SAMPLING RANDOM POLYGONS

BY

EDWARD I. GEORGE

TECHNICAL REPORT NO. 385
DECEMBER 29, 1986

PREPARED UNDER CONTRACT
NO0014-86-K-0156 (NR-042-267)
FOR THE OFFICE OF NAVAL RESEARCH

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STANFORD UNIVERSITY
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1. Introduction.

Each realization of a Poisson line process is a set of lines which subdivides $\mathbb{R}^2$ into a population of nonoverlapping convex polygons. The statistical properties of this polygon population, which we denote by $P^*$, has been of substantial interest to researchers in geometric probability. In particular, much attention has focused on the unknown distributions of the number of sides, the perimeter, and the area of these polygons, Goudsmit (1945), Miles (1964, 1973), Richards (1964), Matheron (1972), Crain and Miles (1976), Solomon (1978), Solomon and Stephens (1980), Tanner (1983a, 1983b).

To obtain better estimates of the unknown aspects of the distributions of polygon characteristics, this paper is concerned with obtaining a 'random sample' from $P^*$. Unfortunately one cannot sample directly from $P^*$ due to the infinitude of polygons. To overcome this difficulty, we develop an alternative stochastic construction of random polygons which have the same distribution as those in $P^*$. This construction, which is based on a sequential stochastic process of random angles and side lengths rather than a Poisson line process, provides a fast and inexpensive simulation method for obtaining a random sample from $P^*$.

For the case where $P^*$ is rotationally invariant or isotropic, we perform independent repetitions of this simulation to obtain a random sample of 2,500,000 polygons. This sample provides the most precise estimates to date of some of the unknown distributional characteristics.

Previous Monte Carlo studies of the polygon distributions were carried out by Dufour (see Solomon (1978), page 54) and Crain and Miles (1976) who obtained approximate random samples from $P^*$ by first simulating a Poisson line process in a fixed bounded region of $\mathbb{R}^2$, and then extracting the polygons circumscribed by the lines in this region. In contrast to samples obtained by our sequential stochastic construction, these approximate random samples contain bias and
dependence which diminish their estimation value. Furthermore, it appears that our stochastic construction requires much less computer effort per polygon obtained. Indeed, we processed 2,500,000 polygons, while Dufour processed 947 polygons and Crain and Miles processed about 200,000 polygons. Rough comparisons suggest that our construction is about 20 times faster per polygon (George 1982). We should mention that Miles (1973) also provides stochastic constructions of random polygons from \( P^* \). However, these constructions are different than ours in that they are based on the partial realization of a Poisson line process. As indicated by Crain and Miles (1976), simulation of these constructions appears to require as much computer effort as their approach.

The outline of this paper is as follows. In Section 2, we review the necessary background and define polygon notation. In Section 3, a sequential stochastic construction of random polygons is developed. In Section 4, the distribution of a 'random polygon from \( P^* \)' is shown to follow directly from this construction. In Section 5, methods for simulating this construction are suggested. In Section 6, the large scale simulation estimates are presented.

2. Background and Notation.

2.1 Poisson Line Processes.

Each line in \( \mathbb{R}^2 \) may be parametrized by \((p, \theta)\) where \( p \in (-\infty, \infty) \) is the signed length of the perpendicular from the line to an origin \( 0 \in \mathbb{R}^2 \) and \( \theta \in [0, \pi) \) is the northeast angle this line makes with the horizontal, see Figure 2.1.

![Figure 2.1. The parametrization \((p, \theta)\).](image-url)
Thus the space of all lines in $\mathbb{R}^2$ corresponds to the set of points on the cylinder

$$C = \{(p, \theta) : p \in (-\infty, \infty), \theta \in [0, \pi)\}.$$ 

A Poisson line process $L$ is a random aggregate of lines in $\mathbb{R}^2$ corresponding to a two-dimensional Poisson point process on $C$. This process is specified by the intensity function $\tau m(\cdot)$ where for any disjoint (measurable) $A_1, \ldots, A_k$ on $C$, the numbers of realized points $N(A_1), \ldots, N(A_k)$ have independent Poisson distributions with means $\tau m(A_1), \ldots, \tau m(A_k)$, where $\tau > 0$ and for some nondegenerate probability measure $G$ on $[0, \pi)$,

$$m(A) = \int_A dp dG(\theta).$$

In the sequel, we sometimes reparametrize $(p, \theta)$ by $(p, \phi)$ where $\theta = \phi \mod \pi$. The distribution of $L$ is then implicitly obtained by extending the domain of $G$ via $dG(\phi) = dG(\phi \mod \pi)$.

It is easy to see that every Poisson line process $L$ is invariant under translations of $\mathbb{R}^2$. When $dG(\theta) = d\theta/\pi$, $L$ is also invariant under rotations of $\mathbb{R}^2$ and is said to be isotropic. Isotropic Poisson line processes are often associated with 'completely random lines' and seem to be the case of most interest, see Crofton (1885), Santalo (1953), Miles (1964, 1973), Davidson (1974).

The following well-known property of $L$ (Miles 1973) can be shown by expressing events concerning $L$ in $\mathbb{R}^2$ as sets on $C$ (George 1982).

**Theorem 2.1** Points of intersection of $L$ with $l' = (\theta', p')$, an arbitrary line in $\mathbb{R}^2$, are a linear Poisson process of constant intensity $\tau \lambda(\theta')$ where

$$\lambda(\theta') = \int_0^\pi |\sin(\theta - \theta')| dG(\theta).$$

Furthermore, the orientation angles of the lines associated with these points of intersection are independent and identically distributed with common conditional
density

\[ dF(\theta | \theta') = \frac{1}{\lambda(\theta')} |\sin(\theta - \theta')| dG(\theta) \]

for \( \theta \in [0, \pi) \).

Because \( \theta' \) is arbitrary, Theorem 2.1 holds conditionally for \( \theta' \in \mathcal{L} \).

