X_{i,j} \) is distributed as a generalized
binomial random variable with associated probability
vector \( p'_i \) with components \( p'_1, \ldots, p'_r \) for \( i=k+1, k+2, \ldots \).

The vectors \( p \) and \( p' \) are assumed unknown along with
the change point \( k \) which is to be estimated.

At any given trial \( \ell \) let us define the following
statistics

\[ (1) \quad S_{m,j}^{(\ell)} = \sum_{i=\ell-m+1}^{\ell} X_{i,j} \]
\[ j=1,2,\ldots,r \]
\[ T_{m,j}^{(\ell)} = \sum_{i=\ell-m+1}^{\ell-1} (m+1-\ell)X_{i+1,j} \]

Let \( S_{m}^{(\ell)} \) and \( T_{m}^{(\ell)} \) be \( rm \times 1 \) vectors whose components are
the \( S_{m,j}^{(\ell)} \) and \( T_{m,j}^{(\ell)} \) respectively.

The \( S \) statistic, for a given \( j \), is just an unweighted
sum of the last \( m \) \( X_{i,j} \)'s up to and including \( X_{\ell,j} \). The \( T \)
statistic for a given \( j \) is a weighted sum of the last
\( m \) \( X_{i,j} \)'s with greater weight attached to the latter obser-
vations. Over increasing \( \ell \) \( S_{m}^{(\ell)} \) and \( T_{m}^{(\ell)} \) will be referred
to an unweighted and weighted moving sums respectively.
The $S_{m}^{(l)}$ and $T_{m}^{(l)}$ may be examined directly over $l$ to uncover evidence of a distributional change. If for a given $j$ $p_{j} < p_{j}'$, then $S_{m,j}^{(l)}$ ought to increase after the change point and $T_{m,j}^{(l)}$ even more so. A similar statement holds when $p_{j} > p_{j}'$. In fact such examination of the $S_{m}^{(l)}$ and $T_{m}^{(l)}$ may be helpful in confirming shifts suggested by the approaches presented in section 3. Of course, by themselves they hardly constitute a precisely defined procedure.

However the intuition incorporated into these statistics argues that we should examine the $X_{i}$'s in blocks sequentially and likely with overlap. Of course to be able to effectively detect a change point we must assume that $k$ is large with respect to the block size $m$. In selecting $m$ we face a trade off. Use of a large $m$ reduces noise, i.e., achieves stability of our $S_{m}^{(l)}$ and $T_{m}^{(l)}$ vectors under no distributional shift while use of a small $m$ makes the moving average more responsive to the incidence of such a shift. In addition, the choice of $m$ must depend (in an obviously monotonic increasing fashion) on the known number of categories, $r$. If $m$ is too small relative to $r$, many of the cell frequencies will be zero, i.e., many of the components of $S_{m}^{(l)}$ and $T_{m}^{(l)}$ will be zero regardless of whether or not a change has occurred. The only shifts we could hope to detect would be in the most probable categories thereby essentially degenerating the problem to a binomial case.
How do we justify the use of $S_{m}^{(l)}$ and $T_{m}^{(l)}$. The $S_{m,j}^{(l)}$ arise rather naturally. They are the raw cell frequencies. Cumulation is suggested by a variety of asymptotic considerations. The $T_{m,j}^{(l)}$ are less intuitive. A more general weighted moving sum would be of the form

$$\sum_{i=l-m+1}^{l-1} c(m,l,i)X_{i+l,j}.$$ 

The selection of $c(m,l,i) = m+i-l$ develops as follows. For the one-dimensional problem in the exponential family Kander and Zacks showed that for testing a single change in the mean (2) is Bayes against a uniform prior on the time of change. Let us consider this approach in our multidimensional situation.

We suppress $l$ and consider a fixed sequence of $m$ observations. We wish to test the hypothesis of no distributional change across the $m$ observations against the alternative of a single change. We have independent $X_{i} \sim GB(p_{i}), i=1,\ldots, m$ with

$$H_{0}: p_{1} = p_{2} = \ldots = p_{m} = p$$

and

$$H_{A}: p_{1} = p_{2} = \ldots = p_{k} = p, \quad p_{k+1} = p_{k+2} = \ldots = p_{m} = p'. $$
Since \( k \) is unknown we will suppose \( k \) is random, distributed according to \( \tau(k) \), \( k=1,2,\ldots,m-1 \). We will specialize \( \tau \) to the discrete uniform later. The conditional distribution of the sample given \( k \) is

\[
f(x_1,\ldots,x_m|k) = \prod_{i=1}^{k} \prod_{j=1}^{r} p_{ij} \prod_{i=k+1}^{m} \prod_{j=1}^{r} p_{ij}^{'x_{ij}}
\]

\[
= \prod_{j=1}^{r} p_{j}^{S_{k,j}} \prod_{j=1}^{r} p_{j}^{S_{m,j}-S_{k,j}} \tau(k).
\]

Thus the unconditional distribution of the sample is

\[
f(x_1,\ldots,x_m) = \sum_{k=1}^{m-1} \prod_{j=1}^{r} p_{j}^{S_{k,j}} \prod_{j=1}^{r} p_{j}^{S_{m,j}-S_{k,j}} \tau(k).
\]

Consider \( g(p) \) such that \( \frac{\partial g}{\partial p_j} \) exists on \( [0,1] \), \( j=1,\ldots,r \). Let \( p_j' = p_j + \delta_j \). We thus have as \( ||\delta|| \to 0 \)

\[
g(p') \approx g(p) + \sum_{j=1}^{r} g^{(j)}(p) \delta_j.
\]

In particular with \( g(p) = \sum c_j \log p_j \)

\[
g(p') \approx \sum_{j=1}^{r} c_j \log p_j + \sum_{j=1}^{r} c_j \frac{\delta_j}{p_j}.
\]

Rewriting (3) as
\[ f(x_1, \ldots, x_m) = \sum_{k=1}^{m-l} \frac{S_{k,j}}{\sum_{j=1}^{r} p_j} \log p_j + \sum_{j=1}^{r} \frac{(S_{m,j}-S_{k,j})}{\sum_{j=1}^{r} p_j} \log p_j \]

and using (4) on the last summation yields

\[ f(x_1, \ldots, x_m) = \sum_{k=1}^{m-l} \frac{S_{k,j}}{\sum_{j=1}^{r} p_j} \log p_j + \sum_{j=1}^{r} \frac{(S_{m,j}-S_{k,j})}{\sum_{j=1}^{r} p_j} \log p_j \]

Under the null hypothesis

\[ f(x_1, \ldots, x_m) = e \]

so that the likelihood ratio \( \lambda(x_1, \ldots, x_m) \) becomes

\[ \lambda(x_1, \ldots, x_m) = \sum_{k=1}^{m-l} \frac{(S_{m,j}-S_{k,j})}{\sum_{j=1}^{r} p_j} \frac{e^{\delta_j}}{p_j} \]

(5) \( \lambda(x_1, \ldots, x_m) = \sum_{k=1}^{m-l} \frac{e^{\delta_j}}{p_j} \frac{e^{c_j \delta_j/p_j}}{e^{c_j \delta_j/p_j}} \)

