APPLIED MATHEMATICS AND STATISTICS LABORATORY

STANFORD UNIVERSITY
CALIFORNIA

ON MINIMIZING AND MAXIMIZING A CERTAIN INTEGRAL
WITH STATISTICAL APPLICATIONS

By
JAGDISH SHARAN RUSTAGI

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Section 1. Summary.

Let $\mathcal{A}$ be the class of cumulative distribution functions (c.d.f:s) $F$ defined over the closed interval $[-X,X]$ of the real line with their first two moments given. Let $S = \{(x,y); -X \leq x \leq X, 0 \leq y \leq 1\}$ be a closed and bounded region. Suppose further that $\varphi$ is a bounded continuous function over the region $S$ and strictly convex and twice differentiable in $y$. We consider here the problem of maximizing and minimizing

$$
\int_{-X}^{X} \varphi(x,F(x)) \, dx
$$

over all c.d.f:s $F$ belonging to the class $\mathcal{A}$.

The minimizing problem is considered in Part I of the paper. We prove the existence of the minimizing and maximizing c.d.f:s and uniqueness of the minimizing c.d.f. and give a characterization of the minimizing c.d.f. Interestingly enough in some cases, the results do not contain $X$, particularly when $X$ is sufficiently large, and hence are the same for c.d.f:s defined over the whole real line.

An important particular case of the minimum problem is the one in which $\varphi(x,y)$ is a function of $y$ alone. This is discussed in Section 5, and the explicit form of the minimizing c.d.f is given.
The maximum problem is considered in Part II of the paper. We find that the maximizing c.d.f. in the above problem is at most a three-point distribution, i.e., a discrete distribution concentrating the whole mass at just three or fewer points of the interval $-X \leq x \leq X$. In some special cases, the maximizing c.d.f. can be further reduced to a two-point distribution.

The results of this paper have many applications, some of which are discussed here. Lower and upper bounds for the expectation of the range and for the expectation of the extreme observations with the above-mentioned restrictions on the c.d.f:s are quite useful to the statistician. We have utilized the results of this paper in deriving the maximizing and minimizing c.d.f:s for the above problems. It seems plausible that our results might be used in deriving many classical inequalities particularly of the Tchebycheff type and in obtaining the bounds for the efficiency of some important non-parametric tests.

Section 2. Introduction.

Various writers, to name a few such as Placket [9], Moriguie [14], Chernoff and Reiter [2], Gumbel [10] and David and Hartley [1], have recently considered problems of the following nature: Minimize and Maximize an expectation of a function of a real random variable subject to side conditions on its mean and variance, and find the minimizing and maximizing c.d.f:s. Karlin [3,4], Hoeffding [11] and Brunk, Ewing and Utz [6] have also considered related maximum and minimum problems.

Karlin [3] has discussed the following problem. Let $f$ and $g$ be
continuous functions defined on $0 \leq t \leq T$, and let $g(t)$ be normalized so that

$$
\int_{0}^{T} g(t) \, dt = 1.
$$

From the class of integrable functions $h(t)$ subject to the restrictions

$$
0 \leq h(t) \leq M,
$$

$$
\int_{0}^{T} h(t)g(t) \, dt = \eta, \quad \eta \leq M,
$$

it is required to select the function $h_0(t)$ for which

$$
\int_{0}^{T} h_0(t)f(t) \, dt = \max_{h} \int_{0}^{T} h(t)f(t) \, dt.
$$

The solution to the above problem exists and is characterized by the following

$$
h_0(t) = 0 \quad \text{if} \quad \frac{f(t)}{g(t)} < k
$$

$$
h_0(t) = M \quad \text{if} \quad \frac{f(t)}{g(t)} > k
$$

$$
0 \leq h_0(t) \leq M \quad \text{if} \quad \frac{f(t)}{g(t)} = k
$$

where $k$ is so determined that the restrictions are satisfied. A more general problem of the same type has also been discussed by Karlin [4]. We shall utilize Karlin's technique of solving the above problem in the solution of our problem.

The problem considered by Chernoff and Reiter [2] is that of
minimizing and maximizing the expectation of a given function \( g(x) \) subject to side-restrictions on the mean and variance of the random variable \( x \).

It is proved there that the minimizing c.d.f. exists and is a two-point distribution. It is also shown that the maximizing c.d.f. does not exist. The method they have used, gives a sort of necessary condition of maximizing the expectation. We compare this technique with that of ours in Section 7, and find that their method applied to our problem gives a result similar to ours.

Rubin and Issacson [13] have also considered the same problem as in [2] but with additional assumptions on the function \( g(x) \) and hence obtain a lower bound on the expectation of \( g(x) \), which is greater than that obtained in [2].

It is really for the following application that our problem has relevance. Let \( x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n \) be \( n \) ordered, independent observations from a population with c.d.f. \( F(x) \) having standard deviation \( \sigma \). Let \( w_n = x_n - x_1 \) denote the sample range. The problem considered by Plackett [3] is that of establishing universal bounds for \( \frac{E(w_n)}{\sigma} \) on the lines similar to Tchebycheff inequalities for moments. Moriguice [4] considered an equivalent case of establishing bounds for the extreme value. Both Plackett [3] and Moriguice [4] assumed that the underlying distribution is symmetrical.

Gumbel [10] uses a variational method to derive maximizing c.d.f:s in maximizing \( E(w_n) \) and \( E(x_n) \) over c.d.f:s with given mean and variance. An assumption of continuity has been made and a sort of sufficiency condition for the maximum has been given. It seems that
the assumption of continuity is unnecessary. We compare, in a later section, our technique with that of Gumbel and obtain a result similar to ours for our problem.

David and Hartely [1] consider the same problem of maximizing $E(x_n)$ as that of Gumbel. They also give the solution to the problem of minimizing and maximizing $E(w_n)$ but assume an additional restriction on the c.d.f., besides prescribed mean and variance, that the c.d.f. is defined on a bounded range. The ingenious method used by them to prove the sufficiency of the condition for the problem of maximizing $E(x_n)$ seems unnecessary, as the condition which seems to be a necessary condition for the maximum given by them, is also a sufficiency condition. They prove that the minimizing c.d.f. for $E(w_n)$ is a two-point distribution. Their intermediary result that the minimizing c.d.f. is at most a three-point distribution, easily applies in more general cases as we shall see in the last section. The process of reduction from three-point to two-point distribution, seems to be very complicated. We suspect that there is a general reason for this reduction, but the specific problem is so complex that we haven't any result in this regard.

We do not make any assumption of continuity of c.d.f.'s and treat the more general bounded case. The aim is to give methods which are widely applicable, even though sometimes somewhat ponderous. The problems treated in [1] will be derived as special cases of the results of our paper.

Section 3. Statement of the Problem and Existence of its Solution.

Let $S = \{(x,y): -X \leq x \leq X, 0 \leq y \leq 1\}$ where $X$ is already specified.
Let \( \varphi \) be a function defined on the closed and bounded region \( S \) such that it satisfies the following conditions:

1. \( \varphi \) is bounded and continuous in \( S \).
2. \( \varphi \) is strictly convex and twice differentiable in \( y \).

We shall minimize (maximize)

\[
(3.1) \quad I(F) = \int_{-X}^{X} \varphi(x,F(x)) \, dx
\]

over all \( F \in \mathcal{A} \) where \( \mathcal{A} \) is the class of all admissible c.d.f:s, i.e., c.d.f:s satisfying the following conditions:

\[
\left\{ \begin{array}{l}
\int_{0}^{1} x \, dF = \mu_1 \\
\int_{0}^{1} x^2 \, dF = \mu_2
\end{array} \right.
\]

\[
(3.2)
\]

\[
F(x) = \begin{cases} 
0 & \text{for } x < -X \\
1 & \text{for } x > X
\end{cases}
\]

Here \( \mu_1 \) and \( \mu_2 \) are such that \( \mu_2 > \mu_1^2 \) and \( \mu_2 < X^2 \). In this case there exist c.d.f:s satisfying (3.2) and hence the class \( \mathcal{A} \) is non-null. However, if \( \mu_2 = \mu_1^2 \) or \( \mu_2 = X^2 \) there is a unique c.d.f. which satisfies (3.2) and hence the problem of minimizing (maximizing) (3.1) subject to restrictions (3.2) becomes a trivial one. Obviously the conditions \( \mu_2 < \mu_1^2 \) or \( \mu_2 > X^2 \) can not be satisfied by any c.d.f. \( F \) on \([-X,X]\).
Integrating by parts the integrals in (3.2), the restrictions become

\[
\begin{align*}
\int_{-X}^{X} F(x) \, dx &= X - \mu_1 \\
\int_{-X}^{X} x F(x) \, dx &= \frac{x^2 - \mu^2}{2}
\end{align*}
\]

(3.3)

It must be remarked here that in integrating (3.2) by parts, continuity of \( F(x) \) is not needed.

We shall first show that an admissible minimizing (maximizing) c.d.f. exists. Let \( \mathcal{F} \) be the class of all c.d.f:s defined on \([-X,X]\). If \( F_1 \) and \( F_2 \) belong to \( \mathcal{F} \), then it is well-known that for a real number \( \lambda \) such that \( 0 \leq \lambda \leq 1 \), \( \lambda F_1 + (1 - \lambda) F_2 \) is also a c.d.f. on \([-X,X]\) and hence belongs to \( \mathcal{F} \), i.e., the set \( \mathcal{F} \) is convex.

