JOINT MOMENTS OF THE MAXIMUM
IN A CRITICAL BRANCHING PROCESS

BY

HOWARD WEINER

TECHNICAL REPORT NO. 397
OCTOBER 15, 1987

PREPARED UNDER CONTRACT
N00014-86-K-0156 (NR-042-267)
FOR THE OFFICE OF NAVAL RESEARCH

Reproduction in Whole or in Part is Permitted
for any purpose of the United States Government

Approved for public release; distribution unlimited.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
JOINT MOMENTS OF THE MAXIMUM
IN A CRITICAL BRANCHING PROCESS

BY

HOWARD WEINER

TECHNICAL REPORT NO. 397
OCTOBER 15, 1987

Prepared Under Contract
N00014-86-K-0156 (NR-042-267)
For the Office of Naval Research

Herbert Solomon, Project Director

Reproduction in Whole or in Part is Permitted
for any purpose of the United States Government

Approved for public release; distribution unlimited.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
1. INTRODUCTION

Let \( \{Z_n\} \), \( n \geq 0 \), with \( Z_0 = 1 \) denote a critical Galton-Watson branching process, and let the \( \sigma \)-field \( F_k = \sigma(Z_1, Z_2, \ldots, Z_k) \) for \( k \geq 1 \). Denote

\[
M_n = \max_{1 \leq k \leq n} Z_k.
\]  

(1.1)


Assume \( E M_n^r < \infty \) for all \( r \geq 1, n \geq 1 \).

Moments of \( M_n \) have been studied by a number of authors (See Athreya [1], Kammerle and Schuh [2], Pakes [3], Weiner [5]). See Kammerle and Schuh for further background on this problem and other references. In this paper the asymptotic order of magnitude of joint moments of \( M_k, M_n \) for \( n \gg k \gg 1 \) are studied. Much use is made of the results of Athreya [1], Kammerle and Schuh [2] to obtain the asymptotic orders of magnitude, and relevant results are given in the next section.

2. PRIOR RESULTS AND NOTATION

The results and notation (2.1) - (2.4) will be used assuming \( Z_0 = 1 \):

(Kammerle and Schuh [2], p. 602).

\[
(Z_n, F_n); n \geq 0 \text{ is a martingale}
\]  

(2.1)

\[
E Z_n = 1, \quad \text{Var} \ Z_n = n \sigma^2, \quad n \geq 1
\]  

(2.2a)

\[
P(Z_n = 0) \to 1 \text{ as } n \to \infty
\]  

(2.2b)
The following notation will be used in this paper:

for \( a_n, b_n > 0 \),

\[
    a_n \sim b_n \iff 
\]

\[
    \lim_{n \to \infty} \frac{a_n}{b_n} = a > 0 \text{ and } \lim_{n \to \infty} \frac{a_n}{b_n} \leq b < \infty \text{ where } a, b > 0 \text{ are constants.}
\]

\[
    a \leq \lim_{n \to \infty} a_n \leq b \iff 
\]

\[
    a \leq \lim a_n \leq \lim a_n \leq b
\]

The following result is critical for the argument:

If \( E(Z_1^2) < \infty \) then if \( Z_0 = \emptyset \),

\[
    E(M_n | Z_0 = \emptyset) / \log n \to \emptyset \text{ as } n \to \infty
\]

(Athreya [1], p. 1).

We note that it was earlier shown (Kammerle and Schuh [2], p.603) that

for \( Z_0 = 1 \),

\[
    \frac{1}{e} \leq \lim_{n \to \infty} E(M_n) / \log n \leq 1.
\]

See also (Pakes [3], Weiner [5]) for earlier results.