Note also that Theorem 2.1 provides an alternative stochastic construction of \( \mathcal{L} \). For other equivalent characterizations of \( \mathcal{L} \), see Miles (1964, 1973), Davidson (1973), Solomon (1978).

The following result is proved in much the same manner as the first part of Theorem 2.1, (George 1982).

**Theorem 2.2** Consider \( T \) an arbitrary triangle with sides \( T_1, T_2, T_3 \) of lengths \( t_1, t_2, t_3 \) and orientations \( \theta_1, \theta_2, \theta_3 \) respectively. Then the number of intersections of \( \mathcal{L} \) with \( T \) which do not intersect side \( T_3 \) has a Poisson distribution with mean

\[ \frac{1}{2} \left( t_1 \lambda(\theta_1) + t_2 \lambda(\theta_2) - t_3 \lambda(\theta_3) \right). \]

Theorems 2.1 and 2.2 are the only essential properties of a Poisson line process which we use in later sections.

### 2.2 A random polygon from \( P^* \)

As discussed in the introduction, this paper is concerned with investigating the distributions of polygon characteristics in \( P^* \), the population of nonoverlapping convex polygons formed by a realization of a Poisson line process \( \mathcal{L} \).

Miles (1964, 1973) showed that such distributions are well-defined by demonstrating the existence of ergodic distributions defined as limits of empirical distributions of polygon characteristics contained in a disc of
radius \( q \) as \( q \rightarrow \infty \). Furthermore, he showed that these ergodic distributions are the same for each realization of \( L \) (w.p.1). Thus, \( P^* \) also refers to the superpopulation of polygons formed by all realizations of \( L \).

Although one cannot directly sample a polygon from \( P^* \) (Cowan 1978), we shall refer to any random polygon whose distribution coincides with the distribution of polygons in \( P^* \), as a 'random polygon from \( P^* \)'. In Section 3, we develop a stochastic construction of such a random polygon from \( P^* \). This construction is based on the sequential realization of random angles and side lengths rather than a Poisson line process. The following notation is used.

We parametrize (up to translation) a random, convex polygon as follows. Let \( N \) be the number of sides of the polygon. Starting with the southernmost vertex (if there are two choose the one on the left), label the vertices consecutively in clockwise direction by \( v_1, \ldots, v_N \). For \( i = 1, \ldots, N \), let \( Z_i \) be the length of the segment \( \overline{v_i v_{i+1}} \), \( (v_{N+1} = v_1) \), and let \( \phi_i \) be the angle between \( \overline{v_i v_{i+1}} \) and the eastern horizontal at \( v_i \). We denote the lines coinciding with \( \overline{v_1 v_2}, \overline{v_2 v_3}, \ldots, \overline{v_{N-1} v_N} \) by \( \ell_1, \ell_2, \ldots, \ell_N \). It will be convenient to let \( \ell_0 = \ell_N \), and let \( \phi_0 = \phi_N + \pi \) be the angle between \( \overline{v_N v_1} \) and the eastern horizontal at \( v_1 \). See Figure 2.2.

![Figure 2.2](image)

Figure 2.2. The notation for a polygon when \( N = 5 \).
For notational ease, the first 2i polygon coordinates \((N \geq i)\), are expressed by,

\[
\phi^i = (\phi_0, \phi_1, Z_1, \phi_2, \ldots, Z_{i-1}, \phi_i), \quad (\phi^1 = (\phi_0, \phi_1)) ,
\]

and the first 2i+1 coordinates by

\[
Z^i = (\phi_0, \phi_1, Z_1, \phi_{2}, Z_2, \ldots, \phi_i, Z_i) .
\]

Using this notation, a random N-sided convex polygon in \(R^2\) may be identified (up to translation) by \(Z^N\). Note that three coordinates of \(Z^N\) will be redundant since

\[
\sum_{i=1}^{N} Z_i \sin \phi_i = \sum_{i=1}^{N} Z_i \cos \phi_i = 0 \quad \text{and} \quad \phi_N = \phi_0 - \pi .
\]

For instance, \(Z^N\) is completely determined by \(\phi^{N-1}\).

In terms of this notation, the perimeter of \(Z^N\) is

\[
S = \sum_{i=1}^{N} Z_i ,
\]

and the area is

\[
A = \frac{1}{2} \sum_{i=1}^{N} (X_i - X_{i-1})(Y_i + Y_{i-1}) ,
\]

where \((X_{i-1}, Y_{i-1})\) are the Cartesian coordinates of \(v_i\) when \(v_1 = (0,0)\), namely

\[
(X_0, Y_0) = (0,0) \quad \text{and} \quad (X_i, Y_i) = (\sum_{j=1}^{i} Z_j \cos \phi_j , \sum_{j=1}^{i} Z_j \sin \phi_j) .
\]

Throughout this paper \(Z^N\) will represent a random polygon from \(P^n\) in the sense described above. We use \(N, Z_i, \phi_i, Z^i, \phi^i, S, A, X_i, Y_i\) to denote random variables or vectors, and the lowercase equivalents \(n, z_i, \phi_i, z^i, \phi^i, s, a, x_i, y_i\) to denote corresponding realizations. For example, a realization of

\[
Z^N = (\phi_0, \phi_1, Z_1, \ldots, \phi_N, Z_N) \quad \text{is denoted} \quad z^N = (\phi_0, \phi_1, z_1, \ldots, \phi_n, z_n) .
\]
3. **Sequential Stochastic Construction of Random Polygons.**

In this section we develop a stochastic construction of $Z^N = (\phi_0, \phi_1, Z_1, \ldots, \phi_N, Z_N)$ based on the sequential realization of its angles and side lengths. We begin by obtaining the marginal distribution of the initial angles $\phi^1 = (\phi_0, \phi_1)$ when these correspond to the southernmost vertex of a random polygon from $P^*$. We then derive conditional distributions of the subsequent side lengths $Z_i$ and angles $\phi_i$ for $i < N$. The construction of $Z^N$ is completed by using the number of sides as a stopping time in the sequence of random angles and side lengths.