Let \( \{F_n\} \) be a sequence of c.d.f:s. We say that \( F_n \) converges to \( F \) in distribution if \( F_n(a) \) converges to \( F(a) \) for every point \( a \) of continuity of \( F \). By Helley's theorem we know that every infinite set of c.d.f:s has a subsequence which converges in distribution. The above results are summarized in the following lemma [7].

\textbf{Lemma 3.1.} \( \mathcal{F} \) is convex and compact.*

---

* Helley's theorem actually implies sequential compactness but here compactness and sequential compactness are equivalent.
Lemma 3.2. The set $\Gamma$ of points
$$
\begin{pmatrix}
\int_{-X}^{X} \varphi(x,F(x)) \, dx, & \int_{-X}^{X} F(x) \, dx, & \int_{-X}^{X} x \, F(x) \, dx
\end{pmatrix}
$$
for $F \in \mathcal{H}$, is closed and bounded set of points in 3-space.

Proof: Define a transformation
$$
T: \mathcal{H} \to \mathbb{R}^3
$$
where $\mathbb{R}^3$ is the three-dimensional Cartesian space, such that
$$
T \circ F = \begin{pmatrix}
\int_{-X}^{X} \varphi(x,F(x)) \, dx, & \int_{-X}^{X} F(x) \, dx, & \int_{-X}^{X} x \, F(x) \, dx
\end{pmatrix}
$$

Obviously $T$ is bounded as $\varphi$ and $F$ are bounded. Now as the transformation $T_1$ given by
$$
T_1 \circ F = \begin{pmatrix}
\int_{-X}^{X} F(x) \, dx, & \int_{-X}^{X} x \, F(x) \, dx
\end{pmatrix}
$$
is linear in $F$, it is continuous in $F$. Again as $\varphi(x,y)$ is continuous in $y$, $\int_{-X}^{X} \varphi(x,F(x)) \, dx$ is also continuous in $F$ by the Helley-Bray Theorem. Hence $T$ is a continuous transformation. But a continuous transformation maps a compact set into a closed and bounded set [8]. Therefore, the set $\Gamma$ is closed and bounded in 3-space.
The restrictions (3.3) define a cross-section $\Gamma_1$ of a closed and bounded set $\Gamma$ and hence $\Gamma_1$ is also closed and bounded. Since $\Gamma_1$ is closed and bounded, the minimizing and maximizing points exist and are given by boundary points of $\Gamma_1$ so long as $\Gamma_1$ is non-null. But $\Gamma_1$ is non-null since, as we have seen before, $A$ is non-null. Hence the minimizing and maximizing admissible c.d.f:s exist.

Remark: The existence of the minimizing and maximizing c.d.f:s depends only on the continuity of the function $\varphi$ and not on its convexity which will be used later on.
Part I
Minimum Problem

Section 4. Reduction of the Minimum Problem to Subsidiary Problems and Uniqueness of its Solution.

In this section we first prove the uniqueness of the minimizing c.d.f. using the property of strict convexity of the function \( \Phi \) in its second argument. In characterizing the solution of the minimizing problem, we use Karlin's method [3] to reduce the main problem of minimizing the integral (3.1) over the class \( A \) of admissible c.d.f.s, to a subsidiary problem of minimizing an integral of a related function over all c.d.f.s \( \int \Phi \). This reduction together with the uniqueness of the minimizing c.d.f. gives us a characterization of the minimizing c.d.f. which we give in the next section.

We shall first prove the uniqueness of the minimizing admissible c.d.f. by the following lemma:

**Lemma 4.1.** There is a unique c.d.f. \( F_0 \) which minimizes (3.1) subject to the side conditions (3.3) when \( \Phi(x,y) \) is strictly convex in \( y \).

**Proof:** Suppose the solution is not unique. Let \( F_0(x) \) and \( F_1(x) \) be two distinct admissible c.d.f.s which minimize (3.1). Let

\[
M = \min_{F \in A} \int_{-X}^{X} \Phi(x,F(x)) \, dx.
\]
As $\varphi$ is strictly convex in $y$, for $0 \leq \lambda \leq 1$,

\[
\int_{-X}^{X} \varphi(x, \lambda F_0(x) + (1 - \lambda) F_1(x)) \, dx < \int_{-X}^{X} \varphi(x, F_0(x)) \, dx + (1 - \lambda) \int_{-X}^{X} \varphi(x, F_1(x)) \, dx
\]

\[= \lambda M + (1 - \lambda) M = M.
\]

But $M$ is the minimum, and hence we have a contradiction.

We shall now prove the following lemmas, with the help of which we shall reduce the main problem to a simpler problem.

**Lemma 4.2.** $F_0(x)$ minimizes (3.1) if and only if

\[
(4.1) \quad \int_{-X}^{X} \left. \frac{\partial \varphi}{\partial y} (x,y) \right|_{y = F_0(x)} F(x) \, dx < \int_{-X}^{X} \left. \frac{\partial \varphi}{\partial y} (x,y) \right|_{y = F_0(x)} F_0(x) \, dx
\]

for all $F \in \mathcal{A}$.

**Proof:** For any other admissible c.d.f. $F(x)$, define

\[
I(\lambda) = \int_{-X}^{X} \varphi(x, \lambda F_0(x) + (1 - \lambda) F(x)) \, dx \quad 0 \leq \lambda \leq 1.
\]
As \( \varphi \) is twice differentiable in \( y \), \( \frac{\partial \varphi}{\partial y} \) exists and is continuous in \( y \), and hence \( I(\lambda) \) is differentiable and is given by

\[
I'(\lambda) = \int_{-X}^{X} \frac{\partial \varphi}{\partial y}(x, \lambda F_0(x) + (1 - \lambda) F(x))(F_0(x) - F(x)) \, dx.
\]

Since \( \varphi \) is strictly convex in \( y \), it follows very easily that \( I(\lambda) \) is a strictly convex function of \( \lambda \). If \( F_0(x) \) minimizes (3.1) then \( I(\lambda) \) achieves its minimum at \( \lambda = 1 \) and this is possible if and only if

\[
I'(\lambda) \bigg|_{\lambda = 1} \leq 0, \text{ i.e.,}
\]

\[
\int_{-X}^{X} \frac{\partial \varphi}{\partial y}(x, y) \bigg|_{y = F_0(x)} (F_0(x) - F(x)) \, dx \leq 0 \quad \text{or,}
\]

\[
\int_{-X}^{X} \frac{\partial \varphi}{\partial y}(x, y) \bigg|_{y = F_0(x)} F_0(x) \, dx \leq \int_{-X}^{X} \frac{\partial \varphi}{\partial y}(x, y) \bigg|_{y = F_0(x)} F(x) \, dx.
\]

Let (4.1) hold true. Then we have \( I'(\lambda) \bigg|_{\lambda = 1} \leq 0 \), and hence by the strict convexity of \( I(\lambda) \) we have,

\[ I(1) < I(0) \]

or,

\[
\int_{-X}^{X} \varphi(x, F_0(x)) \, dx < \int_{-X}^{X} \varphi(x, F(x)) \, dx,
\]
i.e., $P_0(x)$ minimizes (3.1). This proves the lemma.

Let us use the following notation:

$$I_{F_0}(F) = \int_X^{X} \frac{\partial F}{\partial y} (x,F_0(x)) F(x) \, dx.$$ 

With the help of the above lemma, we find that the problem $P_1$ of minimizing $I(F)$ over all admissible c.d.f.s, is related to the problem $P_{2F_0}$ of finding an admissible $F(x)$ which minimizes $I_{F_0}(F)$. In fact, we are interested in finding an $F_0$ such that $F_0$ is a solution of $P_{2F_0}$. This looks like a complicated problem but we now have a problem linear in $F$ which is relatively easy to deal with.

Because $P_1$ has a unique solution, Lemma 4.2 implies that there is one and only one $F_0$ such that $F_0$ solves $P_{2F_0}$. This, however, does not mean that $P_{2F_0}$ has a unique solution.

Let $T: \mathcal{F} \rightarrow \Gamma_2$ be a transformation such that given by

$$T \cdot F = \left( \int_X^{X} \frac{\partial F}{\partial y} (x,F_0(x)) F(x) \, dx, \int_X^{X} F(x) \, dx, \int_X^{X} x F(x) \, dx \right).$$

Obviously $T$ is bounded and is linear and hence continuous in $F$. But as $\mathcal{F}$ is convex and compact in the topology of convergence in distribution by Lemma 3.1, the transformation $T$ maps the convex and compact set $\mathcal{F}$ into a convex and compact set $\Gamma_2$, and hence we have the following result.

**Lemma 4.3.** $\Gamma_2$ is a convex, closed and bounded set in three dimensions.

Solving $P_{2F_0}$ corresponds to finding the minimum among all points of
\[ \Gamma_2 \text{ for which} \]
\[ \int_{-X}^{X} F(x) \, dx = X - \mu_1 \]
\[ \int_{-X}^{X} x F(x) \, dx = \frac{X^2 - \mu_2}{2} \]

and this will be a boundary point of the set \( \Gamma_2 \).