The other results crucial to the argument are as follows (Kammerle and Schuh [2], pp. 607-610):

If \( Z_0 = k \geq 1 \) and \( E(Z_n^r | Z_0 = k) < \infty \) all \( r \geq 1 \), \( n \geq 1 \), then for \( r \geq 2 \),

\[
    \lim_{n \to \infty} \frac{E(Z_n^r | Z_0 = k)}{n^{r-1}} = \left( \frac{q}{2} \right)^{r-1} \frac{r!}{r!}
\]

(2.6)
and also for \( k \geq 1 \).

\[
  k \left( \frac{\sigma^2}{2} \right)^{r-1} \leq \lim_{n \to \infty} \frac{E(M_n^r | Z_0 = k)}{n^{r-1}} \leq k \left( \frac{r-1}{r} \right)^{\frac{\sigma^2}{2}} \frac{r}{r-1}
\]

(2.7)

**LEMMA 1.** For integers \( r, \alpha \geq 1 \), \( \theta \geq 1 \).

\[
  \left\{ \frac{\sigma^2}{2} \right\}^{r+\alpha-1} \leq E(M_n^\alpha | Z_0 = \theta) / n^{r+\alpha-1} \leq \left( \frac{r+\alpha}{r+\alpha-1} \right)^{r+\alpha-1} \left( \frac{\sigma^2}{2} \right)^{r+\alpha-1} (r+\alpha)!
\]

(2.8)

**PROOF.**

\[
  E(Z_n^{r+\alpha} | Z_0 = \theta) \leq E(M_n^\alpha | Z_0 = \theta) \leq E(M_n^r | Z_0 = \theta).
\]

The result follows immediately from (2.6), (2.7). \( \square \)

**LEMMA 2.** For \( 1 \leq k < n \), \( r, \alpha \geq 1 \)

\[
  E(Z_k^\alpha Z_n^\alpha) = E(Z_k^{r+\alpha}) \quad (2.9i)
\]

\[
  E(Z_k^r Z_n) = E(Z_k^{r+1}) \quad (2.9ii)
\]

\[
  E(M_k^r Z_n) = E(M_k^r Z_k) \quad (2.9iii)
\]

**PROOF.** The results follow by similar arguments and only the first result will be indicated.

\[
  E(Z_k^r Z_n^\alpha) = E(E(Z_k^r Z_n^\alpha | F_k)) = E(Z_k^r E(Z_n^\alpha | F_k)) \geq E(Z_k^{r+\alpha})
\]

since \( E(Z_k^{r,0} Z_n^\alpha) \{ Z_n \} \) is a martingale and by Jensen's Conditional inequality

(Rao [4] p. 110). \( \square \)
3. JOINT MOMENTS

THEOREM 1. For $1 \leq k < n$, $r \geq 1$, $Z_0 = 1$.

\[
\frac{2^r}{(2^r)^{(r+1)!}} \leq \lim_{k \to \infty} \frac{E(M_k^r M_n^r)/(k^r \log(n-k))}{\binom{n-k}{r}} \leq \left( \frac{r+1}{r} \right)^r \left( \frac{2^r}{2^r} \right) (r+1)! \tag{3.1}
\]

PROOF.

\[
E(M_k^r M_n^r) = E(E(M_k^r M_n^r | F_k)) = E(M_k^r E(M_n^r | F_k)),
\]

\[
E(M_k^r Z_k I(Z_k \geq 1) \log(n-k) + E(M_k^r M_k I(Z_k = 0)) \] by (2.5) where $I(A)$ is the indicator of $A$. For $n-k \gg 1$, $k \gg 1$,

\[
E(M_k^r M_n^r) \sim \log(n-k) E(M_k^r Z_k) + E(M_k^r I(Z_k = 0)). \tag{3.2}
\]

From

\[
0 \leq E(M_k^r I(Z_k = 0)) \leq E(M_k^r), \tag{3.3}
\]

using $n-k \to \infty$ and $Z_0 = 1$, then (3.2), (3.3), and (2.8) yield the result. \[ \square \]

THEOREM 2. For $1 \leq k < n$, $r \geq 1$, $\alpha \geq 2$.

\[
\frac{2^{r+\alpha-1}}{(2^r)^{(r+1)!}} \leq \lim_{k \to \infty} \frac{E(M_k^r M_k^\alpha n^r \log(n-k)^{\alpha-1}}{\binom{n-k}{r}} \leq \left( \frac{\alpha}{\alpha-1} \right)^{r+1} \left( \frac{2^r}{2^r} \right) (r+1)! \tag{3.4}
\]

PROOF. For $n-k \gg 1$, $k \gg 1$, $Z_0 = 1$.