3.1 **Distribution of the initial angles.**

Because of the bijection between intersections of lines of $L$ and southernmost vertices (if there are two pick the one on the left) of polygons in $P^*$ (Miles 1964), the distribution of $\phi^1 = (\phi_0, \phi_1)$ for $Z^N$, a random polygon from $P^*$, may be obtained from the distribution of angles corresponding to intersection points. Note that only the distribution of intersection angles of $L$ is required; it is not necessary to actually sample a 'random intersection'.

It follows from the second half of Theorem 2.1 (Miles 1964, 1973; George 1982) that at points of intersection of members of $L$, the orientation angles $\theta, \theta'$ of the intersecting lines have joint density

$$dF(\theta, \theta') = \frac{1}{\lambda} |\sin(\theta - \theta')|dG(\theta)dG(\theta')$$

on $\theta, \theta' \in [0, \pi)$, where

$$\lambda = \int_0^\pi \lambda(\theta)dG(\theta).$$

In the isotropic case where $dG(\theta) = d\theta/\pi$ so that $\lambda = 2/\pi$, (3.1) becomes

$$dF(\theta, \theta') = \frac{1}{2\pi} |\sin(\theta - \theta')|d\theta d\theta'$$

on $\theta, \theta' \in [0, \pi]$. Although this result is somewhat counterintuitive because
lines in an isotropic field would seem to meet at uniformly distributed angles, (3.3) reflects the fact that angles far from perpendicular are 'shifted out towards infinity'.

For a polygon with southernmost vertex \( v_1 \) located at the intersection point of two lines in \( L \) with orientation angles \( \theta, \theta' \), the polygon angles \( \phi_0, \phi_1 \) may be obtained as the minimum and maximum of \( \theta, \theta' \) respectively. It follows from (3.1) that the joint distribution of \( \phi^1 = (\phi_0, \phi_1) \) is

\[
dF(\phi_0, \phi_1) = \frac{2}{\lambda} \sin(\phi_1 - \phi_0) dG(\phi_1) dG(\phi_0)
\]

for \( 0 \leq \phi_0 \leq \phi_1 < \pi \).

For isotropic \( P^\infty \), the actual value of \( \phi_0 \) will be irrelevant due to the rotational invariance. In this case, we rotate \( Z^N \) so that \( \phi_0 = 0 \), and begin the sequence of angles and side lengths with \( \phi_1 \). The distribution of \( \phi_1 \) is then obtained from (3.3) as the marginal distribution of the maximum of \((\theta-\theta')\) and \((\theta'-\theta)\), namely

\[
dF(\phi_1) = \frac{1}{\pi} \sin \phi_1 d\phi_1 \int_0^{(\pi-\phi_1)} d\phi = \frac{\pi-\phi_1}{\pi} \sin \phi_1 d\phi_1
\]

for \( 0 \leq \phi_1 < \pi \), (Miles 1973).

3.2 Conditional distribution of the side lengths.

In this section we derive the conditional distribution of \( Z_i \) given \( \phi^i = \phi^i \) \( = (\phi_0, \phi_1, z_1, \ldots, \phi_i) \), and \( i < N \). This derivation is based on the observation that \( Z_i \) is the distance from \( v_1 \) along \( l_i \) to the next intersection by a member of \( L \), namely \( l_{i+1} \). The conditional information here is that \( \phi^i \) is already part of the polygon so that \( l_{i+1} \), which coincides with the next side of the polygon, cannot intersect any of \( \overline{v_1 v_2}, \overline{v_2 v_3}, \ldots, \overline{v_{i-1} v_i} \) or equivalently \( \overline{v_1 v_i} \).

We begin with the simple but illuminating case of \( Z_1 \), the distance from \( v_1 \) along \( l_1 \) to the next intersection by \( l_2 \). By Theorem 2.1, the distribution of
intersections along \( \xi_1 \) is a Poisson process of constant intensity \( \tau \lambda(\phi) \).

Because any orientation angle of \( \xi_2 \) is acceptable, the conditional distribution of \( Z_1 \) given \( \phi^1 = \phi^1 \), \( (N > 1 \text{ always}) \), is exponential, namely

\[
(3.6) \quad dF(z_1|\phi^1) = \tau \lambda(\phi) \exp(-\tau \lambda(\phi)z_1)dz_1 \quad \text{for} \; z_1 \in [0,\infty).
\]

For the case \( i > 1 \), we proceed by first deriving the conditional distribution of \( Z_i \), the distance along \( \xi_i \) from \( v_i \) in the clockwise direction, to the first intersection by some member of \( L \) different from \( \xi_0 \) which does not cross \( \overline{v_1v_i} \). Let \( w_{it} \) be a point on \( \xi_i \) a distance \( t \) in the clockwise direction from \( v_i \). Note that \( w_{i0} = v_i \). Let \( d_i(t) \) be the length of \( \overline{v_iw_{it}} \), and let \( \gamma_i(t) \) be the angle between \( \overline{v_iw_{it}} \) and the eastern horizontal at \( w_{it} \). Consider the triangle \( T \) with vertices \( v_i, v_1 \) and \( w_{it} \). See Figure 3.1.

