ASYMMETRIC WIENER CONTROL

BY

HOWARD WEINER

TECHNICAL REPORT NO. 332
APRIL 12, 1983

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STANFORD, CALIFORNIA
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University of California, Davis and Stanford University

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Asymmetric Wiener Control

by Howard Weiner

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1. Introduction. Let \( W(t) \) denote a standard Wiener process with \( W(0) = 0 \), and corresponding increasing sigma fields \( F(t) = \sigma(W(s), 0 \leq s \leq t) \).

Let \( X(t) \) be the process given by

\[
dX(t) = u(t)dt + dW(t), \quad t \geq 0
\]

and

\[ X(0) = x \text{ (constant)} \]

where

\[
(1.1) \quad u(t) \equiv u(t,X(t)).
\]

The control \( u(t) \) is required to be measurable with respect to \( F(t) \), all \( t \geq 0 \), and to satisfy

\[
(1.2) \quad |u(t) - A| \leq B \quad \text{all } t \geq 0
\]

where \( 0 < |A| < B \) are constants.

The cost function, for \( \alpha \geq 0 \) constant, is given by

\[
(1.3) \quad J(u) \equiv \int_0^\infty e^{-\alpha u} E(X^2(u))du.
\]
The object is to obtain \( u_0(t) \) satisfying (1.1)-(1.2) which minimizes (1.3).

The method is to solve the Bellman equation in two regions and to match solutions appropriately, then demonstrating that a solution so obtained must be optimal.

2. Optimal Control.

Lemma. Denote, for \( D \) a constant,

\[
(2.1) \quad J(D,x) = \int_0^\infty e^{-\alpha y} E(Dy+W(y)+x)^2 \, dy
\]

\[
= \frac{2D^2}{\alpha^3} + \frac{x^2}{\alpha} + \frac{(2Dx+1)}{\alpha^2}.
\]

Then \( J(D,x) \) is a particular solution to the equation

\[
(2.2) \quad x^2 + D V'(x) + \frac{1}{2} V''(x) - \alpha V(x) = 0.
\]

Proof. This is a direct computation.

Theorem. Let the conditions of Section 1 hold. Assume that the constants \( K, L \) given respectively by

\[
(2.3) \quad K = \frac{2}{\alpha} + \frac{4Br}{\alpha^2} \left( \frac{1}{q} - \frac{(A+B)}{\alpha} \right) \geq 0
\]

and

\[
(2.4) \quad L = \frac{2}{\alpha} + \frac{4qB}{\alpha^2} \left( \frac{1}{r} + \frac{(B-A)}{\alpha} \right) \geq 0,
\]
where

\[
(2.5) \quad r = (B-A) - \sqrt{(B-A)^2 + 2\alpha}.
\]

\[
(2.6) \quad q = -(A+B) + \sqrt{(A+B)^2 + 2\alpha}.
\]

Denote the constant \( b \) by

\[
(2.7) \quad b = \frac{\frac{4AB}{r} + \frac{1}{q}\left(\frac{A-B}{r} - \frac{A+B}{q}\right)}{\frac{1}{q} - \frac{1}{r} - \frac{2B}{\alpha}}.
\]

Then the optimal control law \( u_0(t) \) is expressible as

\[
(2.8) \quad u_0(t) = \begin{cases} 
A - B & \text{if } X_0(t) \geq b \\
A + B & \text{if } X_0(t) < b
\end{cases}
\]

where the Itô equation holds:

\[
(2.9) \quad dX_0(t) = u(X_0(t))dt + dW(t)
\]

\[X_0(0) = x.\]
Proof. The Bellman function is

\[ (2.10) \quad V(x) = \inf_{|u-A| \leq 1} \int_0^\infty e^{-\alpha y} E(X^2(y)) dy, \quad X(0) = x. \]

Writing

\[ (2.11) \quad \int_0^\infty e^{-\alpha y} E(X^2(y)) dy = \int_0^h + \int_h^\infty (e^{-\alpha y} E(X^2(y)) dy, \]

and expanding to order \( h \), letting \( h \to 0 \), one obtains the appropriate Bellman equation (see also \([5]\), pp. 329-330, 336-337).

\[ (2.12) \quad x^2 + \inf_{|u-A| \leq 1} uV'(x) + \frac{3}{2} V''(x) - \alpha V(x) = 0. \]

A solution to (2.12) is sought satisfying

\[ (2.13) \quad x^2 + (A-1)V'(x) + \frac{3}{2} V''(x) - \alpha V(x) = 0 \quad \text{if} \ V'(x) > 0, \ x > b \]

\[ (2.14) \quad x^2 + (A+1)V'(x) + \frac{3}{2} V''(x) - \alpha V(x) = 0 \quad \text{if} \ V'(x) < 0, \ x < b. \]

The solution in (2.13) is denoted \( V_1(x) \), and that in (2.14), \( V_2(x) \).