\[
E(M_k^r M_k^\alpha) = E(E(M_k^r M_k^\alpha | F_k)) = E(M_k^r E(M_k^\alpha | F_k)) \sim E(M_k^r Z_k I(Z_k \geq 1)(n-k)^{\alpha-1} + E(M_k^r I(Z_k = 0))
\]

\[
\sim E(M_k^r Z_k)(n-k)^{\alpha-1} + E(M_k^r I(Z_k = 0))^\alpha \sim \alpha - 1 E(M_k^r Z_k). \tag{2.7}, \tag{2.2b}
\]

Then (2.7), (2.8) yields the result. \[ \square \]
4. HIGHER ORDER JOINT MOMENTS

An example of possible higher-order asymptotic joint moment behavior is given. For ease of exposition, only orders of magnitude but not the explicit constants, involved in upper and lower bounds, are given.

THEOREM 3. For \( Z_0 = 1, t \geq 1, \alpha \geq 1, k < r < 1 \) and \( k \gg 1, n-r \gg 1 \), \( r-k \gg 1 \) and in addition

\[
E(M_k^\alpha M_r^t M_n) \sim (k)^{\alpha+1} (r-k)^t \log(n-r). \tag{4.1}
\]

PROOF. For \( k < r < n \), using (3.1)

\[
E(M_k^\alpha M_r^t M_n) = E(M_k^\alpha E(M_r^t M_n | F_k)) \sim E(M_k^\alpha Z_k I(Z_k \geq 1)) (r-k)^t \log(n-r)) +
\]

\[
E(M_k^{\alpha+t+1} I(Z_k = 0) \tag{4.2}
\]

Since \( E(M_k^\alpha Z_k I(Z_k \geq 1)) = EM_k^\alpha Z_k \),

then by (2.2b)

\[
E(M_k^\alpha M_r^t M_n) \sim (r-k)^t \log(n-r) E(M_k^\alpha Z_k) + E(M_k^{\alpha+t+1}). \tag{4.3}
\]

By (2.7), (2.8) applied to (4.3),

\[
E(M_k^\alpha M_r^t M_n) \sim k^\alpha (r-k)^t \log(n-r) + k^{\alpha+t}. \tag{4.4}
\]

Using the hypothesis that \( r-k \gg k \), the result follows. \( \square \)
THEOREM 4. For $Z_0 = 1$, $t \geq 1$, $a \geq 1$, $q \geq 2$ and $k < r < n$ with $k \gg 1$, $n-r \gg 1$, $r-k \gg 1$ and in addition

$$n-r \gg k, \quad r-k \gg k$$

then

$$E(M_k^a M_r^t M_n^0) \sim k^a(n-r)^{q-1}(r-k)^t$$  (4.5)

PROOF. For $k < r < n$, $k \gg 1$, $n-r \gg 1$, $r-k \gg 1$,

$$E(M_k^a M_r^t M_n^0) = E(M_k^a E(M_r^t M_n^0 | F_k)) \sim E(M_k^a Z_k I(Z_k \geq 1))(n-r)^{q-1}(r-k)^t$$

$$+ E(M_k^a + t + q I(Z_k = 0))$$

$$\sim E(M_k^a Z_k)(n-r)^{q-1}(r-k)^t + E(M_k^a + t + q)$$  (4.6)

by (2.2b) and (3.4).

By (2.7), (2.8) applied to (4.6),

$$E(M_k^a M_r^t M_n^0) \sim k^a(n-r)^{q-1}(r-k)^t + k^{a+t+1}$$  (4.7)

Applying $n-r \gg k$, $r-k \gg k$ to (4.7) yields the result. $\Box$
REFERENCES


**Title:** Joint Moments Of The Maximum In A Critical Branching Process

**Author:** Howard Weiner

**Performing Organization:**
- Department of Statistics
- Stanford University
- Stanford, CA 94305

**Controlling Office:**
- Office of Naval Research
- Statistics & Probability Program Code 1111

**Contract or Grant Number:** N00014-86-K-0156

**Report Date:** October 15, 1987

**Number of Pages:** 9

**Security Class. (of this report):** UNCLASSIFIED

**DISTRIBUTION STATEMENT:**
- APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

**KEY WORDS:** Critical branching process, Joint moments.

**ABSTRACT:**
The asymptotic order of magnitude of the joint moments of the maxima in a critical Galton Watson process are given.