![Figure 3.1](image)

By Theorem 2.2, the number of lines intersecting \( T \) which do not cross \( \overline{v_1v_i} \) has a Poisson distribution with mean

\[
(3.7) \quad Q(t; \phi^1) = \frac{\tau}{2} \left[ \tau \lambda(\phi) + d_i(t)\lambda(\gamma_i(t)) - d_i(0)\lambda(\gamma_i(0)) \right].
\]
Hence (Snyder 1975), the distribution of the corresponding intersection points along \( l_i \) in the clockwise direction from \( v_i \) is a nonhomogeneous linear Poisson process with intensity \( q(t; \phi^i) \) satisfying

\[
\int_0^t q(s; \phi^i) ds = Q(t; \phi^i) ,
\]

or

\[
q(t; \phi^i) = \frac{1}{2} \left[ \lambda(\phi^i) + \frac{3}{2} \frac{d_1(t)}{\lambda(\gamma(t))} \right] .
\]

Because \( \bar{Z}_i \) is the distance to the first intersection, it follows that the conditional distribution of \( Z_i \) given \( \phi^i = \phi^i \) and \( i < N \), is

\[
dF(\bar{Z}_i; \phi^i, i < N) = q(\bar{Z}_i; \phi^i) \exp(-Q(\bar{Z}_i; \phi^i)) d\bar{Z}_i \quad \text{for} \quad \bar{Z}_i \in (0, \infty) .
\]

Because the conditional information \( \phi^i = \phi^i \) specifies that \( \xi_0 \) is already part of the polygon, \( Z_i \) may differ from \( \bar{Z}_i \). However, \( Z_i \) is related to \( \bar{Z}_i \) by

\[
Z_i = \min[Z_i, u_i] ,
\]

where

\[
u_i = \begin{cases} 
-\sec(\phi^i - \phi_0) \sum_{j=1}^{i-1} z_j \sin(\phi^i - \phi_0) & \text{for} \quad \phi^i < \phi_0 \\
\infty & \text{for} \quad \phi^i \geq \phi_0 
\end{cases}
\]

is the distance along \( l_i \) from \( v_i \) to \( \xi_0 \). See Figure 3.2. Of course, when \( \phi^i \geq \phi_0 \), \( Z_i = \bar{Z}_i \) since the segment of length \( \bar{Z}_i \) from \( v_i \) along \( l_i \) will never cross \( \xi_0 \). It now follows immediately from (3.10) and (3.11) that the distribution of \( Z_i \) given \( \phi^i = \phi^i \) and \( i < N \) is given by

\[
dF(z_i; \phi^i, i < N) = q(z_i; \phi^i) \exp(-Q(z_i; \phi^i)) dz_i \quad \text{for} \quad z_i \in [0, u_i]
\]

\[
= \exp(-Q(u_i; \phi^i)) \quad \text{for} \quad z_i = u_i
\]
When $\phi_i < \phi_0$, $dF(z_i | \phi^i, i < N)$ is truncated at $u_i = \infty$, where it has a point mass.

Figure 3.2

Before concluding this subsection, we note that given $\phi^i = \phi^i$, the event \( \{Z_i = u_i\} \) determines the number of sides of the polygon as \( \{N = i+1\} \). It follows immediately from (3.13) that

\[
(3.14) \quad P[N = i+1|\phi^i, i < N] = P[Z_i = u_i|\phi^i, i < N] = \exp(-Q(u_i; \phi^i)).
\]

Of course, when $\phi_i \geq \phi_0$ this probability is zero.

3.3 Conditional distribution of the angles.

In this section we derive for $i \geq 1$, the conditional distribution of $\phi_{i+1}$ given $Z^i = z^i$ \( (= (\phi_0, \phi_1, z_1, \ldots, z_i)) \), and $i < N-1$. Our derivation is based on the observation that $\phi_{i+1}$ is the orientation angle (mod $\pi$) of $l_{i+1}$, the line which coincides with the next side of the polygon. The conditional information here is that $z^i$ is already part of the polygon so that $l_{i+1}$ may not intersect any of $\overline{v_i v_2}, \overline{v_2 v_3}, \ldots, \overline{v_i v_{i+1}}$ or equivalently $\overline{v_i v_{i+1}}$. In effect, this
information restricts the range of $\phi_{i+1}$ to be

\begin{equation}
\phi_{i+1} \in [\gamma_i(z_i), \phi_i],
\end{equation}

where $\gamma_i(z_i)$ as defined in Section 3.2 is the angle between $v_i v_{i+1}$ and the eastern horizontal at $v_{i+1}$. See Figure 3.3. Note that in terms of $(x_i, y_i)$, the (realized) coordinates of $v_{i+1}$ from (2.10),

\begin{equation}
\gamma_i(z_i) = \tan^{-1}(y_i/x_i) - \pi.
\end{equation}

![Figure 3.3]

\begin{align*}
\text{It follows from (2.4) that the conditional distribution of } \phi_{i+1} \text{ given } Z^i = z^i \text{ and } i < N-1 \text{ is given by}

(3.17) \quad &dF(\phi_{i+1}|z^i, i < N-1) = dF(\phi_{i+1}|\phi_{i+1} \in [\gamma_i(z_i), \phi_i]) \\
&= \sin(\phi_i - \phi_{i+1})dG(\phi_{i+1})/\gamma_i(z_i) \sin(\phi_i - \phi)dG(\phi)
\end{align*}

for $\phi_{i+1} \in [\gamma_i(z_i), \phi_i]$.

In the Appendix we derive the identity
(3.18) \[ q(t; \phi_i) = \tau \int_{\gamma_i(t)}^{\phi_i} \sin(\phi_i - \phi) dG(\phi), \]

which, loosely speaking, relates the intensity of the nonhomogeneous Poisson process in (3.9) of intersections at \( w_{it} \) which do not cross through \( v_i \), to the Poisson line process measure of the set of lines which are potential candidates for such an event. This identity yields the reexpression of the conditional distribution (3.17),

(3.19) \[ dF(\phi_{i+1} | z_i, i < N-1) = \tau \frac{\sin(\phi_i - \phi_{i+1})}{\sin(\phi_i - \phi_{i+1})} dG(\phi_{i+1})/q(z_i; \phi_i), \]

for \( \phi_{i+1} \in [\gamma_i(z_i), \phi_i] \). The reexpression in (3.19) provides both a convenient simplification of the joint density of \( Z^N \) in Section 4, and a fast procedure for the simultaneous simulation of side lengths and angles in Section 5.