Suppose \( F_0 \) solves \( P_{2F_0} \). Then \( F_0 \) corresponds to a boundary point of \( \Gamma_2 \) and there is a supporting hyperplane of \( \Gamma_2 \) at the minimum point \( \gamma = (u_0, v_0, w_0) \), i.e., for some \( \eta_0, \eta_1, \eta_2 \) and \( \eta_3 (\eta_0, \eta_1, \eta_2 \text{ not all zero}) \)

\[ (4.2) \quad \eta_0 u_0 + \eta_1 v_0 + \eta_2 w_0 + \eta_3 = 0 \]

and

\[ (4.3) \quad \eta_0 u + \eta_1 v + \eta_2 w + \eta_3 \geq 0 \]

for all other points \( (u, v, w) \) belonging to \( \Gamma_2 \), where

\[ u = \int_{-X}^{X} \frac{\partial F}{\partial y} (x, F_0(x)) \, F(x) \, dx , \]

\[ (4.4) \]

\[ v = \int_{-X}^{X} F(x) \, dx , \quad w = \int_{-X}^{X} x \, F(x) \, dx \]

therefore,
\[ (4.5) \quad \eta_0 (u-u_0) + \eta_1 (v-v_0) + \eta_2 (w-w_0) \geq 0. \]

We shall see below that \( \eta_0 \) can be taken positive and hence can be normalized so as to be equal to one. Therefore, by taking \( \eta_0 = 1 \) in (4.5) we have

\[
\int_{-X}^{X} \left[ \frac{\partial F}{\partial y} (x, F_0(x)) + \eta_1 + \eta_2 \right] F(x) \, dx \\
\geq \int_{-X}^{X} \left[ \frac{\partial F}{\partial y} (x, F_0(x)) + \eta_1 + \eta_2 \right] F_0(x) \, dx.
\]

Hence \( F_0(x) \) minimizes

\[ (4.6) \quad \int_{-X}^{X} \left[ \frac{\partial F}{\partial y} (x, F_0(x)) + \eta_1 + \eta_2 \right] F(x) \, dx \]

among the class \( \mathcal{F} \) of all c.d.f.s on \([-X, X]\).

Conversely, if \( F_0(x) \in \mathcal{A} \) minimizes (4.6), we have, retracing the steps, that \( (u-u_0) + \eta_1 (v-v_0) + \eta_2 (w-w_0) \geq 0 \). Suppose \( F(x) \) is admissible and hence \( v = v_0 \) and \( w = w_0 \) and hence \( u-u_0 \geq 0 \), i.e.,

\[
\int_{-X}^{X} \frac{\partial F}{\partial y} (x, F_0(x)) F(x) \, dx \\
\geq \int_{-X}^{X} \frac{\partial F}{\partial y} (x, F_0(x)) F_0(x) \, dx.
\]
In other words, \( F_0 \) minimizes \( I_{F_0} (F) \) over all admissible c.d.f:s \( \mathcal{A} \).

We shall now show that \( \eta_0 \) can be taken positive. Let

\[
\Gamma^* = \{ (u^*, v, w): u^* \geq u, (u, v, w) \in \Gamma_2 \}.
\]

Then \( \Gamma^*_2 \) is obviously convex and \( \Gamma_2 \subseteq \Gamma^*_2 \). \( u_0 \) is the minimum of \( u \) subject to the conditions that \( v = v_0 \) and \( w = w_0 \). This implies that \( (u_0, v_0, w_0) \) is also a minimum point of \( \Gamma^*_2 \) and hence is its boundary point. Hence there is an \( (\eta_0, \eta_1, \eta_2) \neq (0, 0, 0) \) such that

\[
(4.7) \quad \eta_0 (u^* - u_0) + \eta_1 (v - v_0) + \eta_2 (w - w_0) \geq 0
\]

for points \( (u^*, v, w) \) belonging to \( \Gamma^*_2 \). Hence

\[
\eta_0 (u - u_0) + \eta_1 (v - v_0) + \eta_2 (w - w_0) \geq 0
\]

for \( (u, v, w) \in \Gamma_2 \). Suppose \( \eta_0 = 0 \). Then we have

\[
\eta_1 (v - v_0) + \eta_2 (w - w_0) \geq 0
\]

or

\[
\int_{-X}^{X} (\eta_1 + \eta_2 x) F(x) \, dx \geq \int_{-X}^{X} (\eta_1 + \eta_2 x) F_0(x) \, dx,
\]

i.e., \( F_0 \) minimizes \( \int_{-X}^{X} (\eta_1 + \eta_2 x) F(x) \, dx \) over all \( F \in \mathcal{F} \). Now \( \eta_1 + \eta_2 x \) is either non-decreasing or non-increasing according as \( \eta_2 \geq 0 \).
or \( \eta_2 \leq 0 \) and hence the unique minimizing c.d.f. of the above integral is a two-point distribution with its total mass concentrated at \(-X\) and \(X\) so that \( \mu_2 = X^2 \). But such a c.d.f. is not admissible and hence there is a contradiction.

It is easily seen now that \( \eta_0 \) is not negative. Suppose \( \eta_0 \) is negative. Consider then a point \((u_0 + h, v_0, w_0) \in \Gamma_2^*\) for some \( h > 0 \), so that from (4.7) we obtain

\[
\eta_0 h \geq 0
\]

which is again a contradiction. Hence \( \eta_0 \) is positive.

Remark: Another way to show that \( \eta_0 \neq 0 \) would be as follows. \( \eta_0 = 0 \) corresponds to boundary points of the set \( \Gamma_2 \) where the supporting hyperplanes are parallel to the \( u \)-axis and hence \((v_0, w_0)\) corresponds to the boundary of the projection \( \Gamma_2 \) on the \((v, w)\) plane. But the conditions on the first two moments are such that the given point \((v_0, w_0)\) will be interior to the projection set and hence \( \eta_0 \neq 0 \).

The previous argument applies only in the special case of first two moments of \( F(x) \) being given. In general when more moments are specified, the latter argument will apply if we impose conditions on the given moments such that the given point is interior to the moment space which is analogous to the projection of the set \( \Gamma_2 \). It easily follows then that \( \eta_0 > 0 \) in general.

When \( n \) side conditions, e.g.,

\[
\int_{-X}^{X} x^i \, dF(x) = \mu_i, \quad i = 1, 2, \ldots, n
\]
are given with the condition that the quadratic forms

$$\sum_{j,k=0}^{m} \mu_{j+k} \alpha_j \alpha_k \quad \text{and} \quad \sum_{j,k=0}^{m-1} (\mu_{j+k+1} - \mu_{j+k+2}) \beta_j \beta_k$$

when \( n = 2m \); or

$$\sum_{j,k=0}^{m-1} \mu_{j+k+1} \alpha_j \alpha_k \quad \text{and} \quad \sum_{j,k=0}^{m-1} (\mu_{j+k} - \mu_{j+k+1}) \beta_j \beta_k$$

when \( n = 2m+1 \), are positive definite, then it is known (Theorem 16.2 [4]) that the point \( (\mu_1, \mu_2, \ldots, \mu_n) \) is an interior point of the moment space. Hence if the above conditions are satisfied, \( \eta_0 \) will be positive.

Let

$$I_{F_{\eta_1 \eta_2}}(F) = \int_X \left[ \frac{\partial}{\partial y} (x, F_0(x)) + \eta_1 + \eta_2 x \right] F(x) \, dx$$

and let the problem \( F_{\eta_1 \eta_2} \) be that of finding the minimum of

$$I_{F_{\eta_1 \eta_2}}(F) \quad \text{over all} \quad F(x) \in \mathcal{F}.$$

It will be appropriate to mention here that Gumbel's variational method [10] of solving a similar problem leads to a similar result, which is the sufficiency condition for the problem with the difference that the additional restriction of the continuity of the c.d.f. \( F(x) \) has been assumed. We shall discuss this topic further in a later sector.

Recapitulating what we have established before, we have the following problems.
$P_1$: Find a c.d.f. $G \in \mathcal{A}$ such that

$$I(G) = \min_{F \in \mathcal{A}} I(F) = \min_{F \in \mathcal{A}} \int_{-X}^{X} \varphi(x,F(x)) \, dx.$$ 

Lemmas 3.2 and 4.1 state that the solution of $P_1$ exists and is unique. Lemma 4.2 states that $F_0(x)$ solves $P_1$ if and only if $F_0(x)$ solves the problem $P_{2F_0}$.

$P_{2F_0}$: Find $G \in \mathcal{A}$ such that

$$I_{F_0}(G) = \min_{F \in \mathcal{A}} I_{F_0}(F) = \min_{F \in \mathcal{A}} \int_{-X}^{X} \frac{\partial F_0}{\partial y}(x,F_0(x)) F(x) \, dx.$$ 

Finally, we have just established the following result:

**Lemma 4.4.** $F_0$ solves $P_{2F_0}$ if $F_0$ is an admissible c.d.f. which solves $P_{3F_0\eta_1 \eta_2}$ and any $F_0$ which solves $P_{2F_0}$ solves $P_{3F_0\eta_1 \eta_2}$ for some $\eta_1$ and $\eta_2$.