Again as \( ||\delta|| \to 0 \)

\[ e^{c_j \delta_j/p_j} \frac{e^{c_j \delta_j/p_j}}{e^{c_j \delta_j/p_j}} \]

so that (5) becomes, ignoring terms involving \( \delta_j \) and \( \delta_j \),

-8-
\[ \lambda(x_1, \ldots, x_m) \sim \frac{m-1}{r} \sum_{k=1}^{r} \sum_{j=1}^{p_j} (S_{m,j} - S_{k,j}) \delta_j / p_j + 1. \]

If we now let \( \tau(k) = 1/(m-1) \) and interchange the order of summation we obtain

\[ \lambda(x_1, \ldots, x_m) = \frac{1}{m-1} \sum_{j=1}^{r} \sum_{j=1}^{p_j} (S_{m,j} - S_{k,j}) + 1 \]

\[ = \frac{1}{m-1} \sum_{j=1}^{r} \sum_{j=1}^{p_j} T_{m,j} + 1. \]

Thus a distributional shift would be indicated by large values of

(6)

\[ \sum_{j=1}^{p_j} \delta_j T_{m,j}. \]

If \( p \) and \( p' \) were known (6) could be used directly and would provide the Bayes test statistic. With \( p \) and \( p' \) unknown, some \( \delta_j \) will be positive, some negative and some zero but the given linear combination of the \( T_{m,j} \) is not computable. The \( \delta_j \) and \( p_j \) can be estimated from the sample. (We shall examine this point later.) However this would necessitate making \( 2(r-1) \) estimates from \( m \) observations and will provide hopelessly unstable estimators. Hence the most we can conclude at the moment is that the \( T_{m,j} \) seem to be appropriate weighted averages to study over time but exactly how to combine them into a
one-dimensional test statistic remains to be discussed.

In concluding this section let us examine the behavior of $S_m$ and $T_m$. We will continue to suppress $\ell$ for the remainder of this section.

Note that $\sum_{j=1}^{r} S_{m,j} = m$ and thus as we proceed over time, some $S_{m,j}$ will increase while others decrease but their sum, at any fixed time is $m$. Moreover if we continue to assume a uniform distribution for the change point

$$E(S_{m,j}) = E(E(S_{m,j} | k))$$

$$= \frac{1}{m-1} \sum_{k=1}^{m-1} E(\sum_{i=1}^{m} X_{i,j} | k)$$

(7)

$$= \frac{1}{m-1} \sum_{k=1}^{m-1} (kp_j + (m-k)p_j')$$

$$= \frac{m(p_j' + p_j)}{2}$$

which equals $mp_j$ under no change.

$$\text{var}(S_{m,j}) = \text{var}(E(S_{m,j} | k)) + E(\text{var}(S_{m,j} | k))$$

$$= \text{var}(kp_j + (m-k)p_j') + E(\text{var}(\sum_{i=1}^{m} X_{i,j} | k))$$
\[ (p_j - p_j')^2 \text{var}(k) + E(k \ p_j (1-p_j) + (m-k) p_j'(1-p_j')) \]

\[ = (p_j - p_j')^2 \frac{m(m-2)}{12} + \frac{m}{2}(p_j (1-p_j) + p_j'(1-p_j')) \]

which equals \( m \ p_j(1-p_j) \) under no change.

In fact a similar calculation on the individual observations shows that \( X_{ij} \) is a single Bernoulli trial with success probability

\[ p_{ij} = \frac{(i-1)p_j' + (m-i)p_j}{m-1}. \]  

Hence the exact distribution of \( S_{m,j} \) is that of the sum of \( m \) independent but non-identically distributed Bernoulli random variables. Under the null hypothesis it is, of course, \( Bi(m, p_j) \). Moreover from (8) the moments of \( X_{ij} \) can be readily computed although the expressions become rather unwieldy. In particular we can show that

\[ \frac{\sum_{i=1}^{m} E(X_{ij} - E(X_{ij}))^2}{\sum_{i=1}^{m} E(|X_{ij} - E(X_{ij})|^3)}^{1/3} = O(m^{-2/3}). \]

Thus Liapunov's theorem insures that \( S_{m,j} \) is asymptotically normal under either hypothesis.
Turning to $T_m$ we note that upon interchanging order of summation

$$
\sum_{j=1}^{r} T_{m,j} = \frac{m(m-1)}{2}.
$$

Thus as with $S_m$ as we proceed over time, some $T_{m,j}$ will increase while others will decrease but their sum at any fixed time is $\frac{m(m-1)}{2}$. Similarly we may show that

$$
E(T_{m,j}) = \frac{m(m-2)}{6} p_j + \frac{m(2m-1)}{6} p_j',
$$

which becomes $\frac{m(m-1)}{2} p_j$ under no change. Furthermore

$$
\text{var}(T_{m,j}) = \sum_{i=1}^{m} i^2 \text{var} X_{i+1,j} = \sum_{i=2}^{m} (i-1)^2 p_{i,j}(1-p_{i,j}).
$$

This is a rather messy expression which obviously reduces to $\frac{m(m-1)(2m-1)}{6} p_j(1-p_j)$ under no change. Liapunov's theorem again insures that $T_{m,j}$ is asymptotically normal. Kander and Zacks develop this result for the more general exponential family case and also discuss the rate of convergence. They briefly consider the exact distribution of $T_{m,j}$ in a scaled binomial case (p. 1202).

The expressions in (7) and (9) enable us to construct method of moments type estimators of $p_j$, $p_j'$ (and $\delta$, if desired), i.e.,
\[ \hat{p}_j = \frac{m(2m-1)s_{m,j}}{6} - \frac{m^2}{2} m, j = \frac{2(2m-1)s_{m,j}}{m(m+1)} - \frac{6T_{m,j}}{m(m+1)} \]

\[ \hat{p}_j = \frac{m}{2} t_{m,j} - \frac{m(m-2)}{6} s_{m,j} = \frac{6T_{m,j} - 2(m-2)s_{m,j}}{m(m+1)} \]

and

\[ \hat{\delta}_j = \frac{12T_{m,j} - 6(m-1)s_{m,j}}{m(m+1)} \]

All of these estimators are unbiased. However since

\[ \text{cov}(T_{m,j}, S_{m,j}) = \sum_{i=2}^{m} (i-1) \text{var}(X_{ij}) = \sum_{i=2}^{m} (i-1)p_{ij}(1-p_{ij}) \]

none of these estimators is consistent. All three have variance of order \( m \). Thus effective estimation of the coefficients of the \( T_{m,j} \) in (6) is revealed to be hopeless unless the change point is known.

3. The Methods

We now consider a variety of procedures which suggest themselves as plausible methods for detecting a distributional shift. These methods may be loosely classified under three headings -- (1) Law of Large Numbers type approaches, (11) Departures from centrality type
approaches and (iii) Tests of Homogeneity type approaches. We defer comparisons and criticisms of these approaches to the next section.