3.4 The stochastic construction of \( Z^N \).

We are finally ready to specify the sequential stochastic construction of \( Z^N \). The first \( 2N-1 \) coordinates of \( Z^N \), namely \( Z^{N-1} \), are first obtained from a sequential realization of the sequence of angles and side lengths

(3.20) \[ Z_i = (\phi_0, \phi_1, Z_1, \phi_2, Z_2, \ldots, \phi_i, Z_i) \]

from the distributions (3.4), (3.13) and (3.19), by using the stopping time

(3.21) \[ N-1 = \inf\{i : Z_i = u_i\}, \]

where \( u_i \) is given by (3.12). The final coordinates \( \phi_N \) and \( Z_N \) are then determined by \( Z^{N-1} \) through \( \phi_N = \phi_0 - \pi \) and \( Z_1 Z_i \sin \phi_i = 0 \) or \( Z_1 Z_i \cos \phi_i = 0 \) as in (2.7).

4. The joint distribution of \( Z^N \).

In this section, we use the intermediate conditional distributions from the
sequential stochastic construction in the previous section to obtain the joint
density of \( Z^N \). We begin by defining the range of \( Z^N \). Using \( u_i \) in (3.12) and
\( \gamma_i(z_i) \) in (3.16), let
\[
\mathcal{V}^n = \{ z^n : 0 \leq \phi_0 \leq \phi_1 < \pi, \ z_i \in [0, u_i) \},
\]
(4.1)
\[
\phi_{i+1} \in [\gamma_i(z_i), \phi_i], \quad \text{for } i = 1, \ldots, n-2,
\]
\[
\sum_{i=1}^{n} z_i \sin \phi_i = \sum_{i=1}^{n} z_i \cos \phi_i = 0, \quad \phi_n = \phi_0 - \pi
\]
denote the set of all possible configurations (up to translation) of \( n \)-sided
convex polygons \( z^n = (\phi_0, \phi_1, z_1, \ldots, \phi_n, z_n) \). The range of \( Z^N \) is then the set of
all possible polygon configurations
(4.2)
\[
\mathcal{V} = \bigcup_{n=3}^{\infty} \mathcal{V}^n.
\]

Because for \( z^n \in \mathcal{V}^n \), the event \( \{ Z^N = z^n \} \) corresponds to the event
\( \{ \phi^{n-1} = \phi^{n-1}, N = n \} \), the distribution of \( Z^N \) over \( \mathcal{V}^n \) may be expressed as
(4.3)
\[
dF(z^n) = dF(\phi^{n-1}) P[N = n|\phi^{n-1}]
\]
\[
= dF(\phi_0, \phi_1) \left( \prod_{i=1}^{n-2} dF(z_i|\phi_i) dF(\phi_{i+1}|z_i) \right) P[N = n|\phi^{n-1}]
\]
for \( z^n \in \mathcal{V}^n \). Inserting the conditional densities from (3.4), (3.13), (3.14) and
(3.19) into (4.3) yields
(4.4)
\[
dF(z^n) = \frac{2}{\lambda} \sin(\phi_1 - \phi_0) dG(\phi_1) dG(\phi_0)
\]
\[
\cdot \left( \prod_{i=1}^{n-2} \exp(-Q(z_i; \phi_i)) dz_i \tau \sin(\phi_i - \phi_{i+1}) dG(\phi_{i+1}) \right)
\]
\[
\cdot \exp(-Q(u_{n-1}; \phi^{n-1}))
\]
for \( z^n \in \mathcal{V}^n \).

The expression (4.4) can be substantially simplified by noting that for
\[ z^n \in \psi^n, \]

(4.5) \[
Q(u_{n-1}; \phi^{n-1}) + \sum_{i=1}^{n-2} Q(z_i; \phi^i) = \frac{\pi}{2} \sum_{i=1}^{n} z_i \lambda(\phi_i)
\]

which follows from (3.7) by using \( d_{i+1}(0) = d_i(z_i) \), \( \gamma_{i+1}(0) = \gamma_i(z_i) \),
\( u_{n-1} = z_{n-1}, d_{n-1}(z_{n-1}) = z_n \), and \( \gamma_{n-1}(z_{n-1}) = \phi_n \). Substituting (4.5) into (4.4) yields

(4.6)

\[
dF(z^n) = \frac{2\pi^{n-2}}{\lambda} \left( \prod_{i=0}^{n-2} |\sin(\phi_i - \phi_{i+1})| \right) \cdot \exp\left[ -\frac{\pi}{2} \sum_{i=1}^{n} z_i \lambda(\phi_i) \right] \left( \prod_{i=1}^{n-2} dz_i \right) \left( \prod_{i=0}^{n-1} dG(\phi_i) \right)
\]

for \( z^n \in \psi^n \). We remind the reader that the support of (4.6) is \( \psi \) from (4.2).

In the isotropic case, where \( Z_N \) is rotated so that \( \phi_0 = 0 \), the joint density of \( Z_N \) is obtained from (4.3) by replacing \( dF(\phi_0, \phi_i) \) by the marginal density \( dF(\phi_i) \) from (3.5), and using \( dG(\phi) = d\phi/\pi \), \( \lambda(\phi) \equiv 2/\pi \) here,
\( \frac{\pi}{2} \sum_1^N z_i \lambda(\phi_i) = \frac{\pi}{\pi} \sum_1^N z_i = \frac{\pi}{\pi} s \). This density is then the special case of (4.6),

(4.7)

\[
dF(z^n) = \left( \frac{\pi}{\pi} \right)^n \left( \frac{\pi}{\pi} \right)^{n-1} \prod_{i=0}^{n-2} |\sin(\phi_i - \phi_{i+1})| \cdot \exp\left[ -\frac{\pi}{2} s \right] \left( \prod_{i=1}^{n-2} dz_i \right) \left( \prod_{i=1}^{n-1} d\phi_i \right)
\]

with support on \( \{ z^n : z^n \in \psi^n \text{ and } \phi_0 = 0 \} \). By suitable reparametrization the isotropic density (4.7) is the same as the isotropic ergodic density derived by Miles (1973).