$P_{3F_0\eta_1 \eta_2}$: Find $G \in \mathcal{F}$ such that

$$I_{F_0\eta_1 \eta_2}(G) = \min_{F \in \mathcal{F}} I_{F_0\eta_1 \eta_2}(F)$$

$$= \min_{F \in \mathcal{F}} \int_{-X}^{X} \left[ \frac{\partial F_0}{\partial y}(x,F_0(x)) + \eta_1 + \eta_2 x \right] F(x) \, dx$$

$$= \min_{F \in \mathcal{F}} \int_{-X}^{X} \left[ \frac{\partial F_0}{\partial y}(x,F_0(x)) + \eta_1 + \eta_2 x \right] F(x) \, dx.$$
Section 5. **Characterization of the Solution.**

In this section we characterize the solution of the minimum problem in terms of \( F_{\eta_1 \eta_2} (x) \) which is that value of \( y \) for which

\[
\frac{\partial \varphi}{\partial y} (x,y) + \eta_1 + \eta_2 x = 0.
\]

Let \( A(x) = B_{\eta_1 \eta_2} (x,F_0(x)) = \frac{\partial \varphi}{\partial y} (x,F_0(x)) + \eta_1 + \eta_2 x \). Since \( \frac{\partial \varphi}{\partial y} (x,y) \) is continuous in \( y \), \( A(x) \) can have a discontinuity only if \( F_0(x) \) has a jump. But as \( \frac{\partial \varphi}{\partial y}(x,y) \) is increasing in \( y \), the discontinuities of \( A(x) \) are upward jumps. Also since \( \varphi \) is continuous in the region

\[ S = \{(x,y): -X \leq x \leq X, 0 \leq y \leq 1\}, \]

\( A(x) \) is bounded in \( S \). We then have the following theorems which will characterize the solution of our problems.

**Theorem 5.1.** If \( F_0 \) solves \( F_{\eta_1 \eta_2} \), the set \( \{ x: A(x) \neq 0, -X < x < X \} \) has \( F_0 \)-measure zero.

**Proof:** Consider the set \( S_p = \{ x: A(x) > 0, -X < x < X \} \). It is a denumerable union of intervals \( [x_1, x_2] \). We shall show that

\[
F_0(x_2) = F_0(x_1)
\]

and therefore, the interval \( [x_1, x_2] \) has \( F_0 \)-measure zero. Suppose this were not the case. Then as

\[
\int_{x_1}^{x_2} A(x) F_0(x_1) \, dx < \int_{x_1}^{x_2} A(x) F_0(x) \, dx,
\]
there is a contradiction. Consequently, $S_p$ has $F_o$-measure zero.

Consider now the set $S_q = \{ x: A(x) < 0, -X < x < X \}$. Because all discontinuities are upward jumps, $S_q$ is an open set. Hence $S_q$ is denumerable union of intervals $(x_1, x_2)$. Then we shall see that $F_o(x_2) = F_o(x_1) = 0$ and that the $F_o$-measure of the interval $(x_1, x_2)$ is zero. We also prove this by contradiction, as otherwise

$$\int_{x_1}^{x_2} A(x) F_o(x_2) \, dx < \int_{x_1}^{x_2} A(x) F_o(x) \, dx .$$

Since $S_q$ is a denumerable union of such intervals, $S_q$ has also $F_o$-measure zero.

Hence the above arguments show that the set $\{ x: A(x) \neq 0, -X < x < X \}$ has $F_o$-measure zero.

Remarks: 1. It is easy to see that if $A(-X) > 0$, then $F_o(-X) = 0$ and if $A(X) < 0$, $F_o$ is continuous at $X$.

2. The following corollary shows that the integral of $A(x)$ is zero over intervals on which $F$ is constant.

**Corollary:** If $F_o(x)$ be such that $F_o(x) = c$, $0 < c < 1$ for $a_1 < x < b_1$ and $F_o(x) < c$ for $x < a_1$, $F_o(x) > c$ for $x > b_1$, then

$$\int_{a_1}^{b_1} A(x) \, dx = 0 .$$
Proof: Suppose \( \int_{a_1}^{b_1} A(x) \, dx < 0 \) and \( b_1 < X \). Replace \( F_\circ(x) \) on the interval \( [a_1, b_1 + \delta) \) for any small number \( \delta > 0 \), by the constant quantity \( F_\circ(b_1 + \delta) \). Let \( \psi \) be the increase in \( I_{F_\circ \eta_1 \eta_2} (F) \) due to this replacement. Then

\[
\psi = \int_{a_1}^{b_1 + \delta} A(x) F_\circ(b_1 + \delta) \, dx - \int_{a_1}^{b_1 + \delta} A(x) F_\circ(x) \, dx
\]

\[
= (F_\circ(b_1 + \delta) - c) \int_{a_1}^{b_1 + \delta} A(x) \, dx - \int_{a_1}^{b_1 + \delta} A(x)[F_\circ(x) - c] \, dx
\]

\[
= [F_\circ(b_1 + \delta) - c] \int_{a_1}^{b_1 + \delta} A(x) \, dx - \int_{b_1}^{b_1 + \delta} A(x)[F_\circ(x) - c] \, dx
\]

\[
\leq [F_\circ(b_1 + \delta) - c] \left( \int_{a_1}^{b_1} A(x) \, dx + 2M\delta \right) \quad \text{where } |A(x)| < M.
\]

Letting \( \delta \to 0 \), we find that \( \psi \) becomes negative and hence there is a contradiction. The case where \( b_1 = X \) is trivial.

If we suppose that \( \int_{a_1}^{b_1} A(x) \, dx > 0 \) and \( a_1 \) is a point of continuity of \( F_\circ \), then by an argument similar to that above, we get the contradiction when \( a_1 > -X \), by replacing \( F_\circ \) on \( (a_1 - \delta, b_1) \) by \( F_\circ(a_1 - \delta) \).
and letting $\epsilon \to 0$. In case $a_1 = -X$ or there is a jump in $F_o$ at $a_1$, the proof is trivial.

**Remark:** If $K(x)$ is a function satisfying the properties of the function $A(x)$, then the problem $P_K$ (corresponding to $F \in \mathcal{F}_{\eta_1, \eta_2}$) of finding a c.d.f. $F_o$ such that

$$I_{K}(F_o) = \min_{F \in \mathcal{F}} I_{K}(F) = \min_{F \in \mathcal{F}} \int_{-X}^{X} K(x) F(x) \, dx,$$

gives the same results as stated in Theorem 5.1 and its Corollary, i.e., if $F_o$ is the solution of $P_K$,

(a) the set $\{x: K(x) \neq 0, -X < x < X\}$ is of $F_o$-measure zero.

(b) $\int K(x) F(x) \, dx$ is zero over intervals where $F$ is constant.

**Theorem 5.2.** If $F_o$ solves $P_{\mathcal{F}_{\eta_1, \eta_2}}$, then $F_o$ has no jumps on the open interval $(-X, X)$ and hence $A(x)$ is continuous on $(-X, X)$.

**Proof:** Let $F_o$ have a jump at $x_o$, $-X < x_o < X$. Then by Theorem 5.1, $A(x_o) = 0$. But since $\frac{\partial A}{\partial y}$ is strictly increasing in $y$, $x_o$ is the right-hand end point of an interval on which $A(x) < 0$. By the same arguments as in the proof of the Theorem 5.1, we see that on this interval $F_o(x) = F_o(x_o)$ and hence $F_o$ has no discontinuity. But because discontinuities of $A(x)$ arise on account of jumps of $F_o$, there are no discontinuities in $A(x)$ or $A(x)$ is continuous on the interval $(-X, X)$. 
Let $f_{1 \eta_2}(x)$ be defined with $0 \leq f_{1 \eta_2}(x) \leq 1$ such that $B_{1 \eta_2}(x, f_{1 \eta_2}(x)) = 0$. (The function $f_{1 \eta_2}$ is defined on that subset of $[-X, X]$ for which there exists a $y$ between 0 and 1 such that $B_{1 \eta_2}(x, y) = 0$.)

As $\frac{2\phi}{\gamma^2}(x, y)$ is continuous and strictly increasing in $y$, $f_{1 \eta_2}(x)$ is continuous wherever it is defined. If $0 < f_{1 \eta_2}(x_0) < 1$, then $f_{1 \eta_2}$ is defined in some interval about $x_0$ (the interval is one-sided if $x_0 = \pm X$). Graphically $f_{1 \eta_2}$ represents a number of curve segments which terminate when $f_{1 \eta_2}(x)$ is zero or one.

More specifically, $f_{1 \eta_2}(x)$ is defined on the union of closed intervals at the end-points of each of which it is either zero or one. Let $[a_i, b_i]$ and $[a_j, b_j]$ be two such intervals, not separated by any others such that $b_i < a_j$, then $f_{1 \eta_2}(b_i) = f_{1 \eta_2}(a_j)$. If there are an infinite number of intervals $[a_j, b_j]$ in the neighborhood of $b_i$, it follows that

$$f_{1 \eta_2}(b_i) = f_{1 \eta_2}(a_i) = f_{1 \eta_2}(b_j)$$

for $b_j$ sufficiently close to $b_i$. Hence the following definition of a function $g_{1 \eta_2}$ has a meaning.

**Definition:** Define $g_{1 \eta_2}$ to be that unique function for which $g_{1 \eta_2}$ is defined and continuous on $[-X, X]$ such that
$$g_{\eta_1 \eta_2}(x) = \begin{cases} f_{\eta_1 \eta_2}(x) & \text{where } f_{\eta_1 \eta_2} \text{ is defined} \\ 0 \text{ or } 1 & \text{elsewhere} \end{cases}$$

and

$$g_{\eta_1 \eta_2}(x) = 1$$

provided that the subset of \([-X,X]\) for which \(f_{\eta_1 \eta_2}\) is defined, is non-null.