3.1 "Law of Large Numbers" Approaches

Consider the absolute difference

\[ |S_m^{(\ell)} - S_m^{(\ell-1)}| = |X_{\ell,j} - X_{\ell-m,j}| \]

and define

(10) \[ Q_m^{(\ell)} = \sum_{j=1}^{r} |S_m^{(\ell)} - S_m^{(\ell-1)}| . \]

Note that

\[ Q_m^{(\ell)} = \begin{cases} 0 & \text{if } X_{\ell,j} = X_{\ell-m,j}, \ j=1,\ldots,r \\ 2 & \text{otherwise} \end{cases} \]

and thus

\[ P(Q_m^{(\ell)} = 0) = \begin{cases} \sum_{j} p_j^2 & \text{if } \ell < k \\ \sum_{j} p_j' & \text{if } k < \ell \leq k+m \\ \sum_{j} p_j'^2 & \text{if } \ell > k+m \end{cases} \]

with \( P(Q_m^{(\ell)} = 2) = 1 - P(Q_m^{(\ell)} = 0) \). Hence
\[ E(Q_m(\ell)) = \begin{cases} 
2(1 - \Sigma p_j^2) & \text{if } \ell \leq k \\
2(1 - \Sigma p_j p_j') & \text{if } k < \ell \leq k+m \\
2(1 - \Sigma p_j' p_j') & \text{if } \ell > k+m 
\end{cases} \]

\[ \text{var}(Q_m(\ell)) = \begin{cases} 
4 \Sigma p_j^2 (1 - \Sigma p_j^2) & \text{if } \ell \leq k \\
4 \Sigma p_j p_j' (1 - \Sigma p_j p_j') & \text{if } k < \ell \leq k+m \\
4 \Sigma p_j' p_j' (1 - \Sigma p_j' p_j') & \text{if } \ell > k+m 
\end{cases} \]

Similarly, consider the absolute difference

\[ |T_m,j - T_{m,j}^{(\ell-1)}| = |\prod_{i=\ell-m+1}^{\ell-1} (m+1-\ell)X_{i+1,j} - \prod_{i=\ell-m}^{\ell-2} (m+1-\ell+1)X_{i+1,j}| \]

\[ = |m X_{\ell,j} - S_{m,j}(\ell)|. \]

Hence

\[ |T_m,j - T_{m,j}^{(\ell-1)}| = \begin{cases} 
S_{m,j}(\ell) & \text{if } X_{\ell,j} = 0 \\
S_{m,j}(\ell) - m & \text{if } X_{\ell,j} = 1 
\end{cases} \]

and thus

-15-
\[ P(\lceil T_{m,j}^{(\ell)} - T_{m,j}^{(\ell-1)} \rceil = S_{m,j}^{(\ell)}) = \begin{cases} 1-p_j & \text{if } \ell \leq k \\ 1-p_j & \text{if } \ell > k \end{cases} \]

and

\[ P(\lceil T_{m,j}^{(\ell)} - T_{m,j}^{(\ell-1)} \rceil = m-S_{m,j}^{(\ell)}) = 1 - P(\lceil T_{m,j}^{(\ell)} - T_{m,j}^{(\ell-1)} \rceil = S_{m,j}^{(\ell)}). \]

Define

\[ R_m^{(\ell)} = \sum_{j=1}^{r} \lceil T_{m,j}^{(\ell)} - T_{m,j}^{(\ell-1)} \rceil. \]

Then

\[ R_m^{(\ell)} = m - S_{m,j_0}^{(\ell)} + \sum_{j=1}^{r} S_{m,j}^{(\ell)}, \quad \text{if } X_{\ell,j_0} = 1, \]

\[ = 2(m - S_{m,j_0}^{(\ell)}), \quad \text{if } X_{\ell,j_0} = 1. \]

Hence \( 0 \leq R_m^{(\ell)} \leq 2(m-1) \) since \( S_{m,j_0}^{(\ell)} \geq X_{\ell,j_0} = 1 \) and \( R_m^{(\ell)} \) takes on values \( 0, 2, \ldots, 2(m-1) \). Furthermore for \( a=0,1,2,\ldots,m-1 \)

\[ P(R_m^{(\ell)} = 2a) = \sum_{j=1}^{r} P(R_m^{(\ell)} = 2a, X_{\ell,j} = 1) \]

\[ = \sum_{j=1}^{r} P(S_{m,j}^{(\ell)} = m-a, X_{\ell,j} = 1) \]

\[ = \sum_{j=1}^{r} P(\ell-1) = m-a \) P(X_{\ell,j} = 1). \]

-16-
The distribution of $S_{m-1,j}^{(\ell-1)}$ was discussed in the previous section under a uniform distribution over the change point $k$. For the present $k$ is fixed so that if $\ell \leq k+1$,

$S_{m-1,j}^{(\ell-1)} \sim \text{Bi}(m-1,p_j)$. If $\ell > k + m$ \quad $S_{m-1,j}^{(\ell-1)} \sim \text{Bi}(m-1,p'_j)$. If $k+1 \leq \ell < k+m$, $S_{m-1,j}^{(\ell-1)} = W_1 + W_2$ where $W_1$ and $W_2$ are independent with $W_1 \sim \text{Bi}(m+k-\ell,p_j)$ and $W_2 \sim \text{Bi}(\ell-1-k,p'_j)$. Of course

$P(X_{\ell,j} = 1) = p_j$ if $\ell \leq k$ and $= p'_j$ if $\ell > k$. Let

$$n_1(\ell) = \begin{cases} m-1 & \ell \leq k+1 \\ m+k-\ell & k+1 \leq \ell < k+m \\ 0 & \ell \geq k+m \end{cases}$$

$$n_2(\ell) = \begin{cases} 0 & \ell \leq k+1 \\ \ell-1-k & k+1 \leq \ell < k+m \\ m-1 & \ell \geq k+m \end{cases}$$

and

$$\gamma(m,\ell,a,j) = \sum_{i=0}^{m-a-1} n_1(\ell-1)^{i} p_j (1-p_j)^{n_1(\ell)-1} n_2(\ell)^{\ell-a-1}$$

with $(\begin{smallmatrix} c \\ d \end{smallmatrix}) = 0$ if $c < d$ and $(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}) = 1$. Then
\[ P (R_m (\ell) = 2a) = \begin{cases} 
\sum_{j=1}^{r} \gamma (m, \ell, a, j) p_j & \text{if } \ell \leq k \\
\sum_{j=1}^{r} \gamma (m, \ell, a, j) p_j' & \text{if } \ell > k.
\end{cases} \]

Despite the awkwardness of the distribution of \(R_m (\ell)\), its mean and variance are not that difficult to compute. From the argument after expression (11)

\[ E (R_m (\ell)) = \begin{cases} 
\sum_{j=1}^{r} E [2 (m - S_m (\ell_j)) |X_{\ell, j} = 1] p_j & \text{if } \ell \leq k \\
\sum_{j=1}^{r} E [2 (m - S_m (\ell_j)) |X_{\ell, j} = 1] p_j' & \text{if } \ell > k
\end{cases} \]

\[ = \begin{cases} 
\sum_{j=1}^{r} 2 (m - E(S_{m, j} \ell_j - 1)) p_j & \text{if } \ell \leq k \\
\sum_{j=1}^{r} 2 (m - E(S_{m, j} \ell_j - 1)) p_j' & \text{if } \ell > k
\end{cases} \]

\[ = \begin{cases} 
\sum_{j=1}^{r} 2 (m - 1) p_j (1 - p_j) & \text{if } \ell \leq k \\
\sum_{j=1}^{r} 2 (m - 1 - ((m + k - \ell) p_j + (\ell - 1 - k) p_j')) p_j' & \text{if } k + 1 \leq \ell < k + m \\
\sum_{j=1}^{r} 2 (m - 1) p_j' (1 - p_j') & \text{if } \ell \geq k + m
\end{cases} \]
\[
\begin{align*}
E(R_m^2(\ell)) &= \begin{cases} 
2(m-1)(1-\Sigma p_j^2) & \text{if } \ell \leq k \\
2(m+k-\ell)(1-\Sigma p_j p_j') + 2(\ell-1-k)(1-\Sigma p_j^2) & \text{if } k+1 \leq \ell \leq k+m \\
2(m-1)(1-\Sigma p_j^2) & \text{if } \ell > k+m.
\end{cases}
\end{align*}
\]