The matching conditions are

\[ (2.15) \quad V_1(b) = V_2(b) \]

\[ V_1'(b) = V_2'(b) = 0. \]

It may be directly verified that a particular solution to (2.13) is given by
J(A-B,x) and a particular solution to (2.14) is J(A+B,x). A solution to the homogeneous part of (2.13) of at most quadratic growth is

\[(2.16)\quad Ce^{rx}\]

where \(r\) satisfies (2.5). Similarly, an appropriate homogeneous solution for (2.14) is

\[(2.17)\quad De^{qx}\]

where \(q\) satisfies (2.6). Hence one has the trial solutions

\[(2.18)\quad V_1(x) = J(A-B,x) + Ce^{rx}, \quad x \geq b\]

\[V_2(x) = J(A+B,x) + De^{qx}, \quad x < b,\]

for \(b\) determined from the matching conditions (2.15) and is given explicitly by (2.7).

**Claim.**

\[V(x) = \begin{cases} V_1(x), & x \geq b \\ V_2(x), & x < b \end{cases}\]

is a solution to the Bellman equation (2.12).

**Proof of Claim.** Let \(W(x) = V''(x)\). Then (omitting the argument)

\[(2.19)\quad (A-B)W' + \frac{a}{2}W'' - aW = -2 \quad \text{for} \quad x \geq b\]

and
\[(2.20)\] \( (A+B)\dot{W} + \frac{1}{2}W^2 - \alpha W = -2 \quad \text{for } x < b. \]

Suppose there is an \( r > b \) and an \( a, \quad 0 < a < 1 \) such that

\[(2.21)\] \( W(x) < 0, \quad b < r - a < x < r \)

\[(2.22)\] \( W(r) = 0 \)

\[(2.23)\] \( W(x) > 0 \quad \text{for } x > r. \)

Condition (2.23) holds, since by construction of \( V \) in (2.13) - (2.18), it follows that

\[(2.24)\] \( W(x) > 0 \quad \text{as } |x| \to \infty. \)

A direct computation and assumption (2.3) yields

\[(2.25)\] \( V_1''(b) = K \geq 0. \)

A maximum principle ([3], p. 53, Theorem 18) yields, from (2.19), that \( W \) cannot have a negative minimum for \( x > b \). But from (2.24), (2.25), and (2.21), it follows that \( W \) would have a negative minimum for \( x > b \), contradicting the maximum principle. Hence (2.21) cannot hold and \( W(x) \geq 0 \) for \( x > b \). A similar argument using the fact that

\[(2.26)\] \( V_2''(b) = L \geq 0 \)

establishes that \( W(x) \geq 0, \quad x < b, \) so \( W(x) \geq 0 \) all \( x \). This completes the claim.

By construction of the solutions, \( V_1'(b) = V_2'(b) = 0 \), hence \( V''(x) \geq 0 \) for all \( x \) implies that \( V_1'(x) \geq 0, \quad x \geq b \) and \( V_2'(x) \leq 0, \quad x \leq b \), which shows
that the constructed $V(x)$ in the claim solves the Bellman equation (2.12).

It remains to show that $u_0$ is optimal.

Let, for the constructed $V(x)$,

\[(2.27) \quad K(x,t) \equiv e^{-\alpha t} V(x)\]
and so

\[
K(X(t), t) = e^{-\alpha t} V(X(t)),
\]

where \(X(t)\) corresponds to some \(u\) in (1.1)-(1.2) for \(t \geq 0\). By Ito's rule,

\[
\int_0^t e^{-\alpha y} X^2(y) dy + e^{-\alpha t} V(X(t)) - V(x)
\]

\[
= \int_0^t e^{-\alpha y} \left[ -\alpha V(X(y)) + \inf_{|u-A| \leq 1} uV'(X(y)) + \frac{V''(X(y))}{2} + X^2(y) \right] dy
\]

\[
+ \int_0^t e^{-\alpha y} [uV'(X(y)) - \inf_{|u-A| \leq 1} u(X(y), y)V'(X(y))] dy
\]

\[
+ \int_0^t e^{-\alpha y} [V'(X(y))] dW(y).
\]

By construction of \(V\) as a solution to (2.12), the first integral on the right of (2.29) is zero. The second integral on the right of (2.29) is non-negative. Taking expectations, the third integral on the right of (2.29) is zero. By construction of \(V(x)\) as quadratic in \(x\), for some constants \(K, D,\)

\[
E(V(X(t))) \leq K E(x + |A| + 1) t + |W(t)|^2 \leq Dt^2,
\]

so that

\[
e^{-\alpha t} E(V(X(t))) \to 0 \text{ as } t \to \infty.
\]

Hence, letting \(t \to \infty\) in (2.29),
with equality when the $u_0$ of (2.8) is used, hence it is optimal.

3. Remarks 1. The finite integral case of this problem may require the method of [6], but it is not clear how to obtain tractable results.

2. A sufficient condition that $K > 0$, $L > 0$ in (2.3), (2.4) is that $B - A \gg \alpha$. 
REFERENCES


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<td>A one-dimensional Wiener control problem with integral discounted quadratic cost function and asymmetric bounds on the control is considered, with infinite horizon. The optimal control is explicitly found. Bellman equations and Ito integrals are used to show optimality.</td>
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