Although the polygon densities in (4.6) and (4.7) can be used to obtain some partial results on the distributions of \( N \), \( S \) and \( \Lambda \), (see Section 6), they are unfortunately not manageable enough to obtain the full distributions. For this reason we turn to estimating these distributions by using a simulated random sample.
5. **Simulating the sequential stochastic construction.**

In this section we suggest general approaches for simulating the coordinates for the sequential stochastic construction of \(Z_N\). For the isotropic case, these approaches lead to efficient simulation procedures based only on simulated independent uniform \([0,1]\) random variables.

5.1 **Simulating the initial angles.**

For the simulation of the initial angles \((\theta_0, \theta_1)\) with the general distribution (3.4), it will usually be more convenient to obtain \(\theta_0 = \min[\theta, \theta']\) and \(\theta_1 = \max[\theta, \theta']\) where \(\theta, \theta'\) have joint distribution (3.1) as indicated in Section 3.1. The angles \(\theta, \theta'\) can then be obtained sequentially from \(dG(\theta')\) and \(dF(\theta | \theta') = \frac{1}{\lambda(\theta')} |\sin(\theta - \theta')| dG(\theta)\). Depending on the form of \(G\), it may be possible to simulate these univariate distributions by standard simulation techniques such as inversion or the general rejection method (George 1982; Bratley, Fox and Schrage 1983).

In the isotropic case where \(\theta_0 = 0\) and construction of \(Z_N\) begins with the angle \(\theta_1\) which has the distribution (3.5), it may be more convenient to obtain \(\theta_1 = \max[(\theta-\theta'), (\theta'-\theta)]\) where \(\theta, \theta'\) have joint distribution (3.3). Because of the simple form of \(G\), namely \(dG(\theta) = d\theta/\pi\), these angles may be obtained efficiently by inversion using \(\theta = \pi \xi\) and \(\theta' = (\cos^{-1}(1-2\xi') + \theta) \mod \pi\) where \(\xi, \xi'\) are two independent uniform \([0,1]\) random variables.

5.2 **Fast simulation of the side length and angle pairs.**

For the simulation of the side length \(Z_1\) and the angle \(\theta_{i+1}\) with respective distributions (3.13) and (3.19), the following result provides a convenient method for obtaining the pair \(Z_1, \theta_{i+1}\) simultaneously.

**Theorem 5.1.** Let \(S_1, S_2, \ldots\) and \(\theta_1, \theta_2, \ldots\) be two independent sequences of independent and identically distributed random variables with respective
distributions

\begin{equation}
(5.1) \quad dF(s) = \tau \lambda(\phi_1) \exp(-\tau \lambda(\phi_1)s)ds, \quad s \in [0, \infty),
\end{equation}

and

\begin{equation}
(5.2) \quad dF(\theta) = \frac{1}{\lambda(\phi_1)} \sin(\phi_1 - \theta)dG(\theta), \quad \theta \in [\phi_1 - \pi, \phi_1].
\end{equation}

For \( \gamma_i(t) \) as in Section 3.2, (assuming \( \phi_i \) has been specified), define

\[ K = \inf\{k : \theta_k \in (\gamma_i(S_{i+1}, \phi_i)) \}. \]

Then for \( u_i \) as in (3.12), the distribution of \( Z_i = \min[\sum_{j=1}^{K} S_j, u_i] \) is given by (3.10). Furthermore, if \( Z_i < u_i \), then the distribution of \( \phi_{i+1} = \theta_K \) is given by (3.19).

**Proof:** It follows from Theorem 1 of Lewis and Shedler (1979) that \( \sum_{j=1}^{K} S_j \) is the distance to the first arrival of a nonhomogeneous linear Poisson process with intensity

\begin{equation}
(5.3) \quad \psi(t) = \tau \lambda(\phi_1) \mathbb{P}[\theta_k \in (\gamma_i(t), \phi_i)]
\end{equation}

\[ = \tau \int_{\gamma_i(t)}^{\phi_i} \sin(\phi_1 - \phi)dG(\phi) = q(t; \phi_1), \]

where the last equality is given by (3.18). Thus, the distribution of \( \sum_{j=1}^{K} S_j \) is identical to that of \( Z_i \) given by (3.10). The first assertion then follows from (3.11). The second assertion is immediate.

\[ \square \]

Theorem 5.1 may be especially useful because the simulation of \( Z_i, \phi_{i+1} \) involves only repeated simulation of random variables with distributions (5.1) and (5.2) until \( Z_i \) is obtained. Although the sequence \( S_1, S_2, \ldots \) from (5.1) may always be obtained by inversion from simulated independent uniform \([0, 1]\) random variables, the appropriate simulation technique for obtaining \( \theta_1, \theta_2, \ldots \) from
(5.2) depends on the form of \( G \). In the isotropic case, both sequences may be obtained by inversion using \( S_j = -(\pi/\tau) \log \xi_j^S \) and \( \theta_j = \cos^{-1}(1-2\xi_j^\theta) + \phi_1 - \pi \) where \( \xi_1^S, \xi_2^S, \ldots \) and \( \xi_1^\theta, \xi_2^\theta, \ldots \) are independent sequences of independent uniform \([0,1]\) random variables.

