KINGMAN'S SUBADITIVE ERGODIC THEOREM

BY

J. MICHAEL STEELE

TECHNICAL REPORT NO. 324
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Kingman's Subadditive Ergodic Theorem

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The objective of this note is to give a proof of Kingman's subadditive ergodic theorem which is perhaps simpler and more direct than those previously given ([2], [3], [4], [5], [6], [7], [11]).

Theorem. Suppose $T$ is a measure preserving transformation of the probability space $(\Omega, \mathcal{F}, \mu)$ and that $\{g_n, 1 \leq n < \infty\}$ is a sequence of integrable functions which satisfy

$$g_{n+m}(x) \leq g_n(x) + g_m(T^n x). \tag{1}$$

With probability one we then have the existence of the limit

$$\lim_{n \to \infty} g_n(x)/n = g(x) \geq -\infty,$$

where $g(x)$ is an invariant function.

Proof. We first check that $g(x) = \liminf g_n(x)/n$ is an invariant function. Since $g_{n+1}(x)/n \leq g_1(x)/n + g_n(Tx)/n$ we see $g(x) \leq g(Tx)$ which gives $\{g(x) > \alpha\} \subseteq T^{-1}\{g(x) > \alpha\}$. The fact that $T$ is measure preserving then implies $\{g(x) > \alpha\} = T^{-1}\{g(x) > \alpha\}$ up to null sets. This implies $g$ is measurable with respect to the invariant $\sigma$-field and hence is invariant. The function $\phi(x) = \max(t, g(x))$ where $t \in (-\infty, 0)$ is also invariant.
For $\epsilon > 0$, set $A_\epsilon = \{x: g^\infty_\epsilon (x) \leq \ell (\phi (x) + \epsilon)\}$ and note that
\[
\mu (\bigcup_{\epsilon=1}^\infty A_\epsilon) = 1,
\]
so we can choose $N$ such that for $B(N) = (\bigcup_{k=1}^N A_\epsilon)^c$ we have \(\mu (B(N)) \leq \epsilon\).

Now, by Birkhoff's ergodic theorem, \(\frac{1}{n} \sum_{k=1}^n 1_B(N)(T^k x)\) converges a.s. to \(E(1_B(N) | G)\) where $G$ is the invariant field of $T$; so by Chebyshev's inequality
\[
\mu (\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^n 1_B(N)(T^k x) \geq \lambda) \leq \epsilon / \lambda.
\]

Setting
\[
C_M = \{x: \frac{1}{n} \sum_{k=1}^n 1_B(N)(T^k x) \leq 2n\lambda, \quad \forall n \geq M\}
\]
we have for $M$ sufficiently large that \(\mu (C_M) \geq 1 - 2\epsilon / \lambda\).

For any $x \in C_M$ and $n \geq M$ we obtain a decomposition for the integer set $[0, n)$ into three classes of intervals by the following algorithm:

Begin with $k = 0$. If $k$ is the least integer in $[0, n)$ not in an interval already constructed then we consider $T^k x$. If $T^k x \in B(N)^c$ then there is an $\ell \leq N$ so that $g^\ell_k (T^k x) \leq \ell (\phi (T^k x) + \epsilon) \leq \ell (\phi (x) + \epsilon)$ and we take $[k, k+\ell)$ as an element of our decomposition provided $k+\ell \leq n$. If $T^k x \in B(N)$ we take the singleton interval $[k, k+1)$.

This algorithm provides a decomposition of some $[0, n')$ with $n'-N \leq n' \leq n$, and it is extended to a decomposition of $[0, n)$ by adding as many singletons as necessary.
Thus for any \( x \in C_M \) we have a decomposition of \([0,n)\) into a set of \( u \) intervals \([\tau_i, \tau_{i+1}]\), \( 1 \leq \tau_i \leq N \), for which 
\[
g_{0_i}^{\tau_i}(T^{-x}) \leq \ell_i(\phi(x) + \varepsilon)\]

with a set of \( v \) singletons \([\sigma_i, \sigma_{i+1}]\) for which \( 1_{B(N)}(T^{-x}) = 1 \), and a set of \( w \) singletons contained in \((n-N,n)\).

