LONG UNIMODAL SUBSEQUENCES: A PROBLEM OF F.R.K. CHUNG

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I. Introduction.

Let \( p \) denote a permutation of \( \{1, 2, \ldots, n\} \) and call
\[ \{a_1 < a_2 < \ldots < a_t\} \] a unimodal subsequence provided there is a
\( j \) such that

\[
p(a_1) < p(a_2) < \ldots < p(a_j) > p(a_{j+1}) > \ldots > p(a_t)
\]
or

\[
p(a_1) > p(a_2) > \ldots > p(a_j) < p(a_{j+1}) < \ldots < p(a_t).
\]

Let \( \ell(n) \) denote the expected length of the longest unimodal subsequence of a randomly permuted subsequence i.e. \( \ell(n) = \sum p(p)/n! \), where \( p(p) \) denotes the length of the longest unimodal subsequence of the
permutation \( p \).

F.R.K. Chung [1] conjectured that

\[
\lim_{n \to \infty} \frac{\ell(n)}{\sqrt{n}} = C \text{ exists}.
\]

The point of this note is to prove Chung's conjecture and show \( C = 2 \sqrt{2} \).
Actually, Chung's conjecture is slightly more general than this introductory version, and this more general conjecture is obtained by the same proof.
II. Proof of F.R.K. Chung's Conjecture.

Suppose \((X_i, Y_i), 1 \leq i < \infty\) are independent and uniformly distributed in \([0,1]^2\). For any \(A \subset [0,1]\) let

\[
I_n(A) = \max \{k: Y_{i_1} < Y_{i_2} < \ldots < Y_{i_k} \text{ with } X_{i_1} < X_{i_2} < \ldots < X_{i_k}, X_{i_j} \in A \text{ and } i_j \in [1,\ldots,n]\}
\]

and

\[
D_n(A) = \max \{k: Y_{i_1} > Y_{i_2} > \ldots > Y_{i_k} \text{ with } X_{i_1} < X_{i_2} < \ldots < X_{i_k}, X_{i_j} \in A \text{ and } i_j \in [1,2,\ldots,n]\}.
\]

Next set

\[
U_n = \max_{0 \leq t \leq 1} \{\max(I_n([0,t]) + D_n([t,1]), D_n([0,t]) + I_n([t,1]))\}.
\]

The desired proof will be obtained by applying known results to the random variable \(U_n\). To begin it is easy to check that

\[
EU_n = \lambda(n).
\]
Next we note that by the work of Hammersley [2] and Kesten [3] that almost surely and in $L^1$ we have the limits

\[
\lim_{n \to \infty} \frac{I_n(A)}{\sqrt{n}} = C \sqrt{\lambda(A)} \quad \text{and} \quad \lim_{n \to \infty} \frac{D_n(A)}{\sqrt{n}} = C \sqrt{\lambda(A)}
\]

where $\lambda(A)$ is the Lebesgue measure of $A \subset [0,1]$, and $C$ is a universal constant. The work of Logan and Shepp [9] and Vershik and Kerov [5] established that $C = 2$.

For any $N$ and $1 \leq k \leq N$ we define

\[
U_n^N(k) = \max\{I_n(0,k/n) + I_n((k-1)/N,1), D_n(0,k/N) + D_n((k-1)/N,1)\}
\]

and

\[
U_n^N = \max_{1 \leq k \leq N} U_n^N(k).
\]

Clearly, for all $N$, $U_n \leq U_n^N$ and by the above mentioned limit results we have

\[
\lim_{n \to \infty} \frac{U_n^N}{\sqrt{n}} = 2 \max_{1 \leq k \leq N} \left( \sqrt{k/N} + \sqrt{(N-k+1)/N} \right),
\]

where the limit is almost sure and in $L^1$. The arbitrariness of $N$ then shows

\[
\limsup_{n \to \infty} \frac{U_n}{\sqrt{n}} \leq 2 \max_{0 < t < 1} \left( \sqrt{t} + \sqrt{1-t} \right) = 2\sqrt{2} \quad \text{a.s.},
\]

so by Fatou's lemma we get

\[
\limsup_{n \to \infty} \frac{\mathcal{L}(n)}{\sqrt{n}} \leq 2\sqrt{2}.
\]

For the opposite direction note the trivial bound

\[
U_n \geq I_n\left(\left[0,\frac{1}{2}\right)\right] + D_n\left(\left[\frac{1}{2},1\right]\right)
\]

so

\[
\liminf_{n \to \infty} \frac{\mathcal{L}(n)}{\sqrt{n}} \geq \liminf_{n \to \infty} E(I_n\left[0,\frac{1}{2}\right] + D_n\left[\frac{1}{2},1\right]) = 2\sqrt{2}
\]

which completes the proof.
III. The Generalization.

Instead of allowing the subsequence to make "one turn" as in the unimodal case, one can consider subsequences which make \( k \) turns. Explicitly, let \( L_k(n) \) be the expected length of the longest subsequence \( S \) of a random permutation with the following property:

- \( S \) can be decomposed into \( k+1 \) segments which are monotone and which alternate between increasing and decreasing.

The method of the preceding section can be used easily to show

\[
\lim_\limits{n \to \infty} \frac{L_k(n)}{\sqrt{n}} = 2\sqrt{k+1};
\]

all one has to do is define the proper analogue \( U_n(k) \) of \( U_n \) and argue as before. One should also note that the preceding bounds also prove the almost sure and \( L^1 \) convergence of \( U_n(k)/\sqrt{n} \) to \( 2\sqrt{k+1} \).
References


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**Abstract:**
Let $\ell(n)$ be the expected length of the longest unimodal subsequence of a random permutation. It is proved here that $\ell(n)/\sqrt{n}$ converges to $2\sqrt{2}$. This settles a conjecture of F.R.K. Chung.