**Theorem 5.3.** If \(F_o\) solves \(P_{3F_o \eta_1 \eta_2}\), then for \(-X < x < X\), \(F_o\) coincides with \(g_{\eta_1 \eta_2}\) except on intervals on which \(F_o\) is constant.

**Proof:** From Theorems 5.1 and 5.2, we know that \(F_o\) has no jumps on \((-X,X)\) and \(F_o\) cannot increase when \(A(x) \neq 0\). Therefore, \(F_o\) remains constant until it intersects with \(f_{\eta_1 \eta_2}\).

**Remarks:** 1. We can represent a conceivable situation by the following figure:

![Diagram](image-url)

Legend:
- \(\cdashdash\) denotes \(F_o(x)\)
- \(\cdash\) denotes \(f_{\eta_1 \eta_2}(x)\)
- \(-\cdash\) denotes \(g_{\eta_1 \eta_2}(x)\).
2. It must be noted that the corollary to Theorem 5.1 puts a strong restriction on the intervals on which \( F_0 \) is constant.

3. The solution in the general case may not be completely specified, but we shall consider in the following some special cases where the minimizing c.d.f. is completely characterized.

**Special Cases:**

I. When \( \varphi \) is a function of \( y \) alone, i.e., \( \varphi(x,y) = \psi(y) \)

**Lemma 5.1.** If \( \varphi(x,y) = \psi(y) \), then corresponding to \( F_0 \) which is the solution of \( P_1 \), \( \eta_2 \) is negative.

**Proof:** If \( \eta_2 \geq 0, \eta_1 + \eta_2 \) is non-decreasing. Also as \( \psi'(y) \) is non-decreasing in \( x \), the function \( A(x) = \psi'(F_0) + \eta_1 + \eta_2 \) is non-decreasing in \( x \). Therefore, \( f_{\eta_1\eta_2}(x) \) is non-increasing and hence from Theorem 5.3, it follows that \( F_0 \) is constant on \([-X,X]\). But for such \( F_0, \mu_2 = x^2 \) and hence \( F_0 \) is not admissible. Therefore, \( \eta_2 < 0 \).

**Theorem 5.4.** If \( \varphi(x,y) = \psi(y) \), the solution \( F_0 \) of \( P_1 \) is given by \( g_{\eta_1\eta_2} \) for some \( \eta_1, \eta_2 \).

**Proof:** Corresponding to \( F_0 \), there is an \( \eta_1 \) and \( \eta_2 \) such that \( F_0 \) solves \( P_{3\eta_1\eta_2} \). Suppose \( f_{\eta_1\eta_2} \) is not defined at any point of \([-X,X]\). Then either \( A(x) > 0 \) or \( A(x) < 0 \) in \([-X,X]\) and in either case \( F_0 \) is a one-point distribution which is not admissible. Hence \( f_{\eta_1\eta_2} \) is defined on a non-null subset of \([-X,X]\). Since \( \eta_1 + \eta_2 x \) is decreasing, \( f_{\eta_1\eta_2}(x) \) is increasing and hence \( g_{\eta_1\eta_2}(x) \) represents a
c.d.f. in \([-X,X]\).

By Theorem 5.3, \( F_0 \) can be broken up into three parts. \( F_0 \) is constant on the first interval, coincides with \( g_{\eta_1 \eta_2} \) on the second which is to the right of the first and constant on the third interval. As \( A(x) > 0 \) on the first interval \( F_0 = 0 \) there. Similarly \( F_0 = 1 \) on the third interval as \( A(x) < 0 \) there. But \( F_0 \) has no jumps on \((-X,X)\). Hence \( F_0 \) coincides with \( g_{\eta_1 \eta_2} \) on \([-X,X]\).

**Corollary:** In the general case where \( \varphi \) may depend on \( x \), if \( g_{\eta_1 \eta_2} \) is a c.d.f., then \( F_0(x) = g_{\eta_1 \eta_2}(x) \).

II. When \( \frac{\partial^2 \varphi}{\partial x \partial y}(x,y) < 0 \).

**Theorem 5.5.** If \( \frac{\partial^2 \varphi}{\partial x \partial y}(x,y) < 0 \) and \( \eta_2 < 0 \), then \( F_0(x) = g_{\eta_1 \eta_2}(x) \) for \(-X < x < X\).

**Proof:** If \( \frac{\partial^2 \varphi}{\partial x \partial y} < 0 \) and \( \eta_2 < 0 \), \( A(x) \) is a decreasing function of \( x \) and hence \( f_{\eta_1 \eta_2} \) is increasing in \( x \) or \( g_{\eta_1 \eta_2} \) is a c.d.f. We get the result of the theorem, then by the previous corollary.

**Remark:** Unfortunately it is not always true that \( \eta_2 < 0 \). In fact for side conditions corresponding to small variance, one has \( \eta_2 > 0 \). It might still happen that \( f_{\eta_1 \eta_2} \) is non-decreasing and then the above result still holds. In any case Theorem 5.3 with the corollary to Theorem 5.1 gives a useful characterization of the solution of our problem.
Section 6. Comparison of our Technique with that of Gumbel [10] and Chernoff and Reiter [3].

(i) Gumbel's Method. The problem considered by Gumbel is that of maximizing

\[ \int_0^1 x(F) n F^{n-1} dF \]  

with restrictions

\[ \int_0^1 x(F) dF = 0, \quad \int_0^1 x^2(F) dF = 1. \]

A variational technique has been used to derive the form obtained for the maximizing c.d.f. is given by equating to zero, the first variation of

\[ \int \left\{ n x(F) F^{n-1} + \eta_1 X(F) + \eta_2 x^2(F) \right\} dF, \text{ i.e.,} \]

\[ n F^{n-1} + \eta_1 + \eta_2 x(F) = 0 \quad 0 \leq F \leq 1. \]

The above equation gives a sort of sufficiency condition as any admissible \( F \) given by (6.3) does maximize the integral (6.2) and hence maximizes (6.3). David and Hartley [1] have given an ingenious argument to prove the sufficiency of the solution but that seems unnecessary. However, the above equation does not give the necessity of the solution.
since this approach does not provide an argument for proving that the constants $\eta_1$ and $\eta_2$, to make the c.d.f. admissible, always exist.

This method also extends to the case of a bounded random variable as treated in this paper. We could not find out where the assumption that $x$ is a continuous variable, has been utilized. We suspect that in the above method, this assumption of continuity is not needed.

We shall use the above approach for our problem. Integrating by parts, we have

$$\int_{-X}^{X} \varphi(x,F(x)) \, dx = x \varphi(x,F(x)) \bigg|_{-X}^{X} - \int_{-X}^{X} x \frac{d\varphi}{dx}(x,F) \, dx$$

if $\varphi$ is differentiable in $x$. Now

$$d\varphi(x,F(x)) = \frac{\partial \varphi}{\partial x}(x,F(x)) \, dx + \frac{\partial \varphi}{\partial y}(x,F(x)) \, dF.$$  

When $\varphi(x,F(x))$ is a function of $F$ above, say $\psi(F(x))$, $d\psi = \psi'(F) \, dF$ and hence the problem of minimizing

$$\int_{-X}^{X} \psi(F) \, dx$$

is the same as that of maximizing

$$\int_{0}^{1} x(F) \psi'(F) \, dF.$$  

Hence using Gumbel's approach, we maximize
\[ \int_0^1 [x(F) \psi'(F) + \eta_1 x(F) + \eta_2 x^2(F)] \, dF \]

and get the following equation satisfied by the admissible maximizing c.d.f.,

\[ \psi'(F) + \eta_1 + 2\eta_2 x(F) = 0. \]

In the above case, our technique would also lead to a similar equation. But in the general case when we consider \( \varphi(x, F(x)) \), Gumbel's approach does not seem to apply.

(ii) Chernoff and Reiter Method. Chernoff and Reiter [3] consider the problem of minimizing and maximizing

\[ \int g(x) \, dF(x) \]

with side conditions

\[ \int x \, dF(x) = c_1, \quad \int x^2 \, dF(x) = c_2 \]

such that \( c_2 > c_1 \) and \( g(x) \) is a continuous function of \( x \).

In the process of reduction of our main problem, we have an intermediary problem \( P_{2F_0} \) of finding the minimum of

\[ L_{F_0} (F) = \int_{-X}^X \frac{\partial \varphi}{\partial y} (x, F_0(x)) \, F(x) \, dx \]

over all admissible c.d.f:s \( A \). Now as
\[ f(x) = \int_{-X}^{X} \frac{\partial \Phi}{\partial y} (x, F_0(x)) \, dx \]

is continuous in \( x \), integrating by parts \( I_{F_0}(F) \), we have

\[
I_{F_0}(F) = f(x) F(x) \bigg|_X^X - \int_{-X}^{X} f(x) \, dF(x)
\]

\[ = \text{constant} - \int_{-X}^{X} f(x) \, dF(x) \]

or we maximize

\[
\int_{-X}^{X} f(x) \, dF(x)
\]

over all admissible c.d.f.'s \( \mathcal{A} \). Now as \( f(x) \) is continuous, by the methods of Chernoff and Reiter, the necessary condition for the maximum is given by the following.