Similarly we may obtain the variance of \( R_m(\ell) \) by computing

\[
E(R_m^2(\ell)) = \begin{cases} 
\sum_{j=1}^{r} p_j E[(2(m-S_m^{(\ell)}))^2 | X_{\ell,j} = 1] & \text{if } \ell \leq k \\
\sum_{j=1}^{r} p_j E[(2(m-S_m^{(\ell)}))^2 | X_{\ell,j} = 1] & \text{if } \ell > k
\end{cases}
\]

\[
= \begin{cases} 
\sum_{j=1}^{r} p_j (m-1-S_m^{(\ell-1)})^2 & \text{if } \ell \leq k \\
\sum_{j=1}^{r} p_j (m-1-S_m^{(\ell)})^2 & \text{if } \ell > k
\end{cases}
\]

\[
= \begin{cases} 
\sum_{j=1}^{r} p_j [\text{var}(S_m^{(\ell-1)}) + (m-1-E S_m^{(\ell-1)})^2] & \text{if } \ell \leq k \\
\sum_{j=1}^{r} p_j [\text{var}(S_m^{(\ell)}) + (m-1-E S_m^{(\ell)})^2] & \text{if } \ell > k
\end{cases}
\]

From previous discussion we know the mean and variance of \( S_{m-1,j}^{(\ell-1)} \) so that
\[ E(R_m^2(\ell)) = \begin{cases} 
4 \sum_{j=1}^{r} \left[(m-1)p_j^2(1-p_j^\prime) + (m-1)^2(1-p_j^\prime)^2p_j\right] 
& \text{if } \ell \leq k \\
4 \sum_{j=1}^{r} p_j \left[(m+k-\ell)p_j(1-p_j^\prime) + (\ell-1-k)p_j(1-p_j^\prime) \right] 
& 
+ ((m+k-\ell)(1-p_j^\prime) + (\ell-1-k)(1-p_j^\prime))^2] 
& \text{if } k+1 \leq \ell \leq k+m \\
4 \sum_{j=1}^{r} \left[(m-1)p_j^\prime^2(1-p_j^\prime) + (m-1)^2(1-p_j^\prime)^2p_j^\prime\right] 
& \text{if } \ell > k+m. 
\end{cases} \]

Subtracting \((E(R_m(\ell)))^2\) and simplifying yields

\[ \text{var}(R_m(\ell)) = \begin{cases} 
4(m-1)\Sigma p_j^2(1-p_j) + 4(m-1)^2(\Sigma p_j^2 - (\Sigma p_j^2)^2) 
& \text{if } \ell \leq k \\
4(m+k-\ell)\Sigma p_j p_j(1-p_j) + 4(\ell-1-k)\Sigma p_j^2 
& + 4(m+k-\ell)^2(\Sigma p_j^2 - (\Sigma p_j^2)^2) 
& + 4(\ell-1-k)^2(\Sigma p_j^3 - (\Sigma p_j^2)^2) 
& + 4(m+k-\ell)(\ell-1-k)(\Sigma p_j^2 - \Sigma p_j p_j^\prime \Sigma p_j^\prime) 
& \text{if } k+1 \leq \ell \leq k+m \\
4(m-1)\Sigma p_j^\prime^2(1-p_j^\prime) + 4(m-1)^2(\Sigma p_j^\prime^3 - (\Sigma p_j^\prime)^2)^2 
& \text{if } \ell > k+m. 
\end{cases} \]
How may we employ (10) and (11) to develop detection procedures. Note that

\[
E(1 - \frac{Q_m(\ell)}{2}) = E(1 - \frac{R_m(\ell)}{2(m-1)}) = \Sigma p_j^2 \quad \text{if} \quad \ell \leq k
\]

\[= \Sigma p_j'{}^2 \quad \text{if} \quad \ell > k + m \]

and

\[
E(1 - \frac{Q_m(\ell)}{2}) = \Sigma p_j p_j' \quad \text{if} \quad k + 1 \leq \ell \leq k + m
\]

\[
E(1 - \frac{R_m(\ell)}{2(m-1)}) = \frac{(m+k-\ell)\Sigma p_j p_j' + (\ell-1-k)\Sigma p_j'^2}{m-1} \quad \text{if} \quad k + 1 \leq \ell \leq k + m.
\]

It is also apparent that

\[
\text{var}(1 - \frac{R_m(\ell)}{2(m-1)}) \leq \text{var}(1 - \frac{Q_m(\ell)}{2}) \quad \text{for} \quad \ell \leq k
\]

and for \(\ell > k + m\)

i.e., for \(\ell \leq k\)

\[
[\Sigma p_j^2(1 - \Sigma p_j^2)] - [\frac{1}{m-1}\Sigma p_j^2(1 - p_j) + \Sigma p_j^2 - (\Sigma p_j^2)^2]
\]

\[= (\Sigma p_j^2 - \Sigma p_j^3)(\frac{m-2}{m-1}) \geq 0
\]

with a similar argument for \(\ell > k + m\).

Furthermore the Cauchy-Schwarz inequality assures that
\[
\Sigma p_j^\prime \leq \sqrt{\Sigma p_j^2} \Sigma p_j^\prime^2
\]

i.e.,
\[
\Sigma p_j p_j^\prime \leq \max(\Sigma p_j^2, \Sigma p_j')^2
\]

so that we must have one of the following

(i) \( \Sigma p_j p_j^\prime \leq \Sigma p_j^2 \leq \Sigma p_j' \)

(ii) \( \Sigma p_j p_j^\prime \leq \Sigma p_j' \leq \Sigma p_j^2 \)

(iii) \( \Sigma p_j^2 \leq \Sigma p_j p_j^\prime \leq \Sigma p_j' \)

(iv) \( \Sigma p_j' \leq \Sigma p_j p_j^\prime \leq \Sigma p_j^2 \).

Thus if we were to monitor \( 1 - \frac{Q_m(\ell)}{2} \) or \( 1 - \frac{R_m(\ell)}{2(m-1)} \)

over \( \ell \), and observe a fairly steady increase (case (iii)),
a fairly steady decrease (case (iv)), or a fairly well
defined "v", i.e., decrease and then increase (cases (i)
and (ii)) this would provide evidence of a distributional
shift.