By (1) and this decomposition of \([0,n)\) we have on \( C_M \),

\[
g_n(x) \leq \sum_{i=1}^{u} g_{0_i}^{\tau_i}(T^{-x}) + \sum_{i=1}^{v} g_{1}(T^{-x}) + \sum_{i=1}^{w} g_{1}(T^{n-i}x) \leq (\phi(x) + \varepsilon) \sum_{i=1}^{u} \ell_i + \sum_{i=1}^{v} g_{1}(T^kx)1_{B(N)}(T^kx) + \sum_{i=1}^{n} |g_{1}(T^{n-i}x)| .
\]

Since

\[
\sum_{k=1}^{\infty} \mu(|g_{1}(T^kx)| > \delta k) = \sum_{k=1}^{\infty} \mu(|g_{1}(x)| > \delta k) < \infty , \text{ for all } \delta > 0,
\]

the Borel-Cantelli lemma implies \( g_{1}(T^kx)/k \to 0 \) a.s.. From this one easily sees that almost surely

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |g_{1}(T^{n-i}x)| = 0 .
\]

Also, by Birkhoff's ergodic theorem we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} g_{1}(T^kx)1_{B(N)} = E(g_{1}1_{B(N)} | \Omega) .
\]
Finally \( n \geq \sum_{i=1}^{u} \lambda_i \geq n - N - 2 \varepsilon n \) so from (2), (3), (4) we have on \( C_M \) that

\[
\limsup_{n \to \infty} \frac{g_n(x)}{n} \leq \phi(x) + 3\varepsilon + E(g_1 l_{B(N)} | \mathcal{G}) .
\]

For \( N \to \infty, l_{B(N)} \to 0 \) a.s. so by dominated convergence

\[
E(g_1 l_{B(N)} | \mathcal{G}) \to 0 \text{ a.s.}
\]

Therefore, by the arbitrariness of \( \varepsilon, t, \lambda, N, \) and \( M \) we have with probability one that

\[
\limsup_{n \to \infty} \frac{g_n(x)}{n} \leq \liminf_{n \to \infty} \frac{g_n(x)}{n} ,
\]

which completes the proof of convergence.

Remarks. (1). The preceeding proof was motivated by the recent proofs of the Birkhoff ergodic theorem and the Shannon-MacMillan-Breiman theorem given by Paul Shields [8]. That work is in part devoted to the simplification and exposition of some recent work of Ornstein and Weiss [7].

(2). Inspection of the preceeding proof shows that it suffices to assume that just \( g_1^+ \in L^1 \), instead of \( g_n \in L^1 \), for all \( n \). That the subadditive ergodic theorem persists under this condition was already observed in Kingman [5, p. 885].

(3). David Aldous has shown that Kingman's subadditive ergodic theorem can be used to give a very brief proof of the ergodic theorem for Banach space due to Maurier [8]. If \( \{X_i\} \) is a stationary process with values in a Banach space \( F \), we first note there is no loss in assuming \( E(X_1 | \mathcal{G}) = 0 \) where \( \mathcal{G} \) is the invariant \( \sigma \)-algebra. Also, we can find a linear operator \( \theta \) on \( F \) with finite dimensional range such that \( \|X_1 - \theta X_1\| \leq \varepsilon \). Now
Birkhoff's ergodic theorem (applied to linear functionals) shows that
\[ \frac{1}{n} \sum_{i=1}^{n} \theta(X_i) \] converges a.s. and in \( L^1 \) to \( E\theta(X_1) \). The \( L^1 \) convergence guarantees \( \lim \frac{1}{n} \sum_{i=1}^{n} E|S_i - E\theta(X_1)| \leq \varepsilon \) from which it follows that \( \lim E||S_n/n|| = 0 \). But since \( ||S_n|| \) is a subadditive process \( ||S_{n}/n|| \) converges a.s., and now necessarily converges a.s. to zero.

(4) Andrés del Junco has pointed out that there is a useful device of Akcoglu and Sucheston \([1]\) which can be used to circumvent the estimations of the last two terms in equation (2). The idea is that
\[ g_m' = g_m(x) - \sum_{i=0}^{m-1} g_i(T^i x) \] defines a (negative) subadditive process. The last two terms in equation (2) applied to \( g_m' \) would then simply not appear.

The proof given above was retained in order to maximize conceptual simplicity (at the cost of a little extra computation).

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References


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**Abstract:**
A simple proof of Kingman's subadditive ergodic theorem is given.