(a) There is an \( \eta_1 \) and \( \eta_2 \) such that when \( x \) is a point of continuity of \( F(x) \)

\[
B_{\eta_1 \eta_2} (x, F_0(x)) = \frac{\partial \Phi}{\partial y} (x, F_0(x)) + \eta_1 + 2\eta_2 \, x = 0
\]

except on a set of \( F_0 \)-measure zero.

(b) \( F_0 \) has no jump in \(-X < x < X\) otherwise either

\[
B_{\eta_1 \eta_2} (x, F_0(x)) > 0 \quad \text{or} \quad B_{\eta_1 \eta_2} (x, F_0(x^-)) < 0.
\]
Hence we get a result similar to ours obtained in Section 5, i.e., the set
\[ \left\{ x: B_{\eta_1 \eta_2} (x, F_0 (x)) \neq 0 \quad -X < x < X \right\} \]
has $F_0$-measure zero.

Section 7. Examples.

In this section we discuss some examples to illustrate the method of obtaining the minimizing c.d.f. for our problem for some specified function $\varphi$. We also discuss one example of the special case $\varphi(x, y) = \varphi(y)$. There we obtain the solutions to the problems of minimizing the expectation of extreme values and the expectation of the range of a sample of $n$ independent observations on a bounded random variable with given mean and variance.

Example 1. Consider the problem of finding
\[ \min_{F \in \mathcal{A}} \frac{1}{2} \int_{-X}^{X} [F(x) - x]^2 \, dx \]
when $\mathcal{A}$ denotes the admissible class of c.d.f.s as in Section 3.

This is the special case of a more general problem where we minimize
\[ \int_{-X}^{X} \varphi(F(x) - x) \, dx \quad \text{for} \quad F \in \mathcal{A} . \]
Here $\varphi(y-x)^2 = \frac{1}{2} (y, x)^2$, $\varphi$ being strictly convex, bounded and continuous function of its argument. This problem has also been discussed by Birnbaum and Klose [7].
If \( \varphi(x,y) = \psi(y-x) \), \( \varphi \) is strictly convex in \( y \) and it is easy to verify that \( \frac{\partial^2 \varphi}{\partial x \partial y} < 0 \). We know that the solution exists, is unique and the problem is reduced to that of finding \( \eta_1 \) and \( \eta_2 \) so that \( F_0 \) solves \( P \sum_{i \neq 0} \eta_1 \eta_2 \) where \( P \sum_{i \neq 0} \eta_1 \eta_2 \) is the problem of finding \( F \in \mathcal{F} \) which minimizes

\[
\int_{-X}^{X} \left[ \psi'(F_0(x) - x) + \eta_1 + \eta_2 x \right] F(x) \, dx.
\]

Then by the theorems of Section 5, we know that \( F_0(x) \) is given in terms of \( g_{\eta_1 \eta_2}(x) \) where \( g_{\eta_1 \eta_2}(x) \) is uniquely expressed in terms of the function

\[
f_{\eta_1 \eta_2}(x) = x + \psi^{-1}(-\eta_1 \eta_2 x).
\]

Returning to our example, we have the function

\[
f_{\eta_1 \eta_2}(x) = (1-\eta_2) x - \eta_1
\]

\[
f_{\eta_1 \eta_2}(x) = 0 \quad \text{for} \quad x_1 = \frac{\eta_1}{1-\eta_2}, \quad f_{\eta_1 \eta_2}(x) = 1 \quad \text{for} \quad x_2 = \frac{1+\eta_1}{1-\eta_2}.
\]

Case 1. \( \eta_2 < 1 \). Then \( f_{\eta_1 \eta_2} \) is increasing. Define \( g_{\eta_1 \eta_2}(x) \) by the following.

\[
g_{\eta_1 \eta_2}(x) = \begin{cases} 
0 & x < \max(-X,x_1) \\
1 & x \geq \min(x_2,x) \\
(1-\eta_2)x-\eta_1 & \text{elsewhere}
\end{cases}
\]
As $g_{\eta_1 \eta_2}$ is a c.d.f., $F_\circ (x) = g_{\eta_1 \eta_2} (x)$.

Case 2. $\eta_2 \geq 1$, $f_{\eta_1 \eta_2} (x)$ is non-increasing and hence the solution is either a one-point distribution or a two-point distribution with total probability concentrated at $-X$ and $X$. In both cases, then, the solution is not admissible.

Example 2. Consider the same problem as in Example 1 but with an additional restriction on the c.d.f. $F(x)$, i.e., $F(x) \geq x$. Now let $F(x)$ be a c.d.f. on $[0,1]$.

Under this additional restriction, the class $\mathcal{A}^*$ of admissible c.d.f.'s is also compact and convex. Then the solution to this problem exists. It is unique since $\varphi(x,y) = \frac{1}{2} (y-x)^2$ is strictly convex in $y$.

Applying the methods used to prove Lemma 4.2, we see that the problem is the same as that of minimizing

$$\int_0^1 [F_\circ (x) - x] F(x) \, dx$$

over all $F \in \mathcal{A}^*$. It is easy to see that the set analogous to $\Gamma_2$ of Lemma 4.3 here is also convex, closed and bounded and hence applying the method of Lemma 4.4, we reduce the problem to that of finding the c.d.f.'s corresponding to

$$\min_{F \in \mathcal{A}} \int_0^1 [F_\circ (x) - x + \eta_1 + \eta_2 x] F(x) \, dx$$
or

$$\min_{F \in \mathcal{F}} \int_0^1 [F_0(x) + \eta_1 + \eta_2 x] F(x) \, dx, \quad \eta_3 = r_2^{-1}$$

where $\mathcal{F}$ is the class of all c.d.f:s $F$ on $[0,1]$ such that $F(x) \geq x$. We can now apply the methods of Section 5. Define the function $f_{\eta_1 \eta_3}$ with $x \leq f_{\eta_1 \eta_3}^{-1}(x) \leq 1$ such that

$$f_{\eta_1 \eta_3}^{-1}(x) = -\eta_1 - \eta_3 x.$$

Define the function $g_{\eta_1 \eta_3}$ such that

$$g_{\eta_1 \eta_3}(x) = \begin{cases} f_{\eta_1 \eta_3}^{-1}(x) & \text{where } f_{\eta_1 \eta_3}^{-1}(x) \text{ is defined} \\ x \text{ or } 1 & \text{elsewhere on } [0,1] \end{cases}$$

$g_{\eta_1 \eta_3}(x)$ is continuous on $[0,1]$ and $g_{\eta_1 \eta_3}(1) = 1$.

Then $g_{\eta_1 \eta_3}$ gives the solution $F_0$ of the problem if $g_{\eta_1 \eta_3}(x)$ is a c.d.f. We shall give an explicit characterization of $g_{\eta_1 \eta_3}$ in the various possible cases. Let the point of intersection of $y = -\eta_1 - \eta_3 x$ and $y = x$ be denoted by $x^* = -\frac{\eta_1}{1+\eta_3}$. Let $x^{**}$ be such that $-\eta_1 - \eta_3 x^{**} = 1$.

**Case I.** $\eta_3 > -1$.

(a) $x^* \leq 0$
\[ g_{\eta_1 \eta_3}(x) = \begin{cases} 
0 & x < 0 \\
x & 0 \leq x < 1 \\
1 & x \geq 1 
\end{cases} \]

This \( g_{\eta_1 \eta_3} \) is a c.d.f. but it is not admissible.

(b) \( x^* > 1 \)

\[ g_{\eta_1 \eta_3}(x) = \begin{cases} 
0 & x < 0 \\
-\eta_1 \eta_3 x & 0 < x < x^{**} \\
1 & x > x^{**} 
\end{cases} \]

If \( x^{**} > 0 \), \( g_{\eta_1 \eta_3} \) is a c.d.f. and hence \( F_0(x) = g_{\eta_1 \eta_3}(x) \).

If \( x^{**} < 0 \), the solution is a one-point distribution with mass concentrated at \( x = 0 \) which is not admissible.

(c) \( 0 < x^* < 1 \). Consider (l) \(-1 < \eta_3 < 0\)

\[ g_{\eta_1 \eta_3}(x) = \begin{cases} 
0 & x < 0 \\
-\eta_1 \eta_3 x & 0 \leq x < x^* \\
x & x^* \leq x \leq 1 \\
1 & x \geq 1 
\end{cases} \]

This is again a c.d.f. and hence \( F_0(x) = g_{\eta_1 \eta_3}(x) \).