Since the \( Q_m(\ell) \) and \( R_m(\ell) \) may be expected to be rather
unstable let us average them in blocks of \( m \) and define

\[
W_1(\ell) = \frac{1}{m} \sum_{i=\ell-m+1}^{\ell} \left( 1 - \frac{Q_m(i)}{2} \right)
\]

(12)

\[
W_2(\ell) = \frac{1}{m} \sum_{i=\ell-m+1}^{\ell} \left( 1 - \frac{R_m(i)}{2(m-1)} \right).
\]

(13)
(Note that $W_1$ and $W_2$ can not be computed sooner than $\ell=2m$).

For $\ell \leq k$, $W_1$ and $W_2$ are both unbiased estimators of $\Sigma p_j^2$ and for $\ell \leq k+2m$, $W_1$ and $W_2$ are both unbiased estimators of $\Sigma p_j'^2$. Again in the presence of a distributional change, a perturbation of $W_1$ and $W_2$ across $k$ to $k+2m$ should be observed. Although the preceding variance calculations might suggest that $\var(W_2) < \var(W_1)$ this is not necessarily so since $W_1$ is an average of independent random variables (see after (10)) while $W_2$ is an average of dependent variables (see after (11)). While a computation of $\var(W_1)$ and $\var(W_2)$ could be attempted based on previous calculations the results would be hopeless to compare. However since the $R_m(i)$ are positively correlated one might suspect that the inequality would be reversed. In fact our simulation study in the next section reveals that this is not so, i.e., $\var(W_2)$ tends to be much smaller than $\var(W_1)$.

In the binomial case with $X_\ell$ being the result of $\ell$th Bernoulli trial we have

$$W_1(\ell) = m - \sum_{i=\ell-m+1}^{\ell} |X_1 - X_{1-m}|$$

$$W_2(\ell) = m - \frac{1}{m-1} \sum_{i=\ell-m+1}^{\ell} |mX_1 - \sum_{r=1-m+1}^{1} X_r|.$$ 

In order to formulate an estimator of $k$ based on either $W_1$ or $W_2$, if a disturbance is observed commencing at approximately trial $\ell_o$ and stabilizing after approximately trial $\ell_o + 2m$ then $\hat{k} = \ell_o$. 

-23-
3.2 "Departures from Centrality" approaches.

Consider

\begin{equation}
W_3(\ell) = \frac{r}{m} \sum_{j=1}^{n} (S_{m,j}^{(\ell)} - \frac{m}{r})^2 = \frac{r}{m} \sum_{j=1}^{n} (S_{m,j}^{(\ell)})^2 - m^2/r
\end{equation}

(14)

\begin{equation}
W_4(\ell) = \frac{r}{m} \sum_{j=1}^{n} (T_{m,j}^{(\ell)} - \frac{m(m-1)}{2r})^2 = \frac{r}{m} \sum_{j=1}^{n} (T_{m,j}^{(\ell)})^2 - \frac{m^2(m-1)^2}{4r}.
\end{equation}

(15)

Note that \(W_3\) and \(W_4\) can not be computed any sooner than \(\ell=m\).

If \(\ell<k\), \(W_3\) and \(W_4\) indicate "how far" the distribution \(p\) is from the equiprobable cell distribution and if \(\ell\geq k+m\) they indicate "how far" \(p'\) is from the equiprobable cell distribution. As \(\ell\) increases from below \(k\) to above \(k+m\) we ought to be able to observe a change from a fairly stable "distance" to instability and back again to a fairly stable "distance". More precisely

\[E(W_3(\ell)) = \frac{r}{m} \sum_{j=1}^{n} E(S_{m,j}^{(\ell)})^2 - m^2/r\]

\[
= \begin{cases} 
  m(m-1)\Sigma p_j^2 + m - m^2/r & \text{if } \ell<k \\
  (k+m-\ell)(k+m-\ell-1)\Sigma p_j^2 + (\ell-k)(\ell-k-1)\Sigma p_j'p_j^2 + m + 2(k+m-\ell)(\ell-k)\Sigma p_jp_j'^2 - m^2/r & \text{if } k+1 \leq \ell < k+m \\
  m(m-1)\Sigma p_j'^2 + m - m^2/r & \text{if } \ell \geq k+m.
\end{cases}
\]
These may be written a bit more suggestively as

\[
\begin{cases} 
  m^2 \sum (p_j - l/r)^2 + m(1 - \Sigma p_j^2) & \text{if } l \leq k \\
  r \sum [(k+m-l)(p_j - l/r) + (l-k)(p_j' - l/r)]^2 & \\
  + (k+m-l)(1 - \Sigma p_j^2) + (l-k)(1 - \Sigma p_j'^2) & \text{if } k+1 \leq l < k+m \\
  m^2 \sum (p_j' - l/r)^2 + m(1 - \Sigma p_j'^2) & \text{if } l \geq k+m. 
\end{cases}
\]

While the second set of expressions suggests that departure from the center of the r dimensional simplex is being measured, the first set of expressions reveals that \( E(W_3) \) is merely of the form \( a(m) \Sigma p_j^2 + b(m) \) for \( l \leq k \) and of the form \( a(m) \Sigma p_j'^2 + b(m) \) for \( l \geq k+m \). Thus monitoring \( W_3 \) over \( l \) is much like studying \( W_1 \) or \( W_2 \) from the previous section: under a distributional change a departure from relative stability will hopefully be observed across trials \( k \) to \( k+m \) and then a return to relative stability. All of these remarks are appropriate for \( W_4 \). In particular, although the expressions are a bit more complicated it may readily be seen that \( E(W_4) \) is of the form \( c(m) \Sigma p_j^2 + d(m) \) for \( l \leq k \) and of the form \( c(m) \Sigma p_j'^2 + d(m) \) for \( l \geq k+m \).

An argument which lends further support to \( W_3 \) is the following. If \( l \leq k \) then the set \( \{ s_m^{(l)}, 1, \ldots, s_m^{(l)}, r-l \} \) is a complete and sufficient statistic for the most recent
sample of \( m \) observations. It is apparent that

\[
H(S_m^{(\ell)}, \ldots, S_m^{(r-1)}) = \frac{1}{m(m-1)} \sum_{j=1}^{r} S_m^{(\ell)}(S_m^{(\ell)} - 1)
\]

\[
= \frac{1}{m(m-1)} \sum_{j=1}^{r} (S_m^{(\ell)})^2 - \frac{1}{m-1} \text{ is the uniformly minimum variance unbiased estimator of } \Sigma p_j^2. \text{ Similarly, if } \ell \geq k+m \text{ } H \text{ is the UMVU of } \Sigma p_j^2. \text{ Furthermore } H \text{ and } W_3 \text{ are linearly related. Thus } W_3 \text{ will be as effective an index for our purposes as } H \text{ and may be used equivalently. We comment that since } W_1 \text{ and } W_2 \text{ are computed on the basis of a sample of } 2m \text{ their use is not discouraged by the above.}