6. Simulating a Large Random Sample From Isotropic \( P^* \).

To obtain estimates of some of the unknown aspects of the distributions of \( N, S \) and \( A \) in isotropic \( P^* \), we simulated a random sample of 2,500,000 polygons. This sample was obtained by independent repetitions of the sequential stochastic construction outlined in Section 3.4, using the specific methods for the isotropic case described in Section 5.\(^1\) Without loss of generality, we used intensity \( \tau = 1 \) throughout. Letting \( N(\tau), S(\tau) \) and \( A(\tau) \) be random variables denoting number of sides, perimeter and area of a random polygon from isotropic \( P^* \) with intensity \( \tau \), it follows immediately from (4.7) that \( N(\tau) = N(1) \), \( S(\tau) = \tau S(1) \) and \( A(\tau) = \tau^2 A(1) \).

In Tables 1a, 1b and 1c, we present estimates of the cumulative distribution functions of \( N, S \) and \( A \) respectively. These estimates are simply the sample proportions which correspond to the population probabilities. Based on the Kolmogorov statistic, the width of the asymptotic 99% confidence bands for each of the true distribution functions is .00206 (Kendall and Stuart 1973, page 473). For comparison, we note that integration of (4.7) over the appropriate region yields \( P[N = 3] = 2-(\pi^2/6) = .355066 \) (Miles 1964, 1973; George 1982), and \( P[N = 4] = -(1/3) - (7\pi^2)/36 + 4\int_0^{\pi/2} x^2 \cot x \, dx = .381466 \), where this last approximation was obtained by numerical methods, (Tanner 1983a). It is comfort-

\(^1\) The simulation was run on a PDP 10/KI computer using the SAIL programming language. The uniform standard deviates were obtained from the random number generator RAN. Polygons were processed at a rate of 8745 polygons per minute of CPU time.
ing to note the close agreement between our estimates and these known values.

In Table 2a, we present sample moments along with their estimated standard errors. For comparison, some known and numerically approximated values for these moments are presented in Table 2b. These have been obtained using a variety of arguments by Goudsmit (1945), Miles (1964, 1973), D. G. Kendall (unpublished, see Miles 1964), Solomon (1978), and Tanner (1983b). Again, note the close agreement between our estimates and the true values.

There are many more aspects of the joint distribution of $N$, $S$ and $A$ which we have not addressed with this Monte Carlo study, such as investigating some of the unknown cross product moments. However, our intention was not to comprehensively investigate all aspects of this joint distribution, but rather to illustrate how the sequential stochastic construction can be used as a powerful tool in the continuing investigation of the polygon distributions.

7. Acknowledgement. This article is based on the author's Ph.D. dissertation for the Department of Statistics, Stanford University. The author deeply appreciates the guidance and encouragement provided by Professor Herbert Solomon.

Appendix.

We proceed to derive (3.18). Let $(x_t, y_t)$ be the Cartesian coordinates of the point $w_{it}$ on $x_i$. In terms of $(x_{i-1}, y_{i-1})$, the coordinates of $x_i^*$ and $y_i^*$, $x_t = t \cos \phi_i + x_{i-1}$ and $y_t = t \sin \phi_i + y_{i-1}$. For notational convenience let $d_t = d_i(t)$ and $\gamma_t = \gamma_i(t)$. These may expressed in terms of $(x_t, y_t)$ as

\[(A.1) \quad d_t^2 = x_t^2 + y_t^2 \quad \text{and} \quad \gamma_t = \tan^{-1}(y_t/x_t) - \pi\]
It follows that

\[
\sin(\phi - \gamma_t) = -\sin(\phi - \tan^{-1}(y_t/x_t)) = (y_t \cos \phi - x_t \sin \phi)/d_t
\]

which in turn yields

\[
\frac{\partial}{\partial t} \lambda(\gamma_t) = d_t \int_{\gamma_t}^{\gamma_{t+\pi}} \sin(\phi - \gamma_t) dG(\phi) = \int_{\gamma_t}^{\gamma_{t+\pi}} (y_t \cos \phi - x_t \sin \phi) dG(\phi).
\]

By the chain rule,

\[
\frac{\partial}{\partial t} \lambda(\gamma_t) = \int_{\gamma_t}^{\gamma_{t+\pi}} (\sin \phi \cos \phi - \cos \phi \sin \phi) dG(\phi) = \int_{\gamma_t}^{\gamma_{t+\pi}} \sin(\phi - \phi) dG(\phi).
\]

Thus,

\[
q(t; \phi) = \frac{\tau}{2} \left[ \lambda(\phi) + \frac{\partial}{\partial t} \lambda(\gamma_t) \right]
= \frac{\tau}{2} \left[ \int_{\phi}^{\phi_{t+\pi}} \sin(\phi - \phi) dG(\phi) + \int_{\gamma_t}^{\gamma_{t+\pi}} \sin(\phi - \phi) dG(\phi) \right]
= \frac{\tau}{2} \left[ -\int_{\gamma_t}^{\gamma_{t+\pi}} \sin(\phi - \phi) dG(\phi) + \int_{\gamma_t}^{\gamma_{t+\pi}} \sin(\phi - \phi) dG(\phi) \right]
= \tau \int_{\gamma_t}^{\gamma_{t+\pi}} \sin(\phi - \phi) dG(\phi)
\]

which is the desired assertion.

References


### Table 1a. Distribution Estimates For $N$

<table>
<thead>
<tr>
<th>$n$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(N = n)$</td>
<td>.3552</td>
<td>.3814</td>
<td>.1895</td>
<td>.05870</td>
<td>.01275</td>
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<table>
<thead>
<tr>
<th>$n$</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(N = n)$</td>
<td>.002082</td>
<td>.0002712</td>
<td>.0000180</td>
<td>.0000028</td>
<td>.0000004</td>
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</tbody>
</table>

Note: No polygons with $N > 12$ were observed.