(ii) \( \eta_3 > 0 \)

\[ g_{\eta_1 \eta_3}(x) = \begin{cases} 
0 & x < 0 \\
-\eta_1 \eta_3 x & 0 \leq x \leq x^* \\
x & x^* \leq x \leq 1 \\
1 & x \geq 1 
\end{cases} \]
This is not a c.d.f. Here \( A(x) = F_o(x) + \eta_1 + \eta_3 x \). At 
\[
x = 0, \quad A(x) > 0 \text{ if } F_o(x) > -\eta_1 \quad \text{and} \quad A(x) < 0 \text{ if } F_o(x) < -\eta_1.
\]

Suppose there is a jump at \( x = 0 \) such that \( F_o(0) > -\eta_1 \), then \( A(0) < 0 \) and \( F_o(x) \) is then continuous at \( x = 0 \). Hence there is a contradiction. Let there be a jump at \( x = 0 \) such that \( c = F_o(0) < -\eta_1 \), \( A(0) < 0 \) and hence we take as a possible minimizing c.d.f.

\[
G(x) = \begin{cases} 
0 & \text{if } x < 0 \\
c & \text{if } 0 \leq x < c \\
x & \text{if } c \leq x < 1 \\
1 & \text{if } x \geq 1 
\end{cases}
\]

The remark after Theorem 5.1 puts the following restriction on \( c \),

\[
\int_0^c (c + \eta_1 + \eta_3 x) \, dx = 0, \quad \text{i.e.,}
\]

\[
(c + \eta_1)c + c = \frac{\eta_1}{\eta_3} + \frac{\eta_3}{2}c^2 = 0,
\]

since \( c = 0 \) gives an inadmissible c.d.f. Incidentally this shows also, as is evident from the value of \( c \) itself that

\[
x^* < c < -\eta_1.
\]

The unique value of \( c \) which is obtained from the constraints exists if and only if

\[
(7.1) \quad (1 - 2\mu_1)^3 = (1 - 3\mu_2)^2.
\]
This condition is obtained by eliminating $c$ between the equation

$$
\int_0^1 F(x) \, dx = 1 - \mu_1 = c^2 + \frac{1-c^2}{2} = \frac{c^2+1}{2}
$$

$$
\int_0^1 x F(x) \, dx = 1 - \mu_2 = c^3 + \frac{2}{3} (1-c^3) = \frac{c^3+2}{3}
$$

Hence $G(x)$ is admissible and $F_0(x) = G(x)$ if and only if $\mu_1$ and $\mu_2$ are such that (7.1) is satisfied.

**Case II.** $\eta_3 < 0$.

(a) $x^* < 0$. We then have

$$
g_{\eta_1 \eta_3}(x) = \begin{cases} 
0 & x < 0 \\
-\eta_1 \eta_3 x & 0 \leq x < x^* \\
1 & x \geq x^*
\end{cases}
$$

If $x^* < 0$, $g_{\eta_1 \eta_3}$ is a c.d.f. and hence $F_0(x) = g_{\eta_1 \eta_3}(x)$. If $x^* < 0$, the minimizing c.d.f. is a one-point distribution and is not admissible.

(b) $x^* > 1$. The minimizing c.d.f. is the same as in Case I (a) and is not admissible.

(c) $0 < x^* < 1$. Then we have

$$
g_{\eta_1 \eta_3}(x) = \begin{cases} 
0 & x < 0 \\
x & 0 \leq x \leq x^* \\
-\eta_1 \eta_3 x & x^* < x < x^* \\
x & x > x^*
\end{cases}
$$
\[ g_{\eta_1 \eta_3} \] is a c.d.f. and hence \( F_0(x) = g_{\eta_1 \eta_3}(x) \).

**Example 3.** Suppose we want to maximize the third moment of the largest observation from a sample of \( n \) independent ones with the same side conditions on \( F(x) \) as \((3.3)\), i.e., we want to find

\[
\max_{F \in \mathcal{A}} \int_{-X}^{X} x^3 \, d[F(x)]^n.
\]

Integrating by parts the above integral, we see that the above problem is the same as that of finding

\[
\min_{F \in \mathcal{A}} \int_{-X}^{X} x^2 \, F^n(x) \, dx.
\]

For the sake of simplicity we shall consider the case \( n = 2 \).

Now \( \phi(x,y) = x^2 y^2 \). \( \phi \) is clearly strictly convex in \( y \) and has all the desirable properties of the function we are dealing with. Hence the problem has a unique solution which is expressed in terms of \( f_{\eta_1 \eta_2} \) where

\[
f_{\eta_1 \eta_2}(x) = -\frac{\eta_1 + \eta_2}{2x}
\]

\[
f_{\eta_1 \eta_2}(x) = 0 \quad \text{for} \quad x_1 = -\frac{\eta_1}{\eta_2}
\]

\[
f_{\eta_1 \eta_2}(x) = 1 \quad \text{for} \quad x_2 = -\frac{\eta_2}{4} - \frac{1}{4} \sqrt{\eta_2^2 - 8\eta_1} \quad \text{and} \quad x_3 = -\frac{\eta_2}{4} - \frac{1}{4} \sqrt{\eta_2^2 - 8\eta_1}.
\]
We consider the following cases and give the characterization of the solution \( F_0 \) in terms of \( g_{\eta_1 \eta_2} \).

**Case 1.** \( \eta_1 > 0, \eta_2 > 0 \).

Here \( x_1 < 0 \) and \( -\frac{\eta_1 + \eta_2}{2x} x < 0 \) for \( x > x_1 \). Hence if \( x_1 < -X \), there is no admissible solution.

(a) Let \( \eta_2^2 - 8\eta_1 > 0 \). Then \( x_2 < -\frac{2\eta_1}{\eta_2} < x_3 \). Define

\[
G_{\eta_1 \eta_2}^*(x) = \begin{cases} 
0 & x > x_1 \\
1 & x_2 \leq x \leq x_3 \\
& \text{elsewhere}
\end{cases}
\]

\( G_{\eta_1 \eta_2}^*(x) \) coincides with \( G_{\eta_1 \eta_2}^*(x) \) in \([-X,X]\). If we take \( F_0(x) \) such that \( F_0(x) = 1 \) for \( x \geq x_2 \), then \( F_0(x) \) is continuous at \( X \). But \( A(X) \), where \( A(x) = B_{\eta_1 \eta_2}(x,F_0(x)) = 2x^2 F_0(x) + \eta_1 + \eta_2 x \), is positive and hence from the remark after Theorem 5.1, we know that there is a jump at \( X \). Hence we take, if \( x_2 > -X \),

\[
F_0(x) = \begin{cases} 
0 & x < -X \\
f_{\eta_1 \eta_2} & -X \leq x < x_c \\
c & x_c \leq x < X \\
1 & x \geq X
\end{cases}
\]

where \( c = -\frac{\eta_1 + \eta_2}{2x_c} x_c \) and \( x_c \) is determined by the minimum abscissa for which
\[ \int_{x_c}^{X} \left( 2x^2 - c + \eta_1 + \eta_2 x \right) \, dx \leq 0. \]

The above solution is valid if \( c > -\frac{\eta_1 - \eta_2}{2} \). If \( c \leq -\frac{\eta_1 - \eta_2}{2X^2} \), we get a two-point distribution which is not admissible. If \( x_2 < -X \), we again get an inadmissible solution.

(b) \( \eta_2^2 - 8\eta_1 < 0 \). If \( -X \geq -\frac{2\eta_1}{\eta_2} \), we do not have an admissible solution. In case \( -X < -\frac{2\eta_1}{\eta_2} \),

\[
g_{\eta_1 \eta_2}^*(x) = \begin{cases} 
0 & \text{if } x > x_1 \\
\eta_1 \eta_2(x) & \text{elsewhere}
\end{cases}
\]

\( g_{\eta_1 \eta_2}(x) \) coincides with \( g_{\eta_1 \eta_2}^*(x) \) on \([-X, X]\). Also \( F_0 \) has to be continuous for \(-X < x < X\). The characterization of \( F_0 \) is the same as in the Case 1 (a).

Case 2. \( \eta_1 > 0, \eta_2 < 0 \).

Here \( -\frac{\eta_1 + \eta_2 x}{2x^2} < 0 \) for \( x < x_1 \). Hence if \( x_1 > X \), there is no admissible solution.

(a) \( \eta_2^2 - 8\eta_1 > 0 \). In case \( x_2 < X \),
\[ g_{\eta_1 \eta_2}^*(x) = \begin{cases} 
0 & x < x_1 \\
1 & x_2 \leq x \leq x_3 \\
f_{\eta_1 \eta_2}(x) & \text{elsewhere} 
\end{cases} \]

\( g_{\eta_1 \eta_2}^*(x) \) coincides with \( g_{\eta_1 \eta_2}^*(x) \) on \([-X,X]\). By the same argument as in Case 1 (a), the solution is given by

\[ F_0(x) = \begin{cases} 
0 & x < x_1 \\
f_{\eta_1 \eta_2}(x) & x_1 \leq x \leq x_c \\
c & x_c \leq x < X \\
1 & x \geq X 
\end{cases} \]

where \( c \) is uniquely determined by

\[ \int_{x_c}^{X} (2x^2 + c + \eta_1 + \eta_2 x) \, dx = 0 \]

and \( f_{\eta_1 \eta_2}(x_c) = c \).

(b) \( \eta_2^2 - 8\eta_1 < 0 \). Consider (1) \( X > -\frac{2\eta_1}{\eta_2} \)

\[ g_{\eta_1 \eta_2}^*(x) = \begin{cases} 
0 & x < x_1 \\
f_{\eta_1 \eta_2}(x) & \text{elsewhere} 
\end{cases} \]

\( g_{\eta_1 \eta_2}(x) \) coincides with \( g_{\eta_1 \eta_2}^*(x) \) on \([-X,X]\). \( F_0 \) can be characterized exactly in the same way in case (a).