In concluding this subsection we develop a statistic similar to } W_3 \text{ and } W_4 \text{ which is a function of both } S_m^{(\ell)} \text{ and } T_m^{(\ell)}. \text{ Analogously to prior calculations we have}

\[
E(S_m^{(\ell)}) = \begin{cases} 
mp_j & \text{if } \ell \leq k \\
(m+k-\ell)p_j + (\ell-k)p_j' & \text{if } k+1 \leq \ell < k+m \\
mp_j' & \text{if } k > \ell + m
\end{cases}
\]

and

\[
E(T_m^{(\ell)}) = \begin{cases} 
\frac{m(m-1)}{2} p_j & \text{if } \ell \leq k \\
\frac{(m+k-\ell)(m+k-\ell-1)}{2} p_j + \frac{m(m-1)}{2} - \frac{(m+k-\ell)(m+k-\ell-1)}{2} p_j' & \text{if } k+1 \leq \ell < k+m \\
\frac{m(m-1)}{2} p_j' & \text{if } \ell \geq k+m
\end{cases}
\]

so that
\[ E(T_{m,j}^{(\ell)} - \frac{(m-1)}{2} S_{m,j}^{(\ell)}) = 0 \quad \text{if} \quad \ell < k, \ell > k+m \]

\[ = \frac{(m+k-\ell)(\ell-k)}{2} (p_j - p_j') \]

\[ \text{if} \quad k+1 \leq \ell < k+m. \]

Since \( \sum_{j=1}^{r} (T_{m,j}^{(\ell)} - \frac{(m-1)}{2} S_{m,j}^{(\ell)}) = 0 \) for all \( \ell \) we are led to

\[ W_5(\ell) = \sum_{j=1}^{r} \left( T_{m,j}^{(\ell)} - \frac{(m-1)}{2} S_{m,j}^{(\ell)} \right)^2. \]

(Note that \( W_5 \) can't be computed sooner than \( \ell = m \).)

The behavior of \( W_5 \) should be such that below \( k \) and above \( k+m \) it will be small and between \( k \) and \( k+m \) it will tend to be larger.

In the interest of computing \( E(W_5) \) we express (16) as

\[ W_5(\ell) = \sum_{j=1}^{r} \left( \sum_{i=\ell-m+1}^{\ell} \left( \frac{(m-1)}{2} + 1-\ell \right) X_{ij} \right)^2. \]

Thus

\[ E(W_5(\ell)) = \sum_{j=1}^{r} E \left[ \sum_{i=\ell-m+1}^{\ell} \left( \frac{(m-1)}{2} + 1-\ell \right) X_{ij} \right]^2 \]

\[ = \sum_{j=1}^{r} \left[ \var \left( \sum_{i=\ell-m+1}^{\ell} \left( \frac{(m-1)}{2} + 1-\ell \right) X_{ij} \right) \right. \]

\[ \left. + \left( E \left( \sum_{i=\ell-m+1}^{\ell} \left( \frac{(m-1)}{2} + 1-\ell \right) X_{ij} \right) \right)^2 \right] \]
\[
\begin{align*}
\sum_{j=1}^{r} \sum_{i=1}^{\ell} \left( \frac{m-1}{2} + 1 - \ell \right)^2 p_j (1-p_j) \\
+ \sum_{i=1}^{\ell} (\frac{m-1}{2} + 1 - \ell) p_j^2 \right] \text{ if } \ell \leq k \\
+ \sum_{i=k+1}^{\ell} \frac{m-1}{2} + 1 - \ell \right)^2 p_j^2 \right] \text{ if } k+1 \leq \ell < k+m \\
+ \sum_{i=k+1}^{\ell} \frac{m-1}{2} + 1 - \ell \right)^2 p_j^2 \right] \text{ if } \ell \geq k+m
\end{align*}
\]

Although these expressions may be simplified somewhat it is more crucial to note that again $E(W_5)$ is of the form $e(m)\Sigma p_j^2 + f(m)$ for $\ell \leq k$ and of the form $e(m)\Sigma p_j^2 + f(m)$ for $\ell \geq k+m$. Thus $W_5$ may be monitored over $\ell$ as our intuition suggests and analogous to $W_3$ and $W_4$. 

-28-
In the binomial case our expressions for $W_3$, $W_4$ and $W_5$ reduce to

$$W_3(\ell) = 2\left( \sum_{i=\ell-m+1}^{\ell} x_i - \frac{m}{2} \right)^2$$

$$W_4(\ell) = 2\left( \sum_{i=\ell-m+1}^{\ell-1} \frac{(m+1-\ell)x_i + m(m-1)}{4} \right)^2$$

$$W_5(\ell) = 2\left( \sum_{i=\ell-m+1}^{\ell} \frac{(m-1-i-\ell)x_i}{2} \right)^2.$$  

As at the end of section 3.1, $k=\ell_o$ provides an estimator of $k$.

3.3 "Test of Homogeneity" approaches.

A test of homogeneity seems to quite naturally suggest itself for this sort of problem. We are trying to detect for a collection of observations whether one multinomial distribution is generating them or whether two different multinomials are generating them. Discovering which of these two hypotheses is true is precisely the raison d'être for chi-square based test of homogeneity statistics.

However in our present circumstance where we have a continuing sequence of observations there is certainly no unique choice of partitioning into groups to establish comparisons. However since we postulate but one distributional change across any given portion of the sequence
we are examining, a partition into more than two groups seems likely to obscure or confound the detection problem.

If we confine ourselves to two blocks of observations, as an initial attempt, we would compare the first \( m \) observations to the next \( m \) observations by computing the usual \( \chi^2 \) statistic and continue successively comparing the \( i^{th} \) block of \( m \) observations with the \( i+1^{st} \) block, \( i=2,3, \ldots \). Our statistic depends on \( i \) and \( m \) and in the notation developed thus far becomes

\[
V_1(i,m) = \sum_{j=1}^{r} \frac{(S_{m,j}^{(i+1)m} - S_{m,j}^{(im)})^2}{S_{m,j}^{((i+1)m)} + S_{m,j}^{(im)}}.
\]

In monitoring \( V_1 \), if no shift occurred during trials \( im+1 \) to \( i(m+1) \) we expect \( V_1 \) small, otherwise it should be large. As we have indicated previously the selection of \( m \) depends on \( r \). It should be at least \( 5r \) to discourage empty cell problems. More precisely we would like the expected cell frequencies to be at least three to five. With the true distributions unknown we instead suggest \( m \cdot \frac{1}{r} \) at least five. Since we are not concerned with the distribution of our monitoring index we might drop the denominator from \( V_1 \) and simply examine

\[
V_2(i,m) = \sum_{j=1}^{r} (S_{m,j}^{(i+1)m} - S_{m,j}^{(im)})^2.
\]
Another strategy which is an adaptive modification of $V_1$ is the following. If $V_1(1,m)$ is small pool the first $2m$ observations into one sample and compare this group with the third block of $m$ observations. More generally this idea leads to a comparison of the first $im$ observations with the next set of $m$ observations via

$$
V_3(i,m) = r \sum_{j=1}^{r} \frac{(S_{im,j}^{(i)} - 1) S_{m,j}^{((i+1)m)}}{i(S_{im,j}^{(i)} + S_{m,j}^{((i+1)m)})^2}
$$

if $V_3(1,m), V_3(2,m), \ldots, V_3(1-l,m)$ are small. Again the denominator in (19) may be suppressed using instead

$$
V_4(i,m) = r \sum_{j=1}^{r} (S_{im,j}^{(i)} - 1) S_{m,j}^{((i+1)m)}
$$

All of these statistics suffer one drawback. When a distributional change occurs it will occur somewhere within a block of $m$ trials. When our statistics are computed across this change we will be comparing $m$ or more observations from the old distribution with $m$ observations some from the old and some from the new. But the most dramatic difference should be revealed if we compare the $m$ trials just prior to a shift with the $m$ trials just after a shift. Moreover we would prefer a statistic which can be computed at successive trials.