### Table 1b. Distribution Estimates For $S$

<table>
<thead>
<tr>
<th>$s$</th>
<th>.100</th>
<th>.250</th>
<th>.500</th>
<th>.750</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(S \leq s)$</td>
<td>.01128</td>
<td>.02842</td>
<td>.05699</td>
<td>.08528</td>
<td>.1135</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$s$</th>
<th>1.50</th>
<th>2.50</th>
<th>3.75</th>
<th>5.00</th>
<th>6.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(S \leq s)$</td>
<td>.1693</td>
<td>.2764</td>
<td>.3995</td>
<td>.5080</td>
<td>.6013</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$s$</th>
<th>7.50</th>
<th>8.75</th>
<th>10.0</th>
<th>12.5</th>
<th>15.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(S \leq s)$</td>
<td>.6801</td>
<td>.7455</td>
<td>.7987</td>
<td>.8765</td>
<td>.9257</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$s$</th>
<th>17.5</th>
<th>20.0</th>
<th>25.0</th>
<th>30.0</th>
<th>50.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(S \leq s)$</td>
<td>.9561</td>
<td>.9744</td>
<td>.9917</td>
<td>.9974</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Note: 50 polygons with $S > 50$ were observed.
Table 1c. Distribution Estimates For A

<table>
<thead>
<tr>
<th>a</th>
<th>.005</th>
<th>.010</th>
<th>.025</th>
<th>.050</th>
<th>.100</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(A ≤ a)</td>
<td>.04536</td>
<td>.06388</td>
<td>.09968</td>
<td>.1392</td>
<td>.1931</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>a</th>
<th>.250</th>
<th>.500</th>
<th>.750</th>
<th>1.00</th>
<th>1.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(A ≤ a)</td>
<td>.2924</td>
<td>.3926</td>
<td>.4615</td>
<td>.5140</td>
<td>.5929</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>a</th>
<th>2.50</th>
<th>5.00</th>
<th>7.50</th>
<th>10.0</th>
<th>12.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(A ≤ a)</td>
<td>.6944</td>
<td>.8228</td>
<td>.8846</td>
<td>.9201</td>
<td>.9424</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>a</th>
<th>15.0</th>
<th>20.0</th>
<th>30.0</th>
<th>50.0</th>
<th>100.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(A ≤ a)</td>
<td>.9574</td>
<td>.9752</td>
<td>.9902</td>
<td>.9978</td>
<td>.9999</td>
</tr>
</tbody>
</table>

Note: 334 polygons with A > 10 were observed.
Table 2a. Sample Moments

<table>
<thead>
<tr>
<th>k</th>
<th>$EN^k$</th>
<th>$ES^k$</th>
<th>$EA^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.99980</td>
<td>6.2841</td>
<td>3.1406</td>
</tr>
<tr>
<td></td>
<td>(.00061)</td>
<td>(.0034)</td>
<td>(.0039)</td>
</tr>
<tr>
<td>2</td>
<td>16.9336</td>
<td>68.415</td>
<td>48.56</td>
</tr>
<tr>
<td></td>
<td>(.0055)</td>
<td>(.077)</td>
<td>(.205)</td>
</tr>
<tr>
<td>3</td>
<td>76.034</td>
<td>1,028.7</td>
<td>1,718.</td>
</tr>
<tr>
<td></td>
<td>(.040)</td>
<td>(2.1)</td>
<td>(23.)</td>
</tr>
<tr>
<td>4</td>
<td>362.11</td>
<td>19,520.</td>
<td>$1,077 \times 10^2$</td>
</tr>
<tr>
<td></td>
<td>(.28)</td>
<td>(68.)</td>
<td>$(42. \times 10^2)$</td>
</tr>
<tr>
<td>5</td>
<td>1,826.6</td>
<td>$4,449 \times 10^2$</td>
<td>$1,033 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>(1.95)</td>
<td>(27. $\times 10^2$)</td>
<td>(97. $\times 10^4$)</td>
</tr>
<tr>
<td>6</td>
<td>9.735.</td>
<td>$1,180 \times 10^4$</td>
<td>$136. \times 10^7$</td>
</tr>
<tr>
<td></td>
<td>(14.)</td>
<td>(12. $\times 10^4$)</td>
<td>(25. $\times 10^7$)</td>
</tr>
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</table>
Table 2b. Known or Approximated Moments

<table>
<thead>
<tr>
<th>k</th>
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<th>$\text{EA}^k$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>$2\pi$</td>
<td>$\pi$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 6.28319$</td>
<td>$= 3.14159$</td>
</tr>
<tr>
<td>2</td>
<td>$(\pi^2 + 24)/2$</td>
<td>$\pi(\pi^2 + 4)/2$</td>
<td>$\frac{4}{\pi}/2$</td>
</tr>
<tr>
<td></td>
<td>$= 16.9356$</td>
<td>$= 68.4438$</td>
<td>$= 48.7045$</td>
</tr>
<tr>
<td>3</td>
<td>$= 76.036405^*$</td>
<td>$= 1030.4005^*$</td>
<td>$4\pi^7/7$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$= 1725.88$</td>
</tr>
<tr>
<td>4</td>
<td>$= 362.084463^*$</td>
<td>$= 19,586.7133^*$</td>
<td>unknown</td>
</tr>
</tbody>
</table>

*results obtained by numerical integration, (Tanner 1983b).*
Sampling Random Polygons

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Statistics & Probability Program Code 1111

December 29, 1986

UNCLASSIFIED

Also issued as Technical Report No. 43, University of Chicago, School of Business.

geometric probability, Poisson line process.

Every realization of a Poisson line process is a set of lines which subdivides the plane into a population of nonoverlapping convex polygons. To explore the unknown statistical features of this population, an alternative stochastic construction of random polygons is developed. This construction, which is based on an alternating sequence of random angles and side lengths, provides a fast simulation method for obtaining a random sample from the polygon population. For the isotropic case, this construction is used to obtain a random sample of 2,500,000 polygons, providing the most precise estimates to date of some of the unknown distributional characteristics.