(ii) If \( x_1 < X < -\frac{2\eta_1}{\eta_2} \), then
\[ g_{\eta_1 \eta_2}(x) = \begin{cases} 
0 & x < x_1 \\
\ell_{\eta_1 \eta_2}(x) & x_1 \leq x < X \\
1 & x > X 
\end{cases} \]

and \( F_0(x) = g_{\eta_1 \eta_2}(x) \).

**Case 3.** \( \eta_1 < 0, \eta_2 > 0 \).

Let \(-X < x_2\). In case \(-X > x_2\), there is no admissible solution.

\[ g_{\eta_1 \eta_2}^*(x) = \begin{cases} 
0 & x > x_1 \\
1 & x_2 \leq x \leq x_3 \\
\ell_{\eta_1 \eta_2}(x) & \text{elsewhere} 
\end{cases} \]

\( g_{\eta_1 \eta_2} \) coincides with \( g_{\eta_1 \eta_2}^* \) on \([-X, X]\). As seen before, the solution is given by

\[ F_0(x) = \begin{cases} 
0 & x < -X \\
\ell_{\eta_1 \eta_2}(x) & -X \leq x < x_c \\
c & x_c \leq x < X \\
1 & x \geq X 
\end{cases} \]

where \( x_c < x_2 \) is given by \( \ell_{\eta_1 \eta_2}(x_c) = c \) and \( x_c \) is uniquely determined by

\[ \int_{x_c}^{X} A(x) \, dx \leq 0. \]

**Case 4.** \( \eta_1 < 0, \eta_2 < 0 \). Let \(-X < x_2\)
\[ g_{\eta_1 \eta_2}^*(x) = \begin{cases} 
0 & x < x_1 \\
1 & x_2 \leq x \leq x_3 \\
\frac{f_{\eta_1 \eta_2}(x)}{g_{\eta_1 \eta_2}} & \text{elsewhere} 
\end{cases} \]

\( g_{\eta_1 \eta_2}^* \) coincides with \( g_{\eta_1 \eta_2}^* \) on \([-X,X)\).

The solution is given by the following

\[ F_0(x) = \begin{cases} 
0 & x < \max\{-X, x_1\} \\
\frac{f_{\eta_1 \eta_2}(x)}{g_{\eta_1 \eta_2}} & \max\{-X, x_1\} \leq x < x_c \\
c & x_c \leq x < X \\
1 & x \geq X 
\end{cases} \]

where \( x_c \) and \( c \) are determined in the usual manner.

If \(-X > x_2\), there is no admissible solution.

**Example 4.** Let \( x_1 \leq x_2 \leq \ldots \leq x_n \) be \( n \) ordered, independent observations from a c.d.f. \( F(x) \). Consider the problem of maximizing \( E(x_n) \) with restrictions (3.2). The same problem for c.d.f.s defined over the whole real line with restrictions on mean and variance has been discussed by Gumbel [10] and David and Hartley [1].

\[ E(x_n) = \int_{-X}^{X} x \, d\left\{ F(x) \right\}^n. \]

Integrating by parts, the above problem reduces to that of finding

\[ \min_{F \in \mathcal{A}} \int_{-X}^{X} f^n(x) \, dx. \]
Now as \( \varphi(x,y) = y^n \) is strictly convex in \( y \) and is a function of \( y \) alone, the solution \( F_0(x) \) is given by the function \( g_{\eta_1 \eta_2}(x) \) where

\[
g_{\eta_1 \eta_2}(x) = \begin{cases} 
0 & x < \max \left( -\frac{\eta_1}{\eta_2}, -X \right) \\
1 & x \geq \min \left( X, -\frac{n+\eta_1}{\eta_2} \right) \\
\left( -\frac{\eta_1 + \eta_2}{n} \right)^{n-1} & \text{elsewhere}
\end{cases}
\]

Here \( \eta_1 \) and \( \eta_2 \) are determined by the following four cases:

- **Case 1.** \( \max(x_1, -X) \), \( \min(x_2, X) \)
  - \( x_1 \)
  - \( x_2 \)

- **Case 2.** \( -X \), \( x_2 \)
  - \( -X \)
  - \( x_2 \)

- **Case 3.** \( -X \), \( X \)
  - \( x_1 \)
  - \( X \)

- **Case 4.** \( x_1 \), \( X \)
  - \( x_1 \)
  - \( X \)

We give below the equations determining \( \eta_1, \eta_2 \) in the above cases.

**Case 1.**
\[
\mu_1 = -\frac{1}{\eta_2} (1 + \eta_1)
\]

\[
\mu_2 = \frac{1}{\eta_2} \left( \eta_1^2 + 2\eta_1 + \frac{\eta_2^2}{2n-1} \right).
\]

**Case 2.**
\[
\mu_1 = -\frac{n-1}{\eta_2} \zeta \frac{n}{n+1} \zeta \frac{1+\eta_1}{\eta_2} \zeta \frac{1}{n} (-\eta_1 + \eta_2 X)
\]

\( \varsigma \)
\[
\mu_2 = \left( \frac{n+\eta_1}{\eta_2} \right)^2 - \frac{2n}{\eta_2} \left[ \frac{n^2(n-1)}{2(n-1)} + (n-1)\eta_1 - (n-1)\xi \frac{n}{n-1} \left( \eta_1 + \frac{n}{n-1} \xi \right) \right].
\]

Case 3. \( \mu_1 = x + \frac{n-1}{\eta_2} \left( \xi \frac{n}{n-1} - \xi \right) \xi = \frac{1}{n} (-\eta_1 - \eta_2 x) \)

\[
\mu_2 = x^2 - \frac{2n}{\eta_2} \left[ \frac{n-1}{n} \eta_1 - \frac{n}{n-1} \right] + \frac{n(n-1)}{2n-1} \left( \xi \frac{n}{n-1} - \xi \right).
\]

Case 4. \( \mu_1 = x + \frac{n-1}{\eta_2} \xi \frac{n}{n-1} \)

\[
\mu_2 = x^2 - \frac{n}{\eta_2} \left[ \frac{(n-1)\eta_1}{n} - \frac{n}{n-1} \xi \frac{2n-1}{2n-1} \right].
\]

Figure 2 represents \( \eta_1 \) and \( \eta_2 \) in terms of \( \mu_1 \) and \( \mu_2 \) for the case \( x = 1 \).

Similar results can be easily obtained for maximizing the expectations of the range and the first observation with similar restrictions on the underlying c.d.f.s.
$\eta_1$ and $\eta_2$ in terms of $\mu_1$ and $\mu_2$
Part II
Maximum Problem

Section 8. Characterization of the Solution and some Examples.

In this section we find the solution to the problem of maximizing

\[ I(F) = \int_{-X}^{X} \varphi(x, F(x)) \, dx \]

over all admissible c.d.f:s. The existence of the maximizing c.d.f.
has already been established by Lemma 3.2. We shall show now that the
solution is a discrete distribution and in our case, is at most a three-
point distribution, i.e., a distribution concentrating all its mass at
just three or fewer points. Some illustrations have been given at the
end of this section.

**Theorem 8.1.** The solution to the problem of maximizing \( I(F) \) over
the class \( \mathcal{A} \) of admissible c.d.f:s, is at most a three-point distribution.

**Proof:** We shall first show that a convex combination of c.d.f:s
does not maximize \( I(F) \). Let

\[ M = \max_{F \in \mathcal{A}} I(F). \]

Suppose \( F_1 \) and \( F_2 \) are maximizing admissible c.d.f:s. As \( \varphi(x, y) \) is
strictly convex in \( y \), we find that

\[ \int_{-X}^{X} \varphi(x, \lambda F_1(x) + (1-\lambda)F_2(x)) \, dx \]
\[ \lambda \int_{-X}^{X} \varphi(x, F_1(x)) \, dx + (1 - \lambda) \int_{-X}^{X} \varphi(x, F_2(x)) \, dx \]

\[ = \lambda M + (1 - \lambda) M = M. \]

This shows that a convex combination of two maximizing admissible c.d.f:s which is itself also admissible, gives a value which is smaller than the maximum. Hence the maximum of \( I(F) \) occurs for c.d.f:s which correspond to the extreme points of the convex set \( \mathcal{A} \).

By Theorems 21.1 and 21.3 of Karlin and Shapley [5], it is then easy to see that the maximizing c.d.f. is at most a three-point distribution.

Remark: It is important to note here that in some cases, the maximizing admissible c.d.f. can be further reduced to a two-point distribution [1,3].

We shall illustrate the above results by a few examples.

Example 1. Suppose we want to minimize \( E(x_n) \) given in Example 4 of the last section, over all admissible c.d.f:s. The problem is the same as that of maximizing

\[ \int_{-X}^{X} p_n(x) \, dx \]

over all admissible c.d.f:s. As \( \varphi(x, y) = y^n \), is strictly convex in \( y \), the maximizing admissible c.d.f. of the above integral is at most a three-point distribution.

Similarly the minimizing admissible c.d.f. of \( E(w_n) \) is at most a
three-point distribution. David and Hartley [1] show that it can be further reduced to a two-point distribution.

**Example 2.** Suppose we are interested in finding the maximizing c.d.f. of

\[ \frac{1}{2} \int_{-X}^{X} [F(x) - x]^2 \, dx \]

such that \( F(x) \) satisfies side-conditions (3.3).

Now \( \varphi(x,y) = \frac{1}{2} (y,x)^2 \) is strictly convex function of \( y \). Hence the solution of the above problem is at most a three-point distribution.
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