-31-
(such as those of the previous sections) rather than at every m\textsuperscript{th} trial. A statistic which accomplishes both these objectives is

\begin{equation}
W_6(\ell) = \sum_{j=1}^{r} \frac{(S_{m,j}^{(\ell)} - S_{m,j}^{(\ell-m)})^2}{(S_{m,j}^{(\ell)} + S_{m,j}^{(\ell-m)})}
\end{equation}

or deleting the denominator

\begin{equation}
W_7(\ell) = \sum_{j=1}^{r} (S_{m,j}^{(\ell)} - S_{m,j}^{(\ell-m)})^2.
\end{equation}

(Note that \(W_6\) and \(W_7\) can not be computed any sooner than \(\ell=2m\).)

In observing \(W_6\) and \(W_7\) over \(\ell\) we expect them to begin to increase at \(\ell=k+1\) peaking near \(\ell=k+m\) and then decreasing until \(\ell=k+2m\). The occurrence of a spike of this sort suggests that a distributional change has occurred.

Although \(E(W_6)\) is rather hopeless to compute, \(E(W_7)\) is straightforwardly obtained directly from results after (15), i.e.,

\[E(W_7(\ell)) = \sum_{j=1}^{r} E(S_{m,j}^{(\ell)} - S_{m,j}^{(\ell-m)})^2\]

\[= \sum_{j=1}^{r} [E(S_{m,j}^{(\ell)})^2 + E(S_{m,j}^{(\ell-m)})^2 - 2E(S_{m,j}^{(\ell)})E(S_{m,j}^{(\ell-m)})]\]
\[
\begin{align*}
\text{if } \ell &< k: \\
&= 2m(m-l)\Sigma p_j^2 + 2m - 2m^2\Sigma p_j^2 + (k+m-l)(k+m-l-1)\Sigma p_j^2 + (l-k)(l-k-l)\Sigma p_j^2' \\
&+ 2(k+m-l)(l-k)\Sigma p_j p_j' + m(m-l)\Sigma p_j^2 \\
&+ 2m - 2m(m+k-l)\Sigma p_j^2 - 2m(l-k)\Sigma p_j p_j' \\
\text{if } k+1 \leq \ell < k+m: \\
&\text{if } k+m \leq \ell < k+2m: \\
&= 2m(m-l)\Sigma p_j^2 + 2m - 2m^2\Sigma p_j^2 \\
&\text{if } \ell > k+2m.
\end{align*}
\]

After simplification these expressions become

\[
\begin{align*}
\text{if } \ell &< k: \\
&= 2m(1-p_j^2) \\
&\text{if } k+1 \leq \ell < k+m: \\
&\text{if } k+m \leq \ell < k+2m: \\
&= 2m(1-\Sigma p_j^2) \\
&\text{if } \ell > k+2m.
\end{align*}
\]
Thus as with our previous \( W \) statistics, \( E(W_7) \) depends on \( p \) and \( p' \) only through \( \Sigma p_j^2 \), \( \Sigma p_j p'_j \), and \( \Sigma p'_j^2 \). Moreover \( E(W_7) \) depends only on \( \Sigma p_j^2 \) if \( \ell < k \), only on \( \Sigma p'_j^2 \) if \( \ell \geq k+2m \). \( E(W_7) \) also shows that \( W_7 \) may be expected to increase and then decrease across the trials \( k \) to \( k+2m \).

In the binomial case \( W_6 \) and \( W_7 \) reduce to

\[
W_6(\ell) = \frac{2m(\sum_{i=\ell-m+1}^{\ell} X_i - \sum_{i=\ell-2m+1}^{\ell} X_i)^2}{(2m - \sum_{i=\ell-2m+1}^{\ell} X_i)(\sum_{i=\ell-2m+1}^{\ell} X_i)}
\]

\[
W_7(\ell) = 2(\sum_{i=\ell-m+1}^{\ell} X_i - \sum_{i=\ell-2m+1}^{\ell} X_i)^2.
\]

As in the previous sections \( \hat{k} = \ell_0 \) provides an estimator of \( k \).

4. The Results of a Simulation Study.

An extensive simulation study was undertaken to compare the performance of the 7 \( W \) and 4 \( V \) statistics under a variety of circumstances. Table 1 enumerates 10 of the more interesting cases examined. A total of 300 trials were run for each case with \( k = 150 \) and blocks, \( m \), of 20, 30, 40 and 50. The \( r \) values 3, 5, and 7 were selected as the likely range of application. The distributions were selected such that for some, the change should be easy to
discern while for others, it should be more difficult. They also reflect an assortment of different combinations for $\Sigma p_j^2$, $\Sigma p_j p_j'$, and $\Sigma p_j'$.  

We will comment on each case briefly but first some general observations may be made.

(i) The cases revealed the crucial dependence of the selection of $m$ based on $r$. With $r=3$, $m=20$ or 30 were effective while with $r=7$, $m$ at least 40 and usually 50 was necessary. A practical rule of thumb would be to take $m$ at least 7 to 10 times $r$ if feasible.

(ii) In all cases studied $\text{var}(W_3)$ was much smaller than $\text{var}(W_4)$ making $W_3(\ell)$ far more effective than $W_4(\ell)$ in revealing a pattern of distributional shift as opposed to random variation.

(iii) Similarly in all cases studied $\text{var}(W_2)$ was much smaller than $\text{var}(W_1)$ making $W_2(\ell)$ more effective than $W_1(\ell)$.

(iv) None of the $V$ statistics worked as well as either $W_6(\ell)$ or $W_7(\ell)$ so that they will be dispensed with in the remaining discussion.

(v) $W_6(\ell)$ and $W_7(\ell)$ never failed to respond to a distributional shift but occasionally (in rather erratic data sequences) indicated a false shift.

(vi) Because the $W_3$, $W_4$ and $W_5$ statistics will often be of the order of $10^3$ or $10^4$ logarithms were taken to give a more tractable scale and to more clearly reveal patterns.
<table>
<thead>
<tr>
<th>Case #</th>
<th>$r$</th>
<th>Distribution before change</th>
<th>Distribution after change</th>
<th>$\Sigma p_j^2$, $\Sigma p_j p_j'$, $\Sigma p_j'^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>(.6, .3, .1) (.1, .6, .3)</td>
<td></td>
<td>.46, .27, .46</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>(.5, .3, .2) (.8, .1, .1)</td>
<td></td>
<td>.38, .45, .66</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(.3, .3, .4) (.6, .3, .1)</td>
<td></td>
<td>.34, .31, .46</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>(.4, .4, .2) (.4, .2, .4)</td>
<td></td>
<td>.36, .32, .36</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>(.5, .2, .1, .1, .1) (.1, .5, .2, .1, .1)</td>
<td></td>
<td>.32, .19, .32</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>(.3, .2, .2, .1, .1) (.6, .2, .1, .05, .05)</td>
<td></td>
<td>.22, .255, .415</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>(.2, .2, .2, .2, .2) (.4, .3, .1, .1, .1)</td>
<td></td>
<td>.20, .20, .28</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>(.2, .3, .1, .2, .2) (.2, .1, .3, .2, .2)</td>
<td></td>
<td>.22, .18, .22</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
<td>(.4, .1, .1, .1, .1, .1, .1) (.1, .4, .1, .1, .1, .1, .1)</td>
<td></td>
<td>.22, .13, .22</td>
</tr>
<tr>
<td>10</td>
<td>7</td>
<td>(.2, .2, .2, .1, .1, .1, .1) (.3, .3, .2, .05, .05, .05, .05)</td>
<td></td>
<td>.16, .18, .23</td>
</tr>
</tbody>
</table>

Table 1: A Sampling of Simulation Cases Studied
(vii) $W_5(\ell)$ consistently revealed multiple spikes over \(\ell\) thereby clouding the perception of a distributional change. However in virtually every case considered its absolute peak was within 5 trials of $k+m/2$.

We now turn to the individual cases.

Case 1: This should be an easy shift to detect. Nonetheless $W_1$ and $W_2$ were ineffective until $m=40$ and were only really clear at $m=50$. $W_3$ and $W_4$ were reasonably clear throughout. $W_5$ developed a more sharply defined unique peak with increasing $m$. $W_6$ and $W_7$ were excellent even at $m=20$.

Case 2: Again this should be an easy shift to detect. $W_1$ and $W_2$ were both fairly good at $m=20$, 30. By $m=40$, 50, $W_2$ was excellent and $W_1$ quite good. $W_3$ and $W_4$ were ineffective until $m=40$. $W_5$ was erratic but absolute peaks were always correct. $W_7$ and to a lesser extent $W_6$ were bothered by early idiosyncrasies in the sequence which smoothed out by $m=40$.

Case 3: This shift should be a bit more difficult to detect. $W_1$ was ineffective even at $m=50$ although $W_2$ was good by $m=40$. $W_3$ was better than $W_4$ although neither was really sharp until $m=50$. $W_5$ was fairly clear throughout. $W_6$ and $W_7$ were both good even at $m=20$. 

-37-
Case 4: This is clearly the most difficult to detect of the three-cell cases presented. $W_1$ was ineffective throughout. $W_2$ was only effective at $m=50$. $W_3$ and $W_4$ were quite good particularly by $m=40$. $W_5$ was too erratic to be helpful. $W_6$ and $W_7$ were also ineffective for each $m$.

Case 5: This shift, analogous to case 1, should be easy to detect. $W_1$ was not effective until $m=50$. $W_2$ was better being reasonably clear by $m=40$. $W_3$ was good at $m=40$ and excellent at $m=50$ while $W_4$ never responded. $W_5$ was fairly clear throughout. $W_6$ and $W_7$ were excellent surprisingly even at $m=20$.

Case 6: This shift analogous to case 2 should also be easy to detect. $W_2$ was excellent throughout with $W_1$ effective at $m=40$. $W_3$ and $W_4$ were also ineffective until $m=40$. $W_5$ was too erratic to be helpful. $W_6$ was fairly clear at $m=30$ and quite good at $m=40$, 50, while $W_7$ did not become clear until 50. As in case 2 both $W_6$ and $W_7$ seemed somewhat bothered by early idiosyncrasies in the sequence.

Case 7: This case should be more difficult to detect than the previous two. $W_2$ is excellent by $m=40$ with $W_1$ clear by $m=50$. $W_3$ is sharper than $W_4$ but both are good by $m=40$. $W_5$ is fairly clear throughout. $W_6$ and $W_7$ both detected a false
shift at \( \ell = 100 \) which smoothed out somewhat but not completely by \( m = 50 \).

Case 8: This case analogous to case 4 is clearly the most difficult of the five cell cases presented. \( W_1 \) never responded while \( W_2 \) was quite good at \( m = 50 \). \( W_4 \) was erratic throughout while \( W_3 \) was marginally effective by \( m = 40 \). \( W_5 \) was quite good throughout giving better performance relative to the other statistics than in previous cases. \( W_6 \) and \( W_7 \) were good particularly at \( m = 40, 50 \).

Case 9: With 7 cells, \( m = 20 \) and \( m = 30 \) ought not work well for any of the \( W \)'s. Nonetheless \( W_6 \) and \( W_7 \) were excellent by \( m = 30 \). \( W_1 \) was poor throughout with \( W_2 \) finally becoming reasonably clear by \( m = 50 \). \( W_3 \) and \( W_4 \) were poor throughout while \( W_5 \) was fairly good throughout.

Case 10: This case should be much more difficult than Case 9. Nonetheless \( W_1 \) and \( W_2 \) were both quite good by \( m = 40 \). Also \( W_3 \) and \( W_4 \) were effective after \( m = 40 \). \( W_5 \) was too erratic to be effective. \( W_7 \) was poor throughout with \( W_6 \) a bit better by \( m = 50 \).

In summary from the methods of section 3.1, \( W_2(\ell) \) is clearly the better choice. From the methods of section 3.2, \( W_3(\ell) \) is the best choice and from the methods of section 3.3, \( W_6(\ell) \) is the best choice. Amongst \( W_2 \), \( W_3 \), and \( W_6 \) it is not possible to select an overall best
choice. All three should be effective with m large and all three can be monitored concurrently to mutually confirm a detected distribution change.

Acknowledgments: I would like to thank Herbert Solomon for bringing this problem to my attention and John Mahoney for his development of the computer program used in the simulation study.

References


**DETECTING CHANGE POINTS IN SAMPLING FROM MULTINOMIAL DISTRIBUTIONS**

**AUTHOR(S)**
Alan E. Gelfand

**PERFORMING ORGANIZATION NAME AND ADDRESS**
Department of Statistics
Stanford University
Stanford, CA 94305

**CONTRACT OR GRANT NUMBER(S)**
NO0014-76-C-0475

**REPORT DATE**
March 4, 1980

**NUMBER OF PAGES**
40

**DISTRIBUTION STATEMENT (of the Report)**
Approved for Public Release; Distribution is Unlimited.

This report partially supported under U.S. Army Research Office Grant DAAG29-77-G-0031.

**KEY WORDS**
Multinomial distribution, change point, weighted sums, unweighted sums, central limit theory approaches, departure from centrality approaches, test of homogeneity approaches.

**ABSTRACT**
Please see reverse side.
REPORT 49. 282

DETECTING CHANGE POINTS IN SAMPLING FROM
MULTINOMIAL DISTRIBUTIONS

The question of if and when a change in the underlying cell
frequencies has occurred during the observation of a sequence of
categorical variables is analyzed. Several techniques are developed
that may be loosely grouped into three classifications: (i) Central
Limit Theorem approaches, (ii) Departure from Centrality approaches
and (iii) Test of Homogeneity approaches. The approaches monitor the
sequence either at every trial or at every k-th trial. All approaches
construct statistics which are functions of sequential sums either
weighted or unweighted. The behavior of these sums and the statistics
developed from them are discussed in detail. A large scale simulation
study is discussed in an attempt to assess the performance of the
approaches. Three approaches emerge as most promising but a clear
choice among these is